General Relativistic N-body Simulations for Cosmic Large Scale Structure

based on arXiv:1308.6524

and work in progress with C. Clarkson, E. DiDio, R. Durrer, and M. Kunz

Julian Adamek

Heidelberg, Germany, 13.11.2013
GR N-body Simulations for Cosmic LSS

Menu

Entrée: A Case for GR
- Newton vs. Einstein
- The Issue of Backreaction
- Relativistic Sources of Stress-Energy

Plat Principal: The Framework
- Choice of Variables
- Weak Field Approximation
- System of Equations
- Algorithmic Solutions

Dessert: Numerical Results
- A Plane-Symmetric Setup
In order to study the regime of nonlinear structure formation, large N-body simulations are the method of choice.

N-body simulations use Newton’s law of gravity

Works well for nonrelativistic matter (CDM), because of

- Exact correspondence between Newtonian gravity and GR on the background solution (FRW)
- Exact correspondence also on the level of linear (scalar) perturbations
- Nonlinear scale $\ll$ Hubble scale

see, e.g., Green & Wald 2012
The Newtonian picture has several drawbacks, though

- Strong assumption about material content of the Universe
- Misses some degrees of freedom (gravity waves!)
- Gauge issues are not apparent
- Trivial propagation of light beams (relativistic effects have to be put back “by hand”)

A unified relativistic treatment of structure formation would automatically solve these issues. When constructing observables (galaxy catalogs, lensing maps etc.), all geometric effects and gauge issues would be treated in a transparent way.
A Case for GR
The Issue of Backreaction

Long standing question: how important is nonlinear evolution of structure for understanding & interpreting the observed “average” cosmological evolution?


This issue has many facets. Some can be addressed in the Newtonian picture, others require a relativistic treatment (→ perturbation theory, exact solutions . . . ). A unified relativistic treatment of structure formation would be the logical framework to address the issue in full generality.
GR effects are expected to be important for intrinsically relativistic entities

- Cosmic strings
- Dynamical Dark Energy
- Relativistic particles (neutrinos?)
- . . .

In order to test some of the proposed alternatives/extensions to $\Lambda$CDM, general relativistic simulations may be necessary in order to obtain percent accuracy required by future observations (e.g. Euclid)
The Framework
Choice of Variables

Metric of perturbed FRW in “longitudinal gauge”

\[ ds^2 = -(1 + 2\psi) \, dt^2 + a^2(t)[(1 - 2\phi) \, \delta_{ij} + h_{ij}] \, dx^i \, dx^j - 2B_i \, dx^i \, dt \]

Gauge condition: \( \nabla^i B_i = \nabla^i h_{ij} = \delta^{ij} h_{ij} = 0 \)

Stress-energy tensor

\[ T^\mu_\nu = \tilde{T}^\mu_\nu + \delta T^\mu_\nu, \quad \tilde{T}^\mu_\nu = \text{diag}(\bar{\rho}, \bar{P}, \bar{P}, \bar{P}), \quad \bar{P} = w\bar{\rho} \]

Fix “background” equation of state for each constituent → background scale factor \( a \) solves Friedmann’s equations for \( \tilde{T}^\mu_\nu \)
The Framework
Weak Field Approximation

Perturbative approach:

- Metric perturbations $\Psi, \Phi, \ldots$ remain small in cosmological context ($\sim 10^{-5}$) $\rightarrow$ keep only to first order
- Spatial derivatives $\Psi,_{i}, \ldots$ are $\sim v$ ($\sim 10^{-3}$) $\rightarrow$ keep to quadratic order
- Second spatial derivatives $\Delta \Psi, \ldots$ are $\sim \delta$ and therefore non-perturbative

See again Green & Wald 2012
The Framework
System of Equations

"\[ G^0_0 = 8\pi G T^0_0 \]":

\[
\frac{1}{a^2} (1 + 4\Phi) \Delta \Phi - 3H \dot{\Phi} - 3H^2 \Psi + \frac{3}{2a^2} (\nabla \Phi)^2 = -4\pi G \delta T^0_0
\]

"\[ G^i_i - 3G^0_0 - \frac{1}{H} \dot{G}^0_0 = 8\pi G (T^i_i - 3T^0_0 - \frac{1}{H} \dot{T}^0_0) \]":

\[
(1 + 2\Phi - 2\Psi) \Delta \Psi - (\nabla \Psi)^2 - \nabla \Psi \nabla \Phi + \frac{1}{H} \partial_t \left[ \Delta \Phi + 4\Phi \Delta \Phi + \frac{3}{2} (\nabla \Phi)^2 \right] =
\]

\[
4\pi G \frac{a}{H} \left[ \delta T^i_{0,i} - \delta T^i_0 \left( 3\Phi_{,i} - \Psi_{,i} + a \dot{B}_i \right) \right.
\]

\[
- a \dot{\Phi} \left( 3\delta T^0_0 - \delta T^i_i \right) - \frac{a}{2} \delta^{ik} \dot{h}_{jk} \delta T^j_i \right]
\]
The Framework
System of Equations (cont.)

“\(G_0^i = 8\pi G T_0^i\)”:

\[-\frac{4}{a^2} \Delta B_i - \frac{1}{a} \Phi_i - \frac{H}{a} \Psi_i = 4\pi G \delta T_i^0\]

“\(G^i_j - \frac{1}{3} \delta^i_j G^k_k = 8\pi G (T^i_j - \frac{1}{3} \delta^i_j T^k_k)\)”:

\[\ddot{h}_{ij} + 3H \dot{h}_{ij} - \frac{1}{a^2} \Delta h_{ij} + \frac{1}{a} \left( \dot{B}_{(i,j)} + 2H B_{(i,j)} \right)\]
\[+ \frac{1}{a} \ “\text{traceless part} \left[ (1 + 4\Phi) \Phi_{,ij} - (1 + 2\Phi - 2\Psi) \Psi_{,ij} \right. \]
\[+ \left. \Psi_{,i} \Psi_{,j} - 2\Phi (\psi_{,i} \psi_{,j}) + 3\Phi (\psi_{,i} \psi_{,j}) \right] = 8\pi G \left( \delta_{ik} \delta T^k_j - \frac{1}{3} \delta_{ij} \delta T^k_k \right)\]
In order to close the system of equations, one needs evolution equations for all sources of stress-energy.

Geodesic equation for nonrelativistic massive particles

\[ \ddot{v}^i + Hv^i + \delta^{ij} \left( \frac{1}{a} \dot{\Psi}_{,j} - \dot{B}_j - HB_j + \frac{2}{a} B_{[j,k]} v^k \right) = 0 \]

determines the evolution of the particle ensemble and therefore the evolution of the full \( T^{\mu}_\nu \) of CDM.
The Framework
Algorithmic Solutions

Φ: parabolic equation (diffusion type)

• Explicit scheme too inefficient (Courant condition!)
• First-order (in time) implicit scheme shows excellent performance in 1D tests
• ADI (Alternating Direction Implicit) scheme in 3D should perform well (easy to implement & parallelizable)

Ψ: elliptic equation

• Nonlinear Gauß-Seidel / Multigrid solver shows excellent performance in 1D tests
• Same class of solvers is already used for the Poisson equation in modern Newtonian codes
$B_i$: linear elliptic operator

Two possibilities:

- Solve in Fourier space (transverse component can easily be extracted, but incompatible with AMR)
- Use Multigrid solver (gauge condition more difficult to implement)

$h_{ij}$: linear hyperbolic equation (wave equation)

- No conceptual problem, but can be expensive (depending on relevant range of scales)
- $h_{ij}$ does not enter the geodesic equation for massive particles (at our approximation order) → expendable, leave for future work
Numerical Results
A Plane-Symmetric Setup

- Restriction to plane-symmetric configuration ($y-z$-plane) trivializes two dimensions → high resolution possible with cheap computational requirements (no parallelization)
- No vector & tensor perturbations (by construction)
- 32768 particles, 4096 grid points
- Initial conditions: Gaussian random field obtained from semi-realistic initial power spectrum
- Initialized at $z > 1000$ using linear theory (Zel’’dovich approximation)
Numerical Results

A Plane-Symmetric Setup

\[ v/c = \frac{v}{c} \]

Julian Adamek
Université de Genève
Numerical Results
Luminosity Distance

\[ \mu - \mu_{\text{Milne}} \]

\( \Lambda \text{CDM} \)

Einstein-de Sitter
Numerical Results
Newtonian vs. GR Simulation

\[ \frac{|\Delta H|}{H} \]
Summary

- Cosmological simulations within a **GR framework** are feasible
- A unified relativistic treatment is a clear, logical and transparent way to address the **most general observables** with minimal assumptions about the cosmological model
- Technology is useful for simulations with **relativistic sources** (dynamical DE, cosmic strings, neutrinos) – feasibility depends on ability to model the sources accurately
- For CDM simulations, modifications are computationally relatively inexpensive (but may be unnecessary)
- The issue of **backreaction** can be addressed quantitatively within the non-perturbative regime