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A Primer on Cosmology and the Cosmic Microwave Background

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We introduce the theory of cosmic microwave background anisotropies in a self consistent manner.

I. THE HOMOGENOUS BACKGROUND

In general relativity, Einstein's equations relate the geometry of the universe locally to the energy momentum content. The geometry is expressed via the metric $g_{\mu\nu}$ and subsequently through the Ricci Tensor $R_{\mu\nu}$ and the curvature scalar R, while the energy momentum tensor is commonly denoted by $T_{\mu\nu}$. Using the reduced Planck mass $\mathrm{M_P} \equiv (8\pi G)^{-1/2}$, Einstein's equations read[1]

$$T_{\mu\nu} = \mathrm{M_P^2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right). \qquad (1)$$

In order to solve these very complicated, coupled differential equations analytically, one needs to guess the geometry of the space and hence the metric. The most general metric that is isotropic and homogenous on constant time hyper-surfaces is the Robertson-Walker metric. This metric comes in three 'flavors', for the cases of negative, positive or vanishing 3-curvature in the constant time hyper-surfaces. A vanishing of this 3-curvature means a spatially flat universe (we will sloppily call this just a 'flat universe'). Having said this, the Robertson-Walker metric is given by

$$ds^2 \equiv g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \\ -\mathrm{d}t^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\ \mathrm{d}\phi^2\right). \qquad (2)$$

Here $k = -1, 0, +1$ corresponds to open, flat and closed geometries. Using so called conformal time $\mathrm{d}\tau = a^{-1}\mathrm{d}t$, this becomes

$$ds^2 = a(\tau)^2\left(-\mathrm{d}\tau^2 + \frac{dr^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\ \mathrm{d}\phi^2\right). \qquad (3)$$

For flat geometries (the case we will concentrate on), this can be written in terms of the coordinates x^i as

$$ds^2 = a(\tau)^2\left(-\mathrm{d}\tau^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right), \qquad (4)$$

The expression 'conformal time' is well chosen, for the metric (4) is conformally related to the usual Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ by the conformal factor $a(\tau)$. For flat cosmologies, we normalize $a(\tau)$ such that today, we have $a_0 \equiv a(\tau_0) = 1$, where here and in the following a subscript 0 will denote quantities as measured today. As $a(\tau)$ determines the stretching of *physical* length scales,

$$l^2_{\mathrm{physical}}(\tau) = g_{ij}l^il^j = a(\tau)^2\delta_{ij}l^il^j = a(\tau)^2\boldsymbol{l}^2, \qquad (5)$$

it is commonly called the *scale factor*. Please note that 3-vectors are in bold, spatial components are denoted by Latin indices and the 3-vector scalar product is the usual one: $\boldsymbol{x}\,\boldsymbol{y} = \delta_{ij}x^ix^j$.

Exercise 1: Express the Hubble parameter $H \equiv a^{-1}\frac{\mathrm{d}a}{\mathrm{d}t}$ in terms of a derivative w.r.t $\mathrm{d}\tau$ (remember that $\mathrm{d}\tau = \frac{1}{a}\mathrm{d}t$).

It is common practice to describe the matter content of the universe by fluids. Even in cases where this description is no longer valid and one needs to think in terms of distribution functions, we will still *identify* certain parts of these distributions with fluid terminology. For a start, let us briefly forget about cases where the fluid description breaks down and note that the energy momentum tensor for a *perfect* fluid is [19]

$$\bar{T}^\mu_{\ \nu} = \mathrm{diag}(-\bar{\rho}, \bar{p}, \bar{p}, \bar{p}), \qquad (6)$$

where $\bar{\rho}(\tau)$ is the (unperturbed[2]) energy density and $\bar{p}(\tau)$ is the pressure. The relation between $\bar{\rho}$ and $\bar{p}$ is expressed in the equation of state

$$\bar{p}(\tau) = w(\tau)\bar{\rho}(\tau). \qquad (7)$$

For non-relativistic matter, the pressure vanishes, whereas photons and massless neutrinos have $w = 1/3$. From the $0-0$ and $i-i$ part of Einstein's Equation (1), we we get the Friedmann equation

$$3\mathrm{M_P^2}\left(H^2 + \frac{k}{a^2}\right) = \rho(\tau). \qquad (8)$$

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[1] When required, the cosmological constant will be assumed to be part of the energy momentum tensor.

[2] Anticipating perturbation theory, we denote background quantities by a bar.

Here, the Hubble parameter H is related to the scale factor $a(\tau)$ by

$$H \equiv a^{-1}\frac{\mathrm{d}a}{\mathrm{d}t} = a^{-1}\frac{\mathrm{d}a}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} \equiv a^{-2}\dot{a}, \tag{9}$$

where a dot denotes a derivative with respect to conformal time τ throughout this lecture.

The ratio of the energy of some species ρ to the so called critical energy density $\rho_{crit.} \equiv 3\mathrm{M}_\mathrm{P}^2 H^2$ is defined as

$$\Omega \equiv \frac{\rho}{\rho_{crit.}} \tag{10}$$

For a flat universe, Ω is just the fraction a given species contributes to the total energy of the Universe. Careful: while Ω is a function of time, the subscript 0 indicating today's value is frequently omitted in the literature.

Exercise 2: Determine the redshift $z \equiv \frac{1}{a} - 1$ of matter-radiation energy equality ($\rho_m = \rho_\gamma$) in a flat universe, provided you are given Ω_m and Ω_γ today. Use the scaling of energies $\rho_m = \rho_m^0\, a^{-3}$ and $\rho_\gamma = \rho_\gamma^0\, a^{-4}$

Exercise 3: Solve the Friedman Equation for a flat universes dominated by radiation energy which scales as $\rho(\tau) = \rho_r^0 a(\tau)^{-4}$. Having considered this, what is the solution for a universe dominated by pressure-less matter for which $\rho(\tau) = \rho_m^0\, a(\tau)^{-3}$?

Exercise 4: With the solution of the above exercise 3 at hand, what is $\frac{\dot{a}}{a}$ in the case of radiation domination (RD) and what is it for a matter dominated (MD) universe ?

We can rewrite Equation (8) to argue that the curvature of the Universe didn't play a role at early times. We start from

$$3\mathrm{M}_\mathrm{P}^2 H^2 + \frac{3\mathrm{M}_\mathrm{P}^2\, k}{a^2} = \rho(\tau), \tag{11}$$

dividing by $3\mathrm{M}_\mathrm{P}^2 H^2$,

$$\frac{3\mathrm{M}_\mathrm{P}^2\, k}{3\mathrm{M}_\mathrm{P}^2\, H^2\, a^2} = \frac{\rho(\tau)}{3\mathrm{M}_\mathrm{P}^2\, H^2} - 1, \tag{12}$$

using $3\mathrm{M}_\mathrm{P}^2\, H^2 = \rho_{crit.}$ and the definition of Ω, the above equation (12) can be cast in the form

$$\frac{k}{a^2\, H^2} = \Omega_{total} - 1. \tag{13}$$

Now, we know from the exercises that $H = a^{-1}\frac{\dot{a}}{a}$ and $\frac{\dot{a}}{a} = 1/\tau$ in RD and $2/\tau$ in MD. Hence, $a^2\, H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{\tau^2} \rightarrow \infty$ in the early universe and therefore

$$\Omega_{total} - 1 = \frac{k}{a^2\, H^2} \rightarrow 0, \tag{14}$$

in the early universe. As the density today is at least very close to the critical density, we are led to conclude that the spatial curvature of the universe (if it has one) was irrelevant until today and may well continue to be irrelevant in the future. As in addition, the theory of inflation predicts a spatially flat universe, we will from now on work only in flat geometries.

Conservation of the zero component of the energy momentum tensor,$\nabla_\mu \bar{T}^\mu{}_0 = 0$, yields the useful relation

$$\frac{\dot{\bar{\rho}}}{\bar{\rho}} = -3(1+w)\frac{\dot{a}}{a}. \tag{15}$$

Finally, by combining Friedmann's equation (8) with the $i - i$ part of Einstein's equation one obtains

$$\sum_{\text{all species}} \left(\bar{\rho}\left[\frac{1}{3} + w\right]\right) = -2\mathrm{M}_\mathrm{P}^2\, a^{-1}\,\frac{\mathrm{d}^2 a}{\mathrm{d}t^2}. \tag{16}$$

A. Horizons

Within the framework of FRW cosmology, one may ask the question what parts of the universe have been in causal contact since the beginning of time. Assuming that the speed of light c (which in our units is 1, however we keep it here for clarity) is the maximum speed information can travel, the horizon is given by

$$s_{hori.} = \int_0^t \frac{a(t)}{a(t')}\, c\, \mathrm{d}t' \tag{17}$$

where the factor $\frac{a(t)}{a(t')}$ accounts for the fact that a step taken at an earlier time t' has been stretched by exactly this factor at time t.

Exercise 5: Rewrite Equation (17) in terms of conformal time τ

Caution: usually horizons (and other scales) in the early universe are quoted in terms of their size today. Obviously, this size is greater by another factor of $\frac{a(t_0)}{a(t)}$.

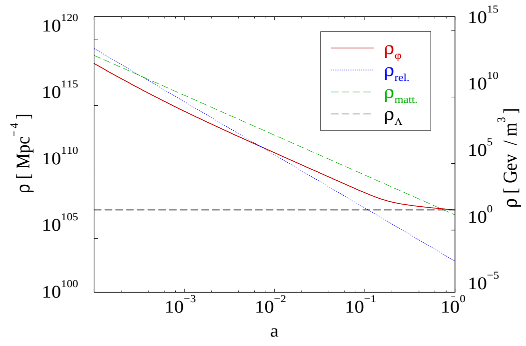


FIG. 1: The energy density of matter, radiation, a cosmological constant and some dark energy model as a function of the scale factor. Please note that today $a_0 = 1$ and the crossing of radiation and matter densities at matter-radiation equality $a_{equ} \approx 10^{-4}$.

For instance, the horizon at recombination (redshift $z \approx 1100$) is said to be $\tau_{ls} \approx 300$Mpc. The physical size at that time, however was almost 1100 times smaller. The reason why one usually quotes the size as seen today is simply that one observes *today* and what used to be small is now much larger.

B. Who's who in the universe ?

As far as particles are concerned that may interact with us (at least at reasonable energy scales of say < 100 Gev), there is radiation (photons and neutrinos) and baryons. Baryons contribute about 5% of the energy density of the Universe today. However, different measurements of the large scale structure, Sne Ia and the CMB indicate that there are two more important ingredients: cold dark matter and dark energy. Cold dark matter (CDM) is supposed to be at least very weakly interacting with us and with itself and in addition it is considered to be massive. CDM makes up roughly 25% of the Universe today. Dark energy is a rather strange substance or vacuum energy that has become important in the late Universe. Too little is known about its time behaviour or equation of state to single out a likely theoretical (string / particle) model describing it. In Figure 1, we plot the energy scaling of baryons, radiation, CDM, a cosmological constant and some form of dark energy.

II. PHOTONS AND HYDROGEN RECOMBINATION

The photons that form the CMB have microwave frequencies. The frequency of a typical wavelength of say 10 cm is

$$f = \frac{c}{\lambda} = \frac{3\times 10^8\,\frac{\mathrm{m}}{\mathrm{s}}}{0.1\,\mathrm{m}} = 3\times 10^9\,\mathrm{Hz}. \qquad (18)$$

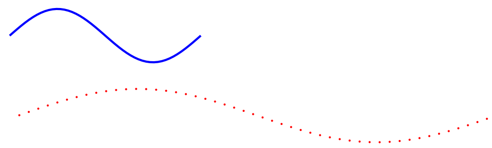
Let us compare this to the Hubble parameter today which is roughly

$$H_0 = 100\,\mathrm{km}\,s^{-1}\mathrm{Mpc}^{-1} = 10^5\mathrm{m}\,s^{-1}/3\times 10^6\mathrm{ly} \approx 2\times 10^{-18}\mathrm{Hz} \qquad (19)$$

Hence, the expansion of the universe is so slow compared to the microwave frequency that it is adiabatic. Therefore the occupation number

$$f_\gamma = (\exp(\hbar\omega/kT) - 1)^{-1} = (\exp(hc/kT\lambda) - 1)^{-1}$$

is preserved. Now, the wavelength λ of radiation stretches as the universe expands:



Hence, to conserve occupation number, the temperature of the CMB scales as $T \propto a^{-1}$.

Despite the low CMB temperature today, we deduce from this argument that there must have been a time before which the CMB photons were powerful enough to ionize hydrogen:

$$e^- + p \leftrightarrow H + \gamma$$

The equilibrium condition for the chemical potential is

$$\mu(e) + \mu(p) = \mu(H) \qquad (20)$$

and for small occupation numbers (and as the protons are non relativistic), we have

$$f_b = \exp\left([\mu - E]/kT\right) \qquad (21)$$

where $E = m_b c^2 + p^2/2m_b$ and $b \in \{p, e^-, H\}$. Integrating over phase space yields the number of particles per unit volume

$$n_b = g_b \frac{(2\pi m_b kT)^{3/2}}{(2\pi\hbar)^3} \exp\left([\mu - m_b c^2]/kT\right). \qquad (22)$$

To avoid any misunderstandings: n_e is the number density of free electrons, n_p is the number density of free protons and n_H is the number density of neutral hydrogen. Furthermore, T is the temperature of the plasma, i.e. the common temperature of e^-, p, H *and* the photons where we assume that the photon temperature evolves as $T_\gamma = T_\gamma^0\, a^{-1}$. If you wonder why the temperature in the plasma should be given by the (unaltered) photon temperature: Even for some time after matter-radiation equality, when the *energy* contribution of photons becomes less important than that of matter, there are still many more photons than matter particles. Each matter particle contributes its large rest mass to the energy budget, whereas photons weigh in by shear number. There is very little kinetic energy in the matter and the energy transfer from photons to baryons needed to keep the baryons at photon temperature is negligible for the photon gas. Now, the binding energy of hydrogen is

$$B = (m_e + m_p - m_H)c^2 = 13.6\,\text{eV}. \tag{23}$$

Exercise 6: Calculate the ratio $n_e n_p : n_H$ using equations (22), (20) and (23).

Using equations (22), (20) and (23) yields

$$\frac{n_e\, n_p}{n_H} = \frac{(2\pi m_e k_B T)^{3/2}}{(2\pi\hbar)^3} \exp\left(-\frac{B}{k_B\, T}\right) \tag{24}$$

which is roughly speaking the Saha equation. Defining the total number density of baryons as $n = n_e + n_H$ and the ionization fraction of electrons as $x_e \equiv n_e/n$, we get from (24)

$$\frac{n_e n_p}{n_H} = \frac{x_e^2 n^2}{n(1-x_e)}$$
$$= n\frac{x_e^2}{1-x_e} = \frac{(2\pi m_e k_B T)^{3/2}}{(2\pi\hbar)^3} \exp\left(-\frac{B}{k_B\, T}\right) \tag{25}$$

Exercise 7: Remember that $\hbar c = 200\,\text{Mev fm}$ and $m_e c^2 \approx 0.5\,\text{Mev}$. Use this and $n \approx 5m^{-3}a^{-3} = 5m^{-3}(T/T_0)^3$ to rewrite Equation (25).

Using $x_e = 0.1$ to define decoupling, one can now calculate the temperature at decoupling.

Exercise 8: Estimate $k_B T_{dec}$, i.e. the energy scale at decoupling. Hint: assume that $k_B T_{dec}$ is of the order of $0.1 \ldots 10\,\text{eV}$ and use the logarithm to estimate the ratio $B/(k_B T)$

So the temperature of decoupling is $T_{dec} = T_0 z_{dec} \approx 3000 - 4000K$ and hence $z_{dec} \approx 1000 \ldots 1500$. Please note that the Temperature T_{dec} is by two orders of magnitude smaller than the Temperature corresponding to $13.6eV$ which is $T \approx 13.6\text{eV}/k_B = 13.6\text{eV} \times 11000\text{K/eV} \approx 150'000\text{K}$ The reason is the vast number of photons per baryon: there are roughly one billion photons per baryon. It should also be remarked that the CMB photons completely dominate the entropy of our Universe: the entropy of the universe per baryon is of the order of 10^{10}.

III. OBSERVATIONS

Observationally, we see that radiation from different directions on the sky has slightly different intensities and polarisation. None less than S. Chandrasekhar wrote a book on radiative transfer and it is due to his influence that astrophysicists still use notation and to large extend his derivations. Let us take a look at a beam from direction $\boldsymbol{n}$, which we can characterize using so called Stokes parameters. These describe phases and intensities of the beam. Pependicular to the beam, on chooses an orthonormal basis $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2$ to describe the electromagnetic wave

$$\boldsymbol{E} = \left(a_1 e^{i\delta_1}\boldsymbol{\epsilon}_1 + a_2 e^{i\delta_2}\boldsymbol{\epsilon}_2\right) e^{ip\boldsymbol{n}\,\boldsymbol{x} - i\omega t}. \tag{26}$$

Let's briefly use the projections $\boldsymbol{E}_1 \equiv \boldsymbol{\epsilon}_1(\boldsymbol{E}\boldsymbol{\epsilon}_1)$ and likewise $\boldsymbol{E}_2 \equiv \boldsymbol{\epsilon}_2(\boldsymbol{E}\boldsymbol{\epsilon}_2)$. The Stokes parameters are then defined by the long time averages of squares of the peak amplitudes

$$I \equiv \langle \boldsymbol{E}\boldsymbol{E}^\star \rangle = a_1^2 + a_2^2, \tag{27}$$
$$Q \equiv \langle \boldsymbol{E}_1\boldsymbol{E}_1^\star - \boldsymbol{E}_2\boldsymbol{E}_2^\star \rangle = a_1^2 - a_2^2, \tag{28}$$
$$U \equiv \left\langle \left|\frac{\boldsymbol{E}_1 + \boldsymbol{E}_2}{\sqrt{2}}\right|^2 - \left|\frac{\boldsymbol{E}_1 - \boldsymbol{E}_2}{\sqrt{2}}\right|^2 \right\rangle \tag{29}$$
$$= 2a_1 a_2 \cos(\delta_1 - \delta_2).$$

There is a fourth one, V which describes cirular polarized light which is not needed here, because Thomson scattering will not produce V. In words, I is the total intensity, $\mathcal{Q}$ is the linear polarization (comparing intensities along the axis $\boldsymbol{\epsilon}_1$ to $\boldsymbol{\epsilon}_2$) and U is the linear polarization (comparing intensities along axis rotated by 45 degrees). An important property of $\mathcal{Q}$ and U is that under a rotation of the coordinate system by an angle ψ

$$\boldsymbol{\epsilon}_1 = \cos(\psi)\boldsymbol{\epsilon}_1 + \sin(\psi)\boldsymbol{\epsilon}_2 \tag{30}$$
$$\boldsymbol{\epsilon}_2 = -\sin(\psi)\boldsymbol{\epsilon}_1 + \cos(\psi)\boldsymbol{\epsilon}_2 \tag{31}$$

they transform as

$$\tilde{\mathcal{Q}} = \cos(2\psi)\mathcal{Q} + \sin(2\psi)U \tag{32}$$
$$\tilde{U} = -\sin(2\psi)\mathcal{Q} + \cos(2\psi)U \tag{33}$$

To see this, rotate the axis by ψ and project $\boldsymbol{E}$ onto these new axis and compute $\tilde{Q}$ and $\tilde{U}$. Alternativly, as a quick way to make it plausible consider a rotation by 90 degrees. Then $\tilde{\boldsymbol{\epsilon}}_1 = \boldsymbol{\epsilon}_2$ and $\tilde{\boldsymbol{\epsilon}}_2 = -\boldsymbol{\epsilon}_1$. Hence $\tilde{\mathcal{Q}} = |\tilde{\boldsymbol{\epsilon}}_1\boldsymbol{E}|^2 - |\tilde{\boldsymbol{\epsilon}}_2\boldsymbol{E}|^2 = |\boldsymbol{\epsilon}_2\boldsymbol{E}|^2 - |-\boldsymbol{\epsilon}_1\boldsymbol{E}|^2 = -\mathcal{Q}$. So a rotation by 90 degrees flips sign and one by 180 degrees goes back to the old state. This tells us that we are looking at a quantity of spin 2. In the late 1960's, Newman, Penrose and Goldberg tackled the problem of functions on the sphere which under a rotation $\psi^\star = -\psi$ [3] around the

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[3] The CMB convention for the rotation is the exact opposite of what Newman and Penrose chose. The CMB community looks

radial unit vector $\hat{\boldsymbol{r}}$ transform as

$$\tilde{\eta} = e^{is\psi^\star}\eta, \tag{34}$$

where s is called spin and is always integer in this formalism. Clearly

$$(\mathcal{Q} \tilde{\pm} iU) = e^{\mp 2i\psi}(\mathcal{Q} \pm iU) \tag{35}$$

and hence these combinations have spin ∓ 2.

Additional reading: Tetrad basis...

When you open a book on General Relativity, you can find a section on the tetra formalism. In short, a tetrad is a set of basis vectors e^a_μ with $a = 0, 1, 2, 3$ and μ the ordinary 4-coordinates. Using such a tetrad, any quantity can be expressed in terms of the components along the basis vectors:

$$v^a = e^a_\mu v^\mu \tag{36}$$

This formalism is particularly useful for field theory in curved spaces. Newman and penrose chose to use 4 complex basis vectors in their paper on the spin formalism for gravitational waves around a black hole. In their later paper about the spin functions in a 3 dimensional space, they chose the radial unit vector $\hat{\boldsymbol{r}}$ and the complex tangent vector on the sphere $\boldsymbol{m} = (\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2)/\sqrt{2}$ and its complex conjungate $\bar{\boldsymbol{m}}$. The complex nature of the basis vectors has a big benefit: The number of equations one has to solve is reduced by a factor of 2. A quantity of some spin can be constructed by contracting a three tensor $\Omega_{ijk...}$ with the complex basis vectors $\boldsymbol{m}, \hat{\boldsymbol{r}}$ and $\bar{\boldsymbol{m}}$. Contracting with $\boldsymbol{m}$ raises the spin by 1, contracting with $\bar{\boldsymbol{m}}$ lowers it by one and contracting with $\hat{\boldsymbol{r}}$ leaves it unchanged. Again, for radiation around a black hole, the projection of e.g. the Field-strength tensor on the tetrad leads to an (almost) decoupling of quantities with definite spin and it is evident in the equations that some of them fall off at infinity much faster than other solutions and hence one can classify the solutions easily.See also the an article on scholarpedia:
`http://www.scholarpedia.org/article/Newman-Penrose_formalism`

For us, the benefit of spin functions is 3-fold:

- We will get a complete set of basis functions for spin 2 quantities on the sphere "for free" from the literature
- We can act with spin lowering and raising operators (see below) on the spin 2 quantities and make them spin 0
- As spin 0 quantities are unambigiously defined (they don't care which coordinate system one chooses), we can compare the polarization at different points on the sky unambigiously

Just like one can expand a scalar function on the sphere in terms of spherical harmonics $Y_l^m(\boldsymbol{n})$ (we will do this frequently later on), one can as well expand a spin s quantity in terms of spin-weighted spherical harmonics ${}_sY_l^m(\boldsymbol{n})$, e.g.

$$(\mathcal{Q}(\boldsymbol{n}) + iU(\boldsymbol{n})) = \sum_{l,m} {}_2a_{lm}\ {}_2Y_l^m(\boldsymbol{n}) \tag{37}$$

On the sky today, one may also expand the temperature anisotropy $\Delta \equiv \Delta T/T$ as

$$\Delta(\boldsymbol{n}) = \sum_{l,m} a_{lm} Y_l^m(\boldsymbol{n}). \tag{38}$$

For our calculation, however, we will not need as many m's as are in the sum (38). Indeed, we can perform our calculations in a "nice" coordinate frame in which we can work (more of this soon) with $m = 0, \pm 1$ and ± 2. We will follow the recent literature and use the functions

$$ {}_sG_l^m(\boldsymbol{x}, \boldsymbol{n}) = (-i)^l \sqrt{\frac{4\pi}{2l+1}}\, {}_sY_l^m(\boldsymbol{n})\, \exp(i\boldsymbol{k}\boldsymbol{x}), \tag{39}$$

to expand a function of spin weight s both in Fourier space (again: more of this soon) and in spin weighted harmonics. A very useful expansion is that of the relative temperature perturbation $\Delta \equiv \Delta T/T$

$$\Delta(\tau, \boldsymbol{x}, \boldsymbol{n}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \sum_l \sum_{m=-2}^{2} \Delta_l(k, \tau)\ {}_0G_l^m(\boldsymbol{x}, \boldsymbol{n}). \tag{40}$$

Likewise, one can expand the spin 2 polarization combinations

$$(\mathcal{Q} \pm i\,U)(\tau, \boldsymbol{x}, \boldsymbol{n}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \sum_{l=2} \sum_{m=-2}^{2} (E_l^m \pm iB_l^m) \times\ {}_{\pm 2}G_l^m(\boldsymbol{x}, \boldsymbol{n}). \tag{41}$$

Don't be intimedated by these spin weighted functions! For $m = 0$ (the most important case)

$$ {}_0Y_l^0(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta) \tag{42}$$

are basically just Legendre Polynomials P_l and so for $m = 0$, the temperature expansion becomes particularly

from earth up and the photon wave is coming in, whereas Newman and Penrose were interested in gravitational and electromagnetic waves going outwards from a black hole. Hence, what is a clockwise rotation on a sphere from the outside is counter clockwise from the inside.

simple

$$\Delta(\tau, \boldsymbol{x}, \boldsymbol{n}) \stackrel{m=0}{=} \int \frac{\mathrm{d}^3k}{(2\pi)^3} \sum_l (-i)^l \Delta_l(k,\tau) P_l(\boldsymbol{n}) \exp(i\boldsymbol{kx}). \tag{43}$$

We will soon see that we can notationally even neglect the Fourier transform part above. Hence, things will become really simple. Promised.

If you don't mind, I would like to remind you (quite out of context) of integration over all directions $\int \mathrm{d}\Omega$ of some function $f(\theta, \phi) = f(\boldsymbol{n})$ on the sky. In spherical coordinates,

$$\int \mathrm{d}\Omega f(\boldsymbol{n}) = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin(\theta) \mathrm{d}\theta f(\boldsymbol{n}) \tag{44}$$

and it is convenient to rewrite this as

$$\int \mathrm{d}\Omega f(\boldsymbol{n}) = \int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{1} \mathrm{d}\cos\theta f(\boldsymbol{n}) \tag{45}$$

and if one called $\cos\theta = \mu$, one could say that

$$\int \mathrm{d}\Omega f(\boldsymbol{n}) = \int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{1} \mathrm{d}\mu f(\boldsymbol{n}). \tag{46}$$

As said, just a reminder ....

IV. PERTURBATIONS

In the previous section, we have seen that using the Robertson-Walker metric, we can solve Einstein's equation. The result is almost miraculously simple. However, the universe is not completely homogenous. On the contrary: it is quite clumpy on the scales of the solar system or even galaxies. Yet, the larger the scale one looks at, the more homogenous it becomes. In addition, the inhomogeneities usually grow due to gravitational infall. Hence, in the early universe, we may expect only small departures from homogeneity. This is where linear perturbation theory enters the stage. Starting from the homogenous FRW universe, one perturbs the metric and the energy momentum tensor. It is convenient to expand these perturbations in (generalized) Fourier modes and to classify physical quantities in the 3-dimensional constant time hyper-surfaces by their transformation properties [12–14, 21]. Eigenfunctions of the 3-dimensional Laplace operator

$$\Delta Q_{\boldsymbol{k}}(\boldsymbol{x}) = -k^2 Q_{\boldsymbol{k}}(\boldsymbol{x}) \tag{47}$$

are used to decompose the metric and energy momentum perturbations into scalar, vector and tensor parts (called *modes*). The benefit of this classification is that different modes do not mix in first order perturbation theory [13]: the perturbation equations decouple. Furthermore, a coupling between perturbations of different *Fourier* modes $\boldsymbol{k}$ and $\boldsymbol{k}'$ involves products of perturbations. These would be of second order and are thus neglected. Hence, also modes with different $\boldsymbol{k}$ decouple and it is not necessary to display the wave vector $\boldsymbol{k}$ of the eigenfunctions Q explicitly. For the same reason, it is not necessary to keep the integration over the Fourier modes explicitly in the equations. One should however keep in mind that, for instance, the energy density is

$$\rho(\tau, \boldsymbol{x}) = \bar{\rho} + \int \frac{\mathrm{d}^3k}{(2\pi)^3} \delta\rho(\tau, \boldsymbol{k}) \, Q_{\boldsymbol{k}}(\boldsymbol{x}), \tag{48}$$

and it is only the decoupling of different $\boldsymbol{k}$ modes that will enable us to compare the *integrands* directly. Finally, there is another benefit of working in Fourier space: much the same as in electrodynamics, differential equations are turned into algebraic equations which are much easier to solve.

If the 3-space is flat (the case we are primarily interested in), then $Q = \exp(i\boldsymbol{kx})$ is the solution of the Laplace equation (47). Now, take for instance some vector V_i. One can decompose it into a gradient and a (divergence-less) rotation part:

$$\boldsymbol{V} = \mathrm{grad}\phi + \mathrm{rot}\boldsymbol{B} \tag{49}$$

The function ϕ is a scalar, yet it contributes to a vector. In general, we can construct the scalar basis functions by deriving Q. Let us define[4]

$$Q_i \equiv -k^{-1} Q_{,i} \tag{50}$$

$$Q_{ij} \equiv k^{-2} Q_{,ij} + \frac{1}{3}\delta_{ij} Q, \tag{51}$$

where Q_{ij} is traceless by construction and gives the scalar contribution towards a symmetric tensor.

Exercise 9: Verify that Q_{ij} is traceless.

In general, the contributions to a vector field $\boldsymbol{B}$ by some scalar function B can thus be written as:

$$B_i(\tau, \boldsymbol{x}) = B(\tau) \, Q_i \tag{52}$$

and for a tensor field, we have

$$H_{ij}(\tau, \boldsymbol{x}) = H_L(\tau) \, Q \, \delta_{ij} + H_T(\tau) \, Q_{ij} \tag{53}$$

Please note that in a coordinate system with $\boldsymbol{k}||\hat{z}$, we get $k^{-1}\boldsymbol{k} \cdot \boldsymbol{n} = \cos\theta$ and hence $Q \propto {}_0G_0^0$, $n^i Q_i = -in^j k_j k^{-1} Q = -i\cos\theta Q \propto {}_0G_1^0$ and $n^i n^j = (-i)^2 k^{-2} k_i k_j n^i n^j Q + \frac{1}{3}\delta_{ij} n^i n^j = \frac{1}{3} - \cos^2\theta \propto {}_0G_2^0$. In

[4] We follow [12], but restrict ourselves to flat universes. Hence the covariant 3-derivative $Q_{|i}$ can be replaced by the partial derivative $Q_{,i}$.

this system, we therefore need only $m = 0$ for scalar perturbations, as promised.

In exactly the same manner, basis functions for vector and tensor type perturbations can be derived. For instance, the divergence-less part of a vector field is expressed [12] via $Q_i^{(V)}(\boldsymbol{x})$ solving the vector Helmholtz equation

$$Q_{i,j}^{(V),j} + k^2 Q_i^{(V)} = 0, \tag{54}$$

and being divergence-less: $Q_{,i}^{(V)i} = 0$. As already mentioned, we will restrict the discussion to scalar perturbations. The most general line element for a perturbed Robertson-Walker metric is [12]

$$ds^2 = a(\tau)^2\big[-(1+2AQ)d\tau^2 - 2B_i d\tau dx^i + (\delta_{ij} + 2H_{ij})dx^i dx^j\big] \tag{55}$$

Where in the scalar case B_i and H_{ij} are given by Equations (52) and (53).

We will concentrate on scalar perturbations in this lecture. Scalar perturbations account by far for the largest contributions of anisotropies measured by today's experiments. Yet, tensor modes are interesting in their own right because inflation predicts a certain level of tensor fluctuations that may be measured in the future.

Exercise 10: The Fourier mode k is usually quoted in units of Mpc^{-1} and the product $k\,\tau$ is a good measure for deciding whether a length scale is inside or outside the horizon τ at any given time. In fact, the transition is at $k\,\tau \approx 1$. The good question is: is a mode with $k\,\tau < 1$ outside or inside the horizon?

---

1. *The Gauge Problem*

General coordinate transformations are a main ingredient of general relativity. Unfortunately, the freedom to choose a coordinate system needs to be used with care in cosmology. Let us see, how this comes about. Consider an infinitesimal coordinate transformation

$$x^\mu \to \tilde{x}^\mu = x^\mu + \epsilon^\mu(\tau, \boldsymbol{x}), \tag{56}$$

where the derivative of ϵ^μ is also assumed to be at most of the order ϵ. We know that some tensor expressed in the new coordinate system will be

$$\tilde{T}^{\mu\dots}_{\nu\dots}(\tilde{x}) = \left(\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\right)\left(\frac{\partial x^\beta}{\partial \tilde{x}^\nu}\right)\dots T^{\alpha\dots}_{\beta\dots}(x), \tag{57}$$

where the transformation matrices are

$$\left(\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\right) = \delta^\mu_\nu + \frac{\partial \epsilon^\mu(\tau,\boldsymbol{x})}{\partial x^\nu} \tag{58}$$

$$\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right) = \delta^\mu_\nu - \frac{\partial \epsilon^\mu(\tau,\boldsymbol{x})}{\partial \tilde{x}^\nu} \tag{59}$$

$$= \delta^\mu_\nu - \frac{\partial \epsilon^\mu(\tau,\boldsymbol{x})}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial \tilde{x}^\nu} \tag{60}$$

$$= \delta^\mu_\nu - \frac{\partial \epsilon^\mu(\tau,\boldsymbol{x})}{\partial x^\nu} + \mathcal{O}(\epsilon^2). \tag{61}$$

The last equation in the above holds, because we have assumed that the derivative of ϵ is also of the order ϵ. Thus, working to order ϵ, the tensor transformation (57) becomes

$$\tilde{T}^{\mu\dots}_{\nu\dots}(\tilde{x}) = T^{\mu\dots}_{\nu\dots}(x) + \bar{T}^{\alpha\dots}_{\nu\dots}(x)\frac{\partial \epsilon^\mu(\tau,\boldsymbol{x})}{\partial x^\alpha} + \cdots - \bar{T}^{\mu\dots}_{\alpha\dots}(x)\frac{\partial \epsilon^\alpha(\tau,\boldsymbol{x})}{\partial x^\nu} - \cdots \tag{62}$$

If we were willing to give up the nice FRW background universe, we could happily use the transformation Equation (62). However, we would like to make the coordinate transformation (56), but *without* paying the price of changing the *background* physics. The reason why we would like to keep the background physics the same regardless of our coordinate transformations is that we would like the background to maintain its (only τ dependent) Robertson-Walker metric, for we have seen that it is this metric that leads to the convenient Friedmann equation. So, in order to stick to the old coordinates for the background, we have to go back from $\tilde{x}$ to x in the argument of $\tilde{T}$:

$$\begin{aligned}\tilde{T}^{\mu\dots}_{\nu\dots}(\tilde{x}) &= \tilde{T}^{\mu\dots}_{\nu\dots}(x+\epsilon)\\ &= \tilde{T}^{\mu\dots}_{\nu\dots}(x) + \epsilon^\alpha\left(\frac{\partial \tilde{T}^{\mu\dots}_{\nu\dots}(\xi)}{\partial \xi^\alpha}\right)_{|\xi=x}\\ &= \tilde{T}^{\mu\dots}_{\nu\dots}(x) + \epsilon^\alpha\left(\frac{\partial T^{\mu\dots}_{\nu\dots}(\xi)}{\partial \xi^\alpha}\right)_{|\xi=x} + \mathcal{O}(\epsilon^2).\end{aligned} \tag{63}$$

Here, we have used the transformation Equation (62). Putting Equations (62) and (63) together, we get the final gauge transformation law

$$\tilde{T}^{\mu\dots}_{\nu\dots}(x) = T^{\mu\dots}_{\nu\dots}(x) - \bar{T}^{\mu\dots}_{\nu\dots,\alpha}(\tau)\epsilon^\alpha + \bar{T}^{\alpha\dots}_{\nu\dots}(\tau)\epsilon^\mu_{,\alpha} + \cdots - \bar{T}^{\mu\dots}_{\alpha\dots}(\tau)\epsilon^\alpha_{,\nu} - \cdots\,. \tag{64}$$

The derivatives above combine to give the Lie derivative $L_\epsilon \bar{T}$ and we can rewrite Equation (64) rather elegantly as $\tilde{T}(x) = T(x) - L_\epsilon \bar{T}$.

Having derived the transformation equation, let us see what this means for the metric. Using Equation (64), we get

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) - \bar{g}_{\alpha\nu}\epsilon^\alpha_{,\mu} - \bar{g}_{\mu\alpha}\epsilon^\alpha_{,\nu} - \epsilon^\alpha\,\bar{g}_{\mu\nu,\,\alpha}. \tag{65}$$

It is the last term in the above equation that would quantify the change in the background, if we allowed one. However, as we stick to the same background, we will interpret this term as a contribution to the change of the perturbation variables due to the coordinate transformation. This is the conceptional difference between the *coordinate* transformation (changing the background, the coordinates and the fluctuations) and the *gauge* transformation (changing only the fluctuations, keeping the old coordinates and background quantities). The transformation four vector ϵ can be decomposed into scalar and vector parts. Following [12], we set

$$\begin{aligned}\tilde{\tau} &= \tau + T(\tau)Q(\boldsymbol{x}) \\ \tilde{x}^i &= x^i + L(\tau)Q^i(\boldsymbol{x}) + L^{(V)}(\tau)Q^{(V)\,i}(\boldsymbol{x}).\end{aligned} \quad (66)$$

The vector contribution will not affect scalar perturbations, just like scalar, vector and tensor perturbations decouple in linear approximation. Using the above transformation (66) in (65), we can calculate for instance the change in the metric perturbation B:

$$\begin{aligned}\tilde{g}_{0i}(x) &= g_{0i}(x) - \bar{g}_{\alpha i}(\tau)\epsilon^\alpha_{,0} - \bar{g}_{0\alpha}(\tau)\epsilon^\alpha_{,i} - \epsilon^\alpha\,\bar{g}_{0i,\,\alpha} \\ &= g_{0i}(x) - \bar{g}_{ji}(\tau)\dot{L}Q^j - \bar{g}_{00}(\tau)TQ_{,i} \\ &= -a^2(\tau)\left(B + \dot{L} + kT\right)Q_i \\ &\equiv -a^2(\tau)\,\tilde{B}\,Q_i\end{aligned} \quad (67)$$

Similar calculations yield the transformation properties of all the metric perturbation variables:

$$\begin{aligned}\tilde{A}(\tau) &= A(\tau) - \frac{\dot{a}}{a}T(\tau) - \dot{T}(\tau) \\ \tilde{B}(\tau) &= B(\tau) + \dot{L} + kT(\tau) \\ \tilde{H}_L(\tau) &= H_L(\tau) - \frac{\dot{a}}{a}T - \frac{k}{3}L(\tau) \\ \tilde{H}_T(\tau) &= H_T(\tau) + kL(\tau)\end{aligned} \quad (68)$$

Exercise 11: Verify the transformation property of A and H_L.

From the transformation properties (68) of the scalar metric fluctuations, it is clear that one can choose the functions $T(\tau)$ and $L(\tau)$ such that two of the perturbation variables vanish. Popular choices are the synchronous gauge defined by $A = B = 0$ and the longitudinal gauge with $H_T = B = 0$.

Having the transformation law (68) at hand, one can construct gauge-invariant combinations, the so called *Bardeen* potentials

$$\Psi \equiv A - \frac{\dot{a}}{a}k^{-1}\sigma - k^{-1}\dot{\sigma} \quad (69)$$

$$\Phi \equiv H_L + \frac{1}{3}H_T - \frac{\dot{a}}{a}k^{-1}\sigma, \quad (70)$$

where $\sigma \equiv k^{-1}\dot{H}_T - B$ vanishes in the longitudinal gauge. Hence, the line element in the longitudinal gauge takes on the particularly convenient form

$$\begin{aligned}ds^2 = a(\tau)^2\big[&-(1+2\Psi\,Q)d\tau^2 \\ &+(1+2\Phi\,Q)\delta_{ij}dx^i dx^j\big],\end{aligned} \quad (71)$$

where we have restricted ourselves to scalar contributions.

Exercise 12: Verify that Ψ and Φ are gauge invariant.

---

2. *The Energy Momentum Tensor*

Having defined the metric, we will now specify the energy momentum tensor for matter and radiation. Even though photons during recombination (and neutrinos) need to be described by a distribution function, it is still convenient to identify certain moments of these distributions as *fluid* perturbations. Here, we are going to derive the perturbation equations for one single species.[5] After deriving the perturbation equations in the fluid description, we will turn to quintessence perturbations. Let us start by defining the energy momentum tensor of a fluid:

$$T^\mu{}_\nu = p\delta^\mu{}_\nu + (p+\rho)u^\mu u_\nu + \pi^\mu{}_\nu \quad (72)$$

Here, the 4-velocity u is the velocity of the matter rest frame with respect to the coordinate frame. Usually, one assumes that the spatial components u^i are first order perturbations. With this in mind, we get from $u^\mu u^\nu g_{\mu\nu} = -1$ the time component

$$u^0 = a(\tau)^{-1}(1 - A(\tau)). \quad (73)$$

Next, we set for the spatial part

$$u^i = a^{-1}v(\tau)Q^i, \quad (74)$$

defining v.

Exercise 13: Calculate u_0 and u_i. Hint: Remember to work to first order in the perturbations only.

Lowering the index, we find for the covariant velocity

$$u_0 = -a(1+A) \qquad u_i = a(v-B)Q_i. \quad (75)$$

[5] In Appendix E we give the full equations (including momentum transfer between baryons and photons) used to calculate the CMB anisotropies.

Using the same conventions as [13, 14], let us set

$$\rho \equiv \bar{\rho}(\tau)\,[1+\delta(\tau)Q], \tag{76}$$

and the spatial trace

$$p\,\delta^i{}_j \equiv \bar{p}(\tau)\,[1+\pi_{\rm L}(\tau)Q]\,\delta^i{}_j, \tag{77}$$

while for the traceless part

$$\pi^i{}_j \equiv \bar{p}\,\Pi\,Q^i_j. \tag{78}$$

This *defines* the perturbations Π and $\pi_{\rm L}$ and δ. Working to first order one gets from these definitions

$$T^0{}_0 = -\bar{\rho}(1+\delta\,Q) \tag{79}$$
$$T^0{}_i = (\bar{\rho}+\bar{p})\,(v-B)\,Q_i \tag{80}$$
$$T^i{}_0 = -(\bar{\rho}+\bar{p})\,v\,Q^i \tag{81}$$
$$T^i{}_j = \bar{p}\left[(1+\pi_{\rm L}\,Q)\delta^i_j+\Pi Q^i_j\right]. \tag{82}$$

3. Gauging the Energy

We will now investigate the gauge dependence of the energy momentum perturbations.

From Equations (64), (66) and (15), we get

$$\tilde{T}^0{}_0(x) = T^0{}_0(x)+\bar{T}^0{}_0\epsilon^0_{,0}-\bar{T}^0{}_0\epsilon^0_{,0}-\bar{T}^0{}_{0,0}\epsilon^0 \tag{83}$$
$$= -\bar{\rho}\left(1+\left[\delta-\frac{\dot{\bar{\rho}}}{\bar{\rho}}T\right]Q\right)$$
$$= -\bar{\rho}\left(1+\left[\delta+3(1+w)\frac{\dot{a}}{a}T\right]Q\right). \tag{84}$$

Hence,

$$\tilde{\delta} = \delta+3(1+w)\frac{\dot{a}}{a}T. \tag{85}$$

Exercise 14: Mimicking the above procedure, what is $\tilde{\pi}_{\rm L}$?

The velocity perturbation transforms as

$$\tilde{v} = v+\dot{L}, \tag{86}$$

which can be seen by either calculating $d\tilde{x}^i/d\tilde{\tau}$ or by transforming $T^i{}_0$. Finally, we get from a calculation similar to the one for δ, that $\pi_{\rm L}$ transforms as

$$\tilde{\pi}_{\rm L} = \pi_{\rm L}-\frac{\dot{\bar{p}}}{\bar{p}}T = \pi_{\rm L}+3(1+w)\frac{c_s^2}{w}\frac{\dot{a}}{a}T, \tag{87}$$

where the sound speed is given by

$$c_s^2 \equiv \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}}. \tag{88}$$

The vanishing of the off diagonal elements $\bar{T}^i{}_j$ ensures that Π is gauge invariant from the start.

4. Gauge Invariant Energy-Momentum Perturbations

We will not need to work in gauge invariant quantities. However, for completeness we present gauge invariant energy momentum perturbations in this section. There are many ways to combine one of the energy-momentum perturbations with the metric fluctuations (or another energy-momentum perturbation) to form gauge-invariant quantities. Following [14], we define

$$V \equiv v-\frac{1}{k}\dot{H}_T = v^{(longit)} \tag{89}$$
$$D_g \equiv \delta+3(1+w)\left(H_L+\frac{1}{3}H_T\right) \tag{90}$$
$$= \delta^{(longit)}+3(1+w)\Phi$$
$$D \equiv \delta^{(longit)}+3(1+w)\frac{\dot{a}}{a}\frac{V}{k} \tag{91}$$
$$\Gamma \equiv \pi_{\rm L}-\frac{c_s^2}{w}\delta, \tag{92}$$

where (*longit*) labels perturbations in the longitudinal gauge, and Γ can be viewed as entropy production rate [22]. This is due to the fact that for adiabatic perturbations for which $\delta p/\delta\rho = \dot{\bar{p}}/\dot{\bar{\rho}}$ and therefore

$$\Gamma = \pi_L-\frac{c_s^2}{w}\delta \tag{93}$$
$$= \frac{\delta p}{p}-\frac{c_s^2}{w}\frac{\delta\rho}{\rho} \tag{94}$$
$$= \frac{\delta p}{p}-c_s^2\frac{\delta\rho}{w\,\rho} \tag{95}$$
$$= \frac{\delta p}{p}-\frac{\delta p}{\delta\rho}\frac{\delta\rho}{p} = 0, \tag{96}$$

in this case.

For a comparison between the cold dark matter power spectrum as inferred from $\delta^{sync.}$ in synchronous gauge and the gauge invariant D^c_g, see Figure 8.

5. Perturbed Einstein's and Conservation Equation

Having defined the metric and the energy momentum tensor, we are now in the position to use Einstein's equation to relate the metric perturbations to the matter perturbations. We will first derive the equations with $\delta,\ v\ldots$ in the longitudinal gauge and in a second step move to the gauge invariant variables. The perturbed

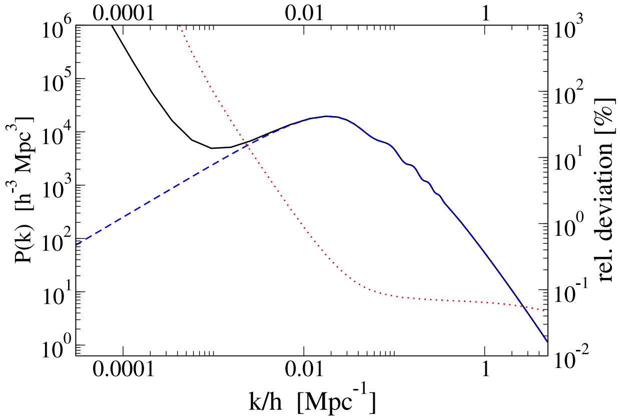

FIG. 2: Cold dark matter power spectrum using $\delta_c^{(sync.)}$ in synchronous gauge (long dashed line) and the gauge invariant D_g^c (straight line). The parameters used were $h = 0.6$, $\Omega_0^\Lambda = 0.6$, $\Omega_0^b = 0.1$, $\Omega_0^c = 1 - \Omega_0^\Lambda - \Omega_0^b$, $n_s = 1$. Also shown is the relative deviation of both spectra (dotted line). Please note the perfect agreement inside the horizon, where both display the same baryon oscillation patterns. On scales approaching and outside the horizon, where the notion of energy density perturbations becomes gauge dependent, they disagree.

part of Einstein's equations yields

$$\mathbf{G^0}_\mathbf{0} \rightarrow \quad a^2\,\bar\rho\,\delta \;=\; 2\mathrm{M}_\mathrm{P}^2\left\{k^2\Phi + 3\frac{\dot a}{a}\left(\dot\Phi - \frac{\dot a}{a}\Psi\right)\right\}, \quad (97)$$

$$a^2\,v\,(\bar\rho + \bar p) \;=\; 2\mathrm{M}_\mathrm{P}^2\,k\left(\frac{\dot a}{a}\Psi - \dot\Phi\right) \quad \leftarrow \mathbf{G^0}_\mathbf{i}, \quad (98)$$

$$\mathbf{G^i}_\mathbf{j} \rightarrow \quad a^2\bar p\,\Pi \;=\; -\mathrm{M}_\mathrm{P}^2\,k^2(\Phi + \Psi). \quad (99)$$

Conservation of the energy component $T^\mu_{\;0;\mu} = 0$, gives

$$(\bar\rho + \bar p)\left[3\dot\Phi + k\,v\right] + 3\frac{\dot a}{a}\bar p\,(\delta + w\pi_L) + \frac{\mathrm{d}}{\mathrm{d}\tau}(\bar\rho\,\delta) = 0 \quad (100)$$

whereas the momentum part $T^\mu_{\;i;\mu} = 0$ yields

$$\bar p\left(\frac{2}{3}\Pi - \pi_L\right) + k^{-1}(\bar\rho + \bar p)\left[4\frac{\dot a}{a}\,v + \dot v\right] + k^{-1}\,v\,(\dot{\bar\rho} + \dot{\bar p}) - (\bar\rho + \bar p)\Psi = 0 \quad (101)$$

Please note that in principle, the equation of state w and the speed of sound c_s could be time dependent. It should also be remarked that we use $c_s^2 = \frac{\dot{\bar p}}{\dot{\bar\rho}}$ as defined by Bardeen. This is the adiabatic sound speed. However, for instance in the case of a scalar field which has two "degrees of freedom", the adiabatic sound speed is not the speed of propagation of perturbations.

Exercise 15: Rewrite Equations (100) and (101) to get evolution equations for δ and v. Replace $\dot{\bar\rho}/\bar\rho$ using Equation (15) and substitute Γ for π_L.

Rewriting (100) and (101), we finally get

$$-\dot\delta = (1+w)\left[k\,v + 3\dot\Phi\right] + 3\frac{\dot a}{a}\,w\,\Gamma + 3\frac{\dot a}{a}\delta\,(c_s^2 - w)\,, \quad (102)$$

and

$$\dot v = \frac{\dot a}{a}\left(3\,c_s^2 - 1\right)\,v + k\Psi + \frac{k\,c_s^2}{1+w}\delta + \frac{k\,w}{1+w}\left[\Gamma - \frac{2}{3}\Pi\right]. \quad (103)$$

Exercise 16: Consider a universe with only radiation present. How do the perturbations δ and v behave ? Hints: In a RD universe, we have already solved Friedmann's equation and hence we know that $\frac{\dot a}{a} = 1/\tau$. Plugging this back into Friedmann's equation, you can immediately read off an expression for $\mathrm{M}_\mathrm{P}^{-2}\,a^2\bar\rho$ which you can substitute into Einstein's equations (97-99). In addition, a radiation fluid has $c_s^2 = w = 1/3$, $\Gamma = \Pi = 0$. Use the product $x \equiv k\,\tau$ and solve for super-horizon $x \ll 1$ and sub-horizon $x \gg 1$ modes separately. Finally, it is useful to formulate the differential equations in terms of $\frac{\mathrm{d}}{\mathrm{d}x} = k^{-1}\frac{\mathrm{d}}{\mathrm{d}\tau}$ which you may indicate using a prime. In order to coordinate our efforts, let's proceed like this:
1. Get an expression for $\mathrm{M}_\mathrm{P}^{-2}\,a^2\bar\rho$
2. Express Φ in terms of Ψ from Einstein's Equations
3. Plug in the values of c_s^2, w, Γ and Π into the e.o.m.'s and re-express these in terms of derivatives w.r.t $x = k\,\tau$
4. Solve the e.o.m's and Poisson's equation (97) in the case $x \ll 1$ and $x \gg 1$

Additional reading: The equations of motion in gauge invariant variables...

In gauge invariant variables, Einstein's equations become

$$a^2\bar{\rho}D = 2\,\mathrm{M}_\mathrm{P}^2\,k^2\Phi \quad (104)$$

$$a^2(\bar{\rho}+\bar{p})V = 2\,\mathrm{M}_\mathrm{P}^2\,k\left(\frac{\dot{a}}{a}\Psi - \dot{\Phi}\right) \quad (105)$$

$$a^2\bar{p}\,\Pi = -\mathrm{M}_\mathrm{P}^2\,k^2(\Phi+\Psi),. \quad (106)$$

the e.o.m's for the density perturbation and velocity are:

$$\dot{D}_g + 3\left(c_s^2 - w\right)\frac{\dot{a}}{a}D_g + k\,V(1+w) + 3\frac{\dot{a}}{a}\,w\,\Gamma = 0, \quad (107)$$

and

$$\dot{V} = \frac{\dot{a}}{a}(3c_s^2-1)V + k\left[\Psi - 3c_s^2\Phi\right] + \frac{c_s^2 k}{1+w}D_g + \frac{wk}{1+w}\left[\Gamma - \frac{2}{3}\Pi\right] \quad (108)$$

V. CMB ANISOTROPIES

In this section, we will outline the derivation of the CMB anisotropy calculation and review the line of sight [15] strategy. The discussion resembles closely that in [14] and deals mainly with scalar fluctuations.

Ultimately, we will calculate the so called C_l spectrum. This spectrum is a plot of the two point correlation of temperature anisotropy on the sky. As we will later see, a multipole l corresponds to an angular separation in the sky according to $l \approx 180°/\theta$. So the higher l, the smaller the angular separation one measures. We postpone the business of the C_l's and their intuitive interpretation until later. First, we will calculate in great detail how the photon distribution function – and as we will see also the temperature anisotropy – evolves as time progresses.

We will mainly use longitudinal gauge to derive our results but express the temperature fluctuation in terms of a gauge invariant quantity. Strictly speaking this is not necessary, because only the monopole C_0 and dipole C_1 are affected by gauge transformation (corresponding to one scalar and one vector degree of freedom in choosing the coordinate system). Hence, any gauge will yield the same C_l spectrum from $l = 2$ on. Having spend quite some time discussing gauge freedom, we will nevertheless stick to gauge invariant expressions, when it is not computationally inconvenient.

A. The Homogenous Photon Background

The expansion of the Universe is slow compared to the microwave frequency of the CMB. It is hence adiabatic, as far as the photons are concerned. Therefore the occupation number of the background

$$\bar{f} = \left[\exp\left(\frac{E}{\bar{T}}\right) - 1\right]^{-1}, \quad (109)$$

is preserved. As the energy of a photon redshifts $\propto a^{-1}$ we see that $T \propto a^{-1}$ in order to keep the ration E/T constant. Writing $T = T_0\,a^{-1}$, we see that $\bar{f}$ is a function of aE only:

$$\bar{f}(aE) = \left[\exp\left(\frac{aE}{\bar{T}_0}\right) - 1\right]^{-1}. \quad (110)$$

Later, we will use the geodesic equation which is cast in terms of the momentum. It is therefore clever to treat E for the momentum. To make the connection, we note that in General Relativity, the energy of a photon is given by

$$E = -u_\mu p^\mu, \quad (111)$$

and as the observers in the unperturbed background are at rest, i.e. $\bar{u}_\mu = a(-1,0,0,0)$, we see that

$$\bar{E} = -ap, \quad (112)$$

where $p \equiv |\boldsymbol{p}| = \sqrt{p^i p^j \delta_{ij}}$ and the above holds, because in the background $p^0 = p$. As $\bar{f}$ solely depends on aE, we see that in terms of momentum, $\bar{f}$ depends solely on

$$\mathcal{P} = a^2 p. \quad (113)$$

Now what we need to do is follow the evolution of the full distribution function $f(x^\mu, p^\mu)$ as time progresses. Excellent articles of the late 60's and early 70' (e.g. [24]) and a very good book [27] discuss relativistic kinetic theory. A small section in the Appendix will outline the derivation of Boltzmann's equation in its standard form using (x^μ, p^μ) as the arguments of f.

While we could certainly use this standard description (and I encourage you to check that it leads to the same result), we will use more convenient arguments for f, for instance $\mathcal{P}$.

B. The Full Boltzmann Equation for Photons

The full distribution function is naively a function of (x^μ, p^μ). Yet, the physics governing the evolution of f respects the mass shell condition $p_\mu p^\mu = m^2$. So one can either evolve $f(x^\mu, p^\mu)$ and at the end of the calculation use the mass-shell condition – or one uses it to obtain one momentum component as a function of the others from the start. It turns out that it is much more economical to regard p^0 as a function of p^i right from the start. Therefore, f is a function $f(x^\mu, p^i)$ only.

As we have already mentioned, we will use $\mathcal{P}$ as an argument of f. In order to do this, let us first split the spatial momentum

$$p^i \equiv p\, n^i, \tag{114}$$

into its modulus p and the unit vector of photon momentum $\boldsymbol{n}$. So $\delta_{ij}n^i n^j = 1$ and $p \equiv |\boldsymbol{p}| = \sqrt{p^i p^j \delta_{ij}}$ just like in the unperturbed case (the definition of p will never change in the following). Now, we are in business! Using $\mathcal{P} = a^2\, p$ to trade p for $\mathcal{P}$, we arrive at our final set of variables for f:

$$f = f(\boldsymbol{x}, \tau, \mathcal{P}, \boldsymbol{n}). \tag{115}$$

In particular, we can again split this into a background plus perturbation part

$$f(\tau, \boldsymbol{x}, \mathcal{P}, \boldsymbol{n}) = \bar{f}(\mathcal{P}) + F(\tau, \boldsymbol{x}, \mathcal{P}, \boldsymbol{n}). \tag{116}$$

In order to be equipped for the later calculation, we derive some results in the next section that will be useful soon.

1. *Useful relations for the photon distribution function*

Being variables on which f depends, the p^i's are free of perturbations. However, p^0 contains perturbation terms. In fact, from $g_{\mu\nu}p^\mu p^\nu = 0$ we get in longitudinal gauge

$$\begin{aligned} 0 &= -(1+2\Psi)(p^0)^2 + \boldsymbol{p}^2 + 2H_{ij}p^i p^j && (117)\\ &= -(1+2\Psi)(p^0)^2 + \boldsymbol{p}^2 + 2\Phi\, Q\delta_{ij}p^i p^j, && (118)\end{aligned}$$

which we multiply with $(1 - 2\Psi)$

$$\begin{aligned} 0 &= (p^0)^2 + \boldsymbol{p}^2\,(1 + 2\Phi\, Q)\,(1 - 2\Psi\, Q) && (119)\\ &= (p^0)^2 + \boldsymbol{p}^2\,(1 + 2\Phi\, Q - 2\Psi\, Q)\,, && (120)\end{aligned}$$

yielding

$$\begin{aligned} p^0 &= \sqrt{\boldsymbol{p}^2\,[1 - 2\Psi\, Q + 2\Phi\, Q]} && (121)\\ &= |\boldsymbol{p}|(1 - \Psi\, Q + \Phi\, Q). && (122)\end{aligned}$$

All in all, p^0 the covariant components p_μ, as well as $\sqrt{-g}$ are given by

$$\begin{aligned} p^0 &= p\,(1 - \Psi\, Q + \Phi\, Q) && (123)\\ p_0 &= -a^2 p\,(1 + [\Psi + \Phi]Q) && (124)\\ p_i &= a^2 p\, n_i\,(1 + 2\Phi Q) && (125)\\ \sqrt{-g} &= a^4(1 + \Psi Q + 3\Phi Q), && (126)\end{aligned}$$

where $p \equiv |\boldsymbol{p}| = \sqrt{p^i p^j \delta_{ij}}$ as always.

Very soon, we will encounter terms of the form $n_i Q^i$ involving dot products $\boldsymbol{n}\cdot\boldsymbol{k}$ between the photon direction and the Fourier vector $\boldsymbol{k}$. It is useful to follow the existing literature to define the direction cosine μ via

$$n_i Q^i = -i\mu Q \tag{127}$$

which in a flat Universe simply translates into

$$\mu = k^{-1}\boldsymbol{k}\cdot\boldsymbol{n}. \tag{128}$$

Finally, a useful formula we need is

$$\frac{\partial p}{\partial p^i} = \frac{\partial\sqrt{\delta_{mn}p^m p^n}}{\partial p^i} = \frac{1}{2}\frac{2p^i}{p} = n^i. \tag{129}$$

2. *Boltzmann's Equation for photons (I): Derivation and Collisionless Part*

After all this preparation, let us finally compute Boltzmann's Equation. We would like to compute both the temperature anisotropies and polariazation. Except for the collisions which mix the components, the evolution is quite similar. The main difference is that while the intensity has a background value $\bar{f}$, the polarization vanishes identically in the background. Hence, the following discussion applies to polarization as well by simply replacing $F \to G$ (we use G to denote the linear polarization distribution function) and $\bar{f} = \bar{f}' \to 0$.

So we would like to calculate the total change in the distribution function $f(\boldsymbol{x}, \tau, \mathcal{P}, \boldsymbol{n})$:

$$\left(\frac{\partial f}{\partial\tau}\right)_{\mathcal{P}} + \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial\tau} + \frac{\partial f}{\partial\mathcal{P}}\frac{\partial\mathcal{P}}{\partial\tau} + \frac{\partial f}{\partial n^i}\frac{\partial n^i}{\partial\tau} = C[f, G], \tag{130}$$

where $C[f, G]$ represents collision terms. The last term in (130) vanishes, because it is of second order in perturbation theory: $\bar{f}$ does not depend on n^i and hence $\frac{\partial f}{\partial n^i}$ is a perturbation. In addition, $\frac{\partial n^i}{\partial\tau}$ is a change in photon direction that can only come from a spatially inhomogeneous scattering process. So all in all the last term is of second order and we can safely discard it.

The most difficult term to compute is the third one on the l.h.s of equation (130). We need an expression for

$$\frac{\partial\mathcal{P}}{\partial\tau} = \frac{\partial}{\partial\tau}a^2 p. \tag{131}$$

To this end, we note that

$$\frac{\partial}{\partial\tau}a^2 p = 2\frac{\dot{a}}{a}a^2 p + a^2\frac{\partial p}{\partial\tau}, \tag{132}$$

and using Equation (129)

$$\frac{\partial p}{\partial\tau} = \frac{\partial p}{\partial p^i}\frac{\partial p^i}{\partial\tau} = n_i\frac{\partial p^i}{\partial\tau}. \tag{133}$$

Luckily, $\frac{\partial p^i}{\partial \tau}$ can be computed from the geodesic equation

$$p^0 \frac{\partial p^i}{\partial \tau} + \Gamma^i_{\alpha\beta} p^\alpha p^\beta = 0. \tag{134}$$

Indeed,

$$n_i \frac{\partial p^i}{\partial \tau} = -\left(p^0\right)^{-1} n_i \, \Gamma^i_{\alpha\beta} p^\alpha p^\beta. \tag{135}$$

Let us collect all terms of $n_i \Gamma^i_{\alpha\beta} p^\alpha p^\beta$. We will do this in great detail keeping both unperturbed and perturbed parts. From $\Gamma^i_{00} p^0 p^0$, and noting that $\bar{\Gamma}^i_{00} = 0$, $\delta\Gamma^i_{00} = -k\Psi Q^i$, we get

$$n_i \Gamma^i_{00} p^0 p^0 = -k\Psi p^2 n_i Q^i \tag{136}$$
$$= i\mu k p^2 \Psi Q. \tag{137}$$

From $\Gamma^i_{0j} p^0 p^j$, and noting that $\bar{\Gamma}^i_{0j} = \frac{\dot{a}}{a} \delta^i_j$, $\delta\Gamma^i_{0j} = \dot{\Phi} \delta^i_j Q$ and that we will get a factor of 2, because the lower indices could be either $0j$ or $j0$,

$$n_i \Gamma^i_{0j} p^0 p^j = 2\frac{\dot{a}}{a} \delta^i_j p^0 n_i p^j + 2\dot{\Phi} Q p^2 n_i n^i \tag{138}$$
$$= 2\frac{\dot{a}}{a} p^0 \, p + 2\dot{\Phi} Q p^2 \tag{139}$$

Finally, from $\Gamma^i_{jl} p^l p^j$, and noting that $\bar{\Gamma}^i_{jl} = 0$, $\delta\Gamma^i_{jl} = k\Phi \left(Q^i \delta_{jl} - Q_j \delta^i_l - Q_l \delta^i_j\right)$, we get

$$n_i \Gamma^i_{jl} p^j p^l = k\Phi n_i \left(Q^i \delta_{jl} - Q_j \delta^i_l - Q_l \delta^i_j\right) p^j p^l \tag{140}$$
$$= k\Phi (n_i Q^i p^2 - 2 n_i Q_j p^j p^i) \tag{141}$$
$$= k\Phi (n_i Q^i p^2 - 2 p_j Q^j n_i p^i) \tag{142}$$
$$= k\Phi (n_i Q^i p^2 - 2 p^2 n_j Q^j) \tag{143}$$
$$= -k\Phi p^2 n_i Q^i \tag{144}$$
$$= i\mu k p^2 \Phi Q. \tag{145}$$

Collecting all terms, we have

$$n_i \Gamma^i_{\beta\gamma} p^\beta p^\gamma = 2\frac{\dot{a}}{a} p^0 \, p + p^2 \Big[i\mu k (\Phi + \Psi) + 2\dot{\Phi} \Big] Q. \tag{146}$$

According to Equation (133), we need to multiply this Equation (146) by $-(p^0)^{-1}$ to find $\frac{\partial p}{\partial \tau}$,

$$\frac{\partial p}{\partial \tau} = -(p^0)^{-1} n_i \Gamma^i_{\beta\gamma} p^\beta p^\gamma \tag{147}$$
$$= -2\frac{\dot{a}}{a} p - p \Big\{ i\mu k [\Phi + \Psi] + 2\dot{\Phi} \Big\} Q \tag{148}$$
$$\tag{149}$$

We are almost there. Equation (132) tell us that

$$\frac{\partial \mathcal{P}}{\partial \tau} = 2\frac{\dot{a}}{a} a^2 p + a^2 \frac{\partial p}{\partial \tau} \tag{150}$$
$$= 2\frac{\dot{a}}{a} \mathcal{P} - \mathcal{P} \left[2\frac{\dot{a}}{a} + \Big\{ i\mu k [\Phi + \Psi] + 2\dot{\Phi} \Big\} Q \right] \tag{151}$$
$$= -\mathcal{P} \Big[i\mu k [\Phi + \Psi] + 2\dot{\Phi} \Big] Q \tag{152}$$

and hence

$$\frac{\partial f}{\partial \mathcal{P}} \frac{\partial \mathcal{P}}{\partial \tau} = -\mathcal{P} \bar{f}' \Big\{ i\mu k [\Phi + \Psi] + 2\dot{\Phi} \Big\} Q. \tag{153}$$

The final piece in Boltzmann's equation is the spatial part, i.e.

$$\frac{\partial f}{\partial x^i} \frac{\partial x^i}{\partial \tau} = \frac{\partial F}{\partial x^i} n^i \tag{154}$$
$$= F(\tau, \mathcal{P}) Q_{,i} n^i \tag{155}$$
$$= -k F(\tau, \mathcal{P}) n_i Q^i \tag{156}$$
$$= i\mu k F(\boldsymbol{x}, \tau, \mathcal{P}, \boldsymbol{n}) \tag{157}$$

which holds, because $\bar{f}$ does not depend on x^i and $\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = u^i = p^i/p = n^i$.

Let's pause here a little to collect all terms involving the background only. It turns out that this collection doesn't leave us with much, namely

$$\left(\frac{\partial \bar{f}}{\partial \tau} \right)_{\mathcal{P}} = 0. \tag{158}$$

There is not even a collision term, because the collisions average out in a homogenous plasma (as we shall see shortly). Equation (158) is the proof that we have been right in our conjecture that $\bar{f}$ is a function of $\mathcal{P}$ only: The change in a distribution function of massless particles which depends solely on $\mathcal{P}$ is zero! Now that's a fancy derivation of the preservation of the background black body spectrum!

As far as the perturbed distribution is concerned, it's much more exciting:

$$\left(\frac{\partial F}{\partial \tau} \right)_{|\mathcal{P}} + ik\mu F - \mathcal{P} \bar{f}' \Big\{ i\mu k [\Phi + \Psi] + 2\dot{\Phi} \Big\} Q = C[f, G]. \tag{159}$$

Finally, making the substitution $F \to G$, $\bar{f}' \to 0$, we get the simple evolution equation for the linear polarization G

$$\left(\frac{\partial G}{\partial \tau} \right)_{|\mathcal{P}} + ik\mu G = C_G[f, G], \tag{160}$$

where C_G is the collision term for G.

3. *Boltzmann's Equation for photons (II): The collision term*

Electrons and photons scatter in the plasma of the early universe. At the relatively low energies of a few eV before recombination, the only efficient scattering process is Thomson scattering. From electrodynamics, we know the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{3\sigma_T}{8\pi} |\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}'|^2 \tag{161}$$

of Thomson scattering an incident beam polarized along some axis $\boldsymbol{\epsilon}'$ into the outgoing polarization $\boldsymbol{\epsilon}$. Unfortunately, the above formula for Thomson scattering is valid in the Minkowski rest frame of the electron. In this frame, Thomson scattering is elastic, i.e. $\tilde{p}^0 = \tilde{p}^{0\prime}$. In the FRW frame, however, the electrons might move relative to the photon fluid. This introduces a doppler shift which will automatically come out in our discussion later on. For the time being, however let us take one step after the other and compute the change in some Minkowski rest frame intensity $\tilde{f}$ and polarization $\tilde{G}$ and later go back into the FRW frame.

To make life as simple as possible (don't worry, it still won't be super easy :-), let choose our coordinate system best as possible. We orient the z-axis such that $z||\boldsymbol{k}$. In addition, we choose the x-axis such that the outgoing direction $\boldsymbol{n}$ is in the $x-z$ plane. To compute both intensity and polarization, we need a basis which we choose as

$$\boldsymbol{\epsilon}_1 = \boldsymbol{\epsilon}_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)^T \quad (162)$$
$$\boldsymbol{\epsilon}_2 = \boldsymbol{\epsilon}_\phi = (-\sin\phi, \cos\phi, 0)^T. \quad (163)$$

So for the outgoing beam with $\phi = 0$, this gives

$$\boldsymbol{\epsilon}_\theta = (\cos\theta, 0, -\sin\theta)^T, \quad (164)$$
$$\boldsymbol{\epsilon}_\phi = (0, 1, 0)^T. \quad (165)$$

We will briefly work with intensities along the θ and ϕ unit vectors,

$$\tilde{f}_\theta(\tilde{x}, \tilde{p}', \boldsymbol{n}') \equiv \frac{\tilde{f}(\tilde{x}, \tilde{p}', \boldsymbol{n}') + \tilde{G}(\tilde{x}, \tilde{p}', \boldsymbol{n}')}{2} \quad (166)$$
$$\tilde{f}_\phi(\tilde{x}, \tilde{p}', \boldsymbol{n}') \equiv \frac{\tilde{f}(\tilde{x}, \tilde{p}', \boldsymbol{n}') - \tilde{G}(\tilde{x}, \tilde{p}', \boldsymbol{n}')}{2} \quad (167)$$

In terms of these, the Thomson cross section says that the number of particles scattered into the beam $\boldsymbol{n}$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\tilde{f}_\theta^+ = \frac{3\sigma_T\, n_e}{8\pi}\int \tilde{f}_\theta(\tilde{x}, \tilde{p}', \boldsymbol{n}')|\boldsymbol{\epsilon}_\theta\boldsymbol{\epsilon}_\theta'|^2 + \tilde{f}_\phi(\tilde{x}, \tilde{p}', \boldsymbol{n}')|\boldsymbol{\epsilon}_\theta\boldsymbol{\epsilon}_\phi'|^2 \mathrm{d}\Omega' \quad (168)$$

and likewise

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\tilde{f}_\phi^+ = \frac{3\sigma_T\, n_e}{8\pi}\int \tilde{f}_\theta(\tilde{x}, \tilde{p}', \boldsymbol{n}')|\boldsymbol{\epsilon}_\phi\boldsymbol{\epsilon}_\theta'|^2 + \tilde{f}_\phi(\tilde{x}, \tilde{p}', \boldsymbol{n}')|\boldsymbol{\epsilon}_\phi\boldsymbol{\epsilon}_\phi'|^2 \mathrm{d}\Omega' \quad (169)$$

To compute these, let's compute $|\boldsymbol{\epsilon}_\theta\boldsymbol{\epsilon}_\theta'|^2$ etc.

$$|\boldsymbol{\epsilon}_\theta\boldsymbol{\epsilon}_\theta'|^2 = \cos^2\theta\cos^2\theta'\cos^2\phi' + \sin^2\theta\sin^2\theta' + 2\cos\phi'\cos\theta\cos\theta'\sin\theta\sin\theta' \quad (170)$$
$$|\boldsymbol{\epsilon}_\theta\boldsymbol{\epsilon}_\phi'|^2 = \sin^2\phi'\cos^2\theta \quad (171)$$
$$|\boldsymbol{\epsilon}_\phi\boldsymbol{\epsilon}_\theta'|^2 = \sin^2\phi'\cos^2\theta' \quad (172)$$
$$|\boldsymbol{\epsilon}_\phi\boldsymbol{\epsilon}_\phi'|^2 = \cos^2\phi' \quad (173)$$

In our cleverly chosen coordinate system, we can write $\mu = k^{-1}\boldsymbol{kn} = \cos\theta$, $\lambda = k^{-1}\boldsymbol{kn}' = \cos\theta'$. In addition, the distributions will not depend on the azimutal angle ϕ, because ${}_0G_l^0$ is independent of ϕ. We can therefore easily perform the $\int \mathrm{d}\phi'$ part of the $\mathrm{d}\Omega'$ integration above. Let's keep in mind that $\int_0^{2\pi}\cos^2\phi\,\mathrm{d}\phi = \int_0^{2\pi}\sin^2\phi\,\mathrm{d}\phi = \pi$, $\int_0^{2\pi}\cos\phi\,\mathrm{d}\phi = 0$ and $\int_0^{2\pi}\mathrm{d}\phi = 2\pi$. Then Equations (170) to (173) yield

$$\int_0^{2\pi}\mathrm{d}\phi'|\boldsymbol{\epsilon}_\theta\boldsymbol{\epsilon}_\theta'|^2 = \pi\left(3\mu^2\lambda^2 - 2\lambda^2 - 2\mu^2 + 2\right) \quad (174)$$
$$\int_0^{2\pi}\mathrm{d}\phi'|\boldsymbol{\epsilon}_\theta\boldsymbol{\epsilon}_\phi'|^2 = \pi\mu^2 \quad (175)$$
$$\int_0^{2\pi}\mathrm{d}\phi'|\boldsymbol{\epsilon}_\phi\boldsymbol{\epsilon}_\theta'|^2 = \pi\lambda^2 \quad (176)$$
$$\int_0^{2\pi}\mathrm{d}\phi'|\boldsymbol{\epsilon}_\phi\boldsymbol{\epsilon}_\phi'|^2 = \pi. \quad (177)$$

It's about time to move back from $\tilde{f}_\theta$ and $\tilde{f}_\phi$ to $\tilde{f}$ and $\tilde{G}$. Let me for the sake of notationally simplicity drop the arguments $\tilde{x}$ and $\tilde{p}'$ of $\tilde{f}$ and $\tilde{G}$. In addition, $\tilde{f}$ and $\tilde{G}$ depend on $\boldsymbol{n}'$ only through λ. The total intesity scattered into the beam is then

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\tilde{f}^+ = \frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\tilde{f}_\theta^+ + \frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\tilde{f}_\phi^+ \quad (178)$$
$$= \frac{3\sigma_T\, n_e}{8}\int_{-1}^{1}\mathrm{d}\lambda\frac{\tilde{f}(\lambda) - \tilde{G}(\lambda)}{2}(\mu^2+1) + \frac{\tilde{f}(\lambda) + \tilde{G}(\lambda)}{2}\left(3\mu^2\lambda^2 - \lambda^2 - 2\mu^2 + 2\right) \quad (179)$$
$$= \frac{3\sigma_T\, n_e}{8}\int_{-1}^{1}\mathrm{d}\lambda\tilde{f}(\lambda)\left[3\mu^2\lambda^2 - \lambda^2 - \mu^2 + 3\right] + \tilde{G}(\lambda)\left[3\mu^2\lambda^2 - \lambda^2 - 3\mu^2 + 1\right] \quad (180)$$

The polynomials above in the square brackets should trigger a thought in your brain right now: hey, don't those look like Legendre Polynomials multiplied together ? Indeed they do. Remember that $P_2(\lambda) = \frac{1}{2}(3\lambda^2 - 1)$. After some minor shuffling around, you'll find (try it!)

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\tilde{f}^+ = \frac{\sigma_T\, n_e}{4}\int_{-1}^{1}\mathrm{d}\lambda\tilde{f}(\lambda)\left[P_2(\lambda)P_2(\mu) + 2\right] + \tilde{G}(\lambda)P_2(\mu)[P_2(\lambda) - 1]. \quad (181)$$

Very similar along the same lines, one can obtain (again: try it, it's fairly simple)

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\tilde{G}^+ = \frac{\sigma_T\, n_e}{4}\int_{-1}^{1}\mathrm{d}\lambda\tilde{f}(\lambda)\left[P_2(\lambda)[P_2(\mu) - 1]\right] + \tilde{G}(\lambda)[P_2(\mu) - 1][P_2(\lambda) - 1]. \quad (182)$$

Radiation leaving the beam direction $\boldsymbol{n}$ is much easier to compute, because every scattering event will lead to the radiation leaving the beam:

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\tilde{f}^{-} = n_e\sigma_T\tilde{f}(\tilde{x},\tilde{p},\boldsymbol{n}) \quad (183)$$

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\tilde{G}^{-} = n_e\sigma_T\tilde{G}(\tilde{x},\tilde{p},\boldsymbol{n}) \quad (184)$$

(no primes here!).

We are almost there! The above holds in the rest frame of the electron. Yet, what we really want to calculate is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}f(x^\mu,p^i), \quad (185)$$

i.e. the total change in the distribution function due to collisions. As f is a scalar (which does not transform under coordinate transformations), we can rewrite this as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}f(x^\mu,p^i) = \frac{\mathrm{d}\tilde{t}}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\tilde{f}(\tilde{x},\tilde{p})$$
$$= \frac{\mathrm{d}\tilde{t}}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\tilde{f}_{+} - \frac{\mathrm{d}\tilde{t}}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\tilde{f}_{-}, \quad (186)$$

As already mentioned, we may replace $\tilde{f}(\tilde{x},\tilde{p}',\boldsymbol{n}')$ by $f(x,\mathcal{P}',\boldsymbol{n}')$, because f is a scalar. So

$$\tilde{f}(\tilde{x},\tilde{p}',\boldsymbol{n}') = f(x,\mathcal{P}',\boldsymbol{n}'). \quad (187)$$

Let us power expand the r.h.s in $\Delta\mathcal{P} \equiv \mathcal{P}' - \mathcal{P} = a^2p' - a^2p$:

$$\tilde{f}(\tilde{x},\tilde{p}',\boldsymbol{n}') = f(x,\mathcal{P},\boldsymbol{n}') + \frac{\partial f}{\partial\mathcal{P}}\Delta\mathcal{P}. \quad (188)$$

In an exercise at the end of Appendix A, you can compute

$$\Delta\mathcal{P} = i\,\mathcal{P}(\mu-\lambda)Qv_b, \quad (189)$$

hence $\Delta\mathcal{P}$ is a perturbation and we can write

$$\tilde{f}(\tilde{x},\tilde{p}',\boldsymbol{n}') = \bar{f}(\mathcal{P}) + \frac{\partial\bar{f}}{\partial\mathcal{P}}\Delta\mathcal{P} + F(\boldsymbol{x},\tau,\mathcal{P},\boldsymbol{n}'). \quad (190)$$

Please note that it does not matter whether the argument of F is $\mathcal{P}$ or $\mathcal{P}'$, because the difference is of second order in perturbation theory. There is no such complication for G, because it vanishes in the background, hence

$$\tilde{G}(\tilde{x},\tilde{p}',\boldsymbol{n}') = G(\boldsymbol{x},\tau,\mathcal{P},\boldsymbol{n}'). \quad (191)$$

Before we proceed to take the final steps, we need to pause a bit. We didn't yet put all parts together, but what we derived so far is the Boltzmann equation for F and G. At this point, you surely have wondered already how we are going to connect this to the familiar language of density perturbations, velocities etc., which couple to the metric perturbations. One way would be to take moments of the entire Boltzmann equation, i.e. multiply the entire equation by some combination of momenta p and integrate over momenta. There are cases in which one has to do this, however in our case, we are lucky. Neither graviation nor Thomson scattering is the least interested in the momentum of our photons. We can therefore separate the momentum dependence from the rest. Let's do it!

4. *Connecting the perturbed distribution to temperature*

The relative temperature perturbation Δ is defined as

$$T(\tau,\boldsymbol{x},\boldsymbol{n}) = \bar{T}(\tau)[1+\Delta(\tau,\boldsymbol{x},\boldsymbol{n})]. \quad (192)$$

F and Δ are connected. Just observe what happens to $\bar{f}$ if one uses $\frac{\mathcal{P}}{1+\Delta}$ as the argument of $\bar{f}$ and expands in Δ:

$$f = \bar{f}\left(\frac{\mathcal{P}}{1+\Delta}\right) = \bar{f} + \frac{\partial\bar{f}}{\partial\mathcal{P}}\left[\frac{\mathcal{P}}{1+\Delta} - \mathcal{P}\right] + \ldots \quad (193)$$
$$= \bar{f} + \frac{\partial\bar{f}}{\partial\mathcal{P}}\mathcal{P}\left(\frac{1}{1+\Delta} - 1\right) \quad (194)$$
$$= \bar{f} + \frac{\partial\bar{f}}{\partial\mathcal{P}}\mathcal{P}(1-\Delta-1) \quad (195)$$
$$= \bar{f} - \mathcal{P}\bar{f}'\Delta. \quad (196)$$

Here and in the following, $\bar{f}'$ denotes the derivative w.r.t $\mathcal{P}$. Comparing Equations (196) to (116), we get

$$F(\tau,\boldsymbol{x},\mathcal{P},\boldsymbol{n}) = -\mathcal{P}\bar{f}'\Delta(\tau,\boldsymbol{x},\boldsymbol{n}). \quad (197)$$

The linear polarization G (which vanishes in the background) carries the same p-dependence as the intensity: Thomson scattering does not alter the momentum in the rest frame and the doppler shift can be neglected as it would be a second order correction. So

$$G(\tau,\boldsymbol{x},\mathcal{P},\boldsymbol{n}) = -\mathcal{P}\bar{f}'\mathcal{Q}(\tau,\boldsymbol{x},\boldsymbol{n}). \quad (198)$$

5. *Boltzmann Equation continued*

Let us use Equation (197) to simplify our Boltzmann equation. We start with the colisionless part. Replacing F by $\Delta(\tau,\boldsymbol{x},\boldsymbol{n})$ using Equation (197), we have[6]

$$-\mathcal{P}\bar{f}'\dot{\Delta}Q - \mathcal{P}\bar{f}'ik\mu\Delta Q$$
$$-\mathcal{P}\bar{f}'\left\{i\mu k[\Phi+\Psi] + 2\dot{\Phi}\right\}Q = C[f,G], \quad (199)$$

and dividing out $-\mathcal{P}\bar{f}'Q$, we get

$$\dot{\Delta} + ik\mu\Delta = -i\mu k[\Phi+\Psi] - 2\dot{\Phi} + \hat{C}[f,G], \quad (200)$$

where $\hat{C}[f,G] \equiv -C[f]/(\mathcal{P}\bar{f}'Q)$. For this collision term, we need to finally perform the $\int \mathrm{d}\lambda$ integration. In appendix C, you can find the detailed derivation of the following useful expressions for the first few moments of

[6] $\left(\frac{\partial\bar{f}}{\partial\tau}\right)_{\mathcal{P}}$ acts on the explicit time dependence only. Hence $\left(\frac{\partial\bar{f}}{\partial\tau}\right)_{\mathcal{P}} = -\mathcal{P}\bar{f}'\dot{\Delta}$.

the distribution function

$$\delta = 4\Phi + \frac{1}{\pi}\int \Delta(\boldsymbol{n})\mathrm{d}\Omega \qquad (201)$$

$$V_\gamma = \frac{3i}{4\pi}\int \mu\Delta(\boldsymbol{n})\mathrm{d}\Omega \qquad (202)$$

$$\Pi_\gamma = -\frac{3}{\pi}\int P_2(\mu)\Delta(\boldsymbol{n})\mathrm{d}\Omega. \qquad (203)$$

where as usual $\mu = k^{-1}\boldsymbol{k}\cdot\boldsymbol{n}$ is the direction cosine and $\Delta(\boldsymbol{n})$ is a function of either $\Delta(\boldsymbol{n},\boldsymbol{x},\tau)$ or in Fourier space $\Delta(\boldsymbol{n},\boldsymbol{k},\tau)$. Equipped with the above moments and the orthogonality relation $\int \mathrm{d}\lambda P_l(\lambda)P_n(\lambda) = 2\delta_{ln}/(2l+1)$, we now have to perform the λ integration of the collision terms. Our starting point is to put (181), (183) and (186) together:

$$C[f,G] = a\frac{\sigma_T\, n_e}{4}\int_{-1}^{1}\mathrm{d}\lambda\Big\{\left[\bar f(\mathcal{P}) + \bar f'\Delta\mathcal{P} + F(\boldsymbol{x},\tau,\mathcal{P},\boldsymbol{n}')\right]$$
$$[P_2(\lambda)P_2(\mu)+2] + G(\lambda)P_2(\mu)[P_2(\lambda)-1]\Big\}$$
$$- a n_e\sigma_T[\bar f(\mathcal{P}) + F(\boldsymbol{x},\tau,\mathcal{P},\boldsymbol{n})]. \qquad (204)$$

Please remind yourself that the integration is over $\boldsymbol{n}'$. Furthermore, the background part above cancels, because $\bar f(\mathcal{P}) = \bar f(\mathcal{P})1 = \bar f(\mathcal{P})P_0(\lambda)$ and therefore $\int \mathrm{d}\lambda \bar f(\mathcal{P})2 = 4$ yields the only non-vanishing term for the background in the integration. This nicely cancels the contribution scattered out of the beam. So the lowest order expression $\mathrm{d}\tilde t/\mathrm{d}\tau = a$ was sufficient, as promised. Having eliminated the background piece, let's plug in $\Delta\mathcal{P}$ from (189) and our expression of F in terms of Δ and G in terms of $\mathcal{Q}$ and proceed slowly

$$C[f,G] = a\frac{\sigma_T\, n_e}{4}\bar f'\mathcal{P}\int_{-1}^{1}\mathrm{d}\lambda\Big\{\left[i(\mu-\lambda)Qv_b - \Delta(\tau,\boldsymbol{x},\boldsymbol{n}')\right]$$
$$[P_2(\lambda)P_2(\mu)+2] - \mathcal{Q}(\tau,\boldsymbol{x},\boldsymbol{n}')P_2(\mu)[P_2(\lambda)-1]\Big\}$$
$$+ a n_e\sigma_T\bar f'\mathcal{P}\Delta(\tau,\boldsymbol{x},\boldsymbol{n}). \qquad (205)$$

The term proportional to λv_b cannot contribute, because $\lambda = P_1(\lambda)$ and there is no other dipole in the integration. In fact, the only non-vanishing term proportional to μv_b yields $\int \mathrm{d}\lambda i\mu v_b Q2 = 4i\mu v_b Q$. In addition, $P_2(\lambda)-1 = \frac{3}{2}\lambda^2 - \frac{1}{2} - 1 = \frac{3}{2}(\lambda^2-1) = -\frac{3}{2}\sin^2\theta = -2\sqrt{6\pi/5}\,{}_2Y_2^0$ and so

$$C[f,G] = a n_e\sigma_T\bar f'\mathcal{P}\Big\{i\mu v_b Q + \Delta(\tau,\boldsymbol{x},\boldsymbol{n})$$
$$-\frac{1}{4}\int_{-1}^{1}\Delta(\tau,\boldsymbol{x},\boldsymbol{n}')\left[P_2(\lambda)P_2(\mu)+2\right]\mathrm{d}\lambda$$
$$-\frac{1}{4}\int_{-1}^{1}\mathcal{Q}(\tau,\boldsymbol{x},\boldsymbol{n}')P_2(\mu)[-2\sqrt{6\pi/5}\,{}_2Y_2^0(\lambda)]\mathrm{d}\lambda\Big\}. \qquad (206)$$

Now comes the big entry of our multipole expansion for Δ and $\mathcal{Q}\pm iU$. As $U=0$ in our frame, and dropping the k integration, they are

$$\Delta(\tau,\boldsymbol{x},\boldsymbol{n}) = \sum_l (-i)^l\Delta_l(k,\tau)P_l(\hat{\boldsymbol{k}}\boldsymbol{n})Q \qquad (207)$$

and

$$(\mathcal{Q}\pm i\,U)(\tau,\boldsymbol{x},\boldsymbol{n}) = \sum_{l=2}(-i)^l(E_l^0 \pm iB_l^0)$$
$$\times\sqrt{\frac{4\pi}{2l+1}}\,{}_{\mp 2}Y_l^0(\boldsymbol{n})\,Q \qquad (208)$$

As B_l^0 vanishes and the λ integration picks out ${}_2Y_2^0$ by virtue of the orthonormality of spin-weigthed harmonics, we get (writing $E_l^0 = E_l$ for ease of notation)

$$C[f,G] = a n_e\sigma_T\bar f'\mathcal{P}\Big\{i\mu v_b Q + \Delta(\tau,\boldsymbol{x},\boldsymbol{n})$$
$$-\frac{1}{4}\int_{-1}^{1}\Delta(\tau,\boldsymbol{x},\boldsymbol{n}')\left[P_2(\lambda)P_2(\mu)+2\right]\mathrm{d}\lambda$$
$$-\frac{1}{4}P_2(\mu)(-2)\sqrt{\frac{6\pi}{5}}\frac{1}{2\pi}(-i)^2E_2\sqrt{\frac{4\pi}{5}}Q\Big\}. \qquad (209)$$

Likewise, the λ integration for $\Delta(\lambda)$ picks out the monopole and the quadrupole

$$C[f,G] = a n_e\sigma_T\bar f'\mathcal{P}\Big\{i\mu v_b Q + \Delta(\tau,\boldsymbol{x},\boldsymbol{n})$$
$$-\frac{1}{4}\left[\frac{2}{5}(-i)^2\Delta_2P_2(\mu)Q + 4\Delta_0 Q\right] - \frac{\sqrt{6}}{10}E_2P_2(\mu)Q\Big\}. \qquad (210)$$

So

$$C[f,G] = a n_e\sigma_T\bar f'\mathcal{P}\Big\{i\mu v_b + \Delta(\tau,\boldsymbol{k},\boldsymbol{n})$$
$$+\frac{1}{10}\Delta_2P_2(\mu) - \Delta_0 - \frac{\sqrt{6}}{10}E_2P_2(\mu)\Big\}Q. \qquad (211)$$

So finally, we find the full Bolzmann equation by dividing out $-(\mathcal{P}\bar f'Q)$

$$\dot\Delta + ik\mu\Delta + \dot\kappa\Delta = -i\mu k[\Phi+\Psi] - 2\dot\Phi$$
$$+\dot\kappa\Big\{\Delta_0 - i\mu v_b + \frac{1}{10}P_2(\mu)\left[\sqrt{6}E_2 - \Delta_2\right]\Big\} \qquad (212)$$

where we defined the differential optical depth $\dot\kappa \equiv a n_e\sigma_T$. Very similiarly, we can find the evolution equation for G. In fact, this is much more simple, because the background and Doppler shift part of f do not contribute because of the $P_2(\lambda)$ that is integrated over in the first term of Equation (182). The result is

$$\dot{\mathcal{Q}}+ik\mu\mathcal{Q}+\dot{\kappa}\mathcal{Q} = \frac{1}{10}\dot{\kappa}\left[P_2(\mu)-1\right]\left[\sqrt{6}E_2-\Delta_2\right] \quad (213)$$

We can express the monopole Δ_0 nicely by observing the relations

$$\begin{aligned} \delta &= 4\Phi+4\Delta_0 && (214)\\ v_\gamma &= \Delta_1 && (215)\\ \Pi_\gamma &= \frac{12}{5}\Delta_2; && (216) \end{aligned}$$

which follow from inserting the multipole decomposition (207) into Equations (C18-C20). So in terms of longitudinal gauge perturbations our final expression is

$$\dot{\Delta}+ik\mu\Delta+\dot{\kappa}\Delta = -i\mu k[\Phi+\Psi]-2\dot{\Phi} + \dot{\kappa}\left\{\frac{1}{4}\delta_\gamma-\Phi-i\mu v_b+\frac{1}{10}P_2(\mu)\left[\sqrt{6}E_2-\Delta_2\right]\right\} \quad (217)$$

and

$$\dot{\mathcal{Q}}+ik\mu\mathcal{Q}+\dot{\kappa}\mathcal{Q} = \frac{\dot{\kappa}}{10}\left\{P_2(\mu)-1\right\}\left[\sqrt{6}E_2-\Delta_2\right], \quad (218)$$

where you may substitute $P_2(\mu)-1 = -2\sqrt{6\pi/5}\,{}_2Y_2^0(\mu)$.

At this point, you may want to read Appendix B, in which we define the gauge invariant temperature perturbation $\mathcal{M}=\Delta+2\Phi$. It's multipole decompisition looks exactly like that for Δ:

$$\mathcal{M}(\tau,\boldsymbol{x},\boldsymbol{n}) = \sum_l (-i)^l\mathcal{M}_l(k,\tau)P_l(\boldsymbol{n})Q \quad (219)$$

Using $\mathcal{M}$ and D_g^γ and V_b instead of δ_γ and v_b, you can easily verify (do it!) that

$$\dot{\mathcal{M}}+ik\mu\mathcal{M}+\dot{\kappa}\mathcal{M} = i\mu k[\Phi-\Psi] + \dot{\kappa}\left\{\frac{1}{4}D_g^\gamma-i\mu v_b+\frac{1}{10}P_2(\mu)\left[\sqrt{6}E_2-\mathcal{M}_2\right]\right\} \quad (220)$$

A standard way to solve this equation is to project out multipoles. In the old days, one needed multipole moments up to $l\approx 3000$. Since the introduction of the fast line-of-sight method, one needs far less (usually $l\approx 10$). Still we need to know the evolution of the multipole moments. The procedure is as follows: For each Legendre polynomial P_l ...

- multiply equation (220) by $P_l(\mu)$.
- replace $\mathcal{M}(\tau,\mu)$ by its multipole expansion (219)
- integrate both l.h.s and r.h.s of the new equation over μ: $\int_{-1}^{1}\mathrm{d}\mu$.
- use the orthogonality relation $\int_{-1}^{1}\mathrm{d}\mu P_l(\mu)P_n(\mu) = 2\delta_{ln}/(2l+1)$.

As an example, consider $l=0$. In this case P_l is particularly simple, because $P_0=1$. So according to the recipe (220) stays unaltered and we integrate it over μ. Let's consider each term separately. First

$$\begin{aligned} \int \mathrm{d}\mu P_0\dot{\mathcal{M}} &= \int \mathrm{d}\mu P_0\frac{\mathrm{d}}{\mathrm{d}\tau}\sum_l(-i)^l M_l(\tau)P_l(\mu)\\ &= \sum_l(-i)^l\dot{M}_l(\tau)\int \mathrm{d}\mu P_l(\mu)P_0(\mu)\\ &= \sum_l(-i)^l\dot{M}_l(\tau)\frac{2\delta_{l0}}{2l+1} && (221)\\ &= 2\dot{\mathcal{M}}_0(\tau), && (222) \end{aligned}$$

furthermore (remember that $P_1=\mu$)

$$\begin{aligned} \int \mathrm{d}\mu P_0 i\mu k\mathcal{M} &= \int \mathrm{d}\mu P_1 ik\sum_l(-i)^l(2l+1)M_l(\tau)P_l(\mu)\\ &= \sum_l(-i)^l ik\int \mathrm{d}\mu P_1(\mu)P_l(\mu)\\ &= \sum_l(-i)^l ik\frac{2\delta_{l1}}{2l+1}\\ &= i(-i)k\,\frac{2}{3}\mathcal{M}_1(\tau) && (223)\\ &= \frac{2}{3}kM_1(\tau), && (224) \end{aligned}$$

then $\dot{\kappa}\mathcal{M}$ gives a similar result as $\dot{\mathcal{M}}$:

$$\int \mathrm{d}\mu P_0\dot{\kappa}\mathcal{M} = 2\dot{\kappa}\mathcal{M}_0, \quad (225)$$

the $i\mu k[\Phi-\Psi]$ term on the r.h.s vanishes, because

$$\int \mathrm{d}\mu P_0 i\mu k[\Phi-\Psi] = \int \mathrm{d}\mu iP_1P_0k[\Phi-\Psi] = 0, \quad (226)$$

the term $\dot{\kappa}D_g^\gamma/4$ gives the same result as $\dot{\kappa}M$ (in fact the terms cancel)

$$\int \mathrm{d}\mu\dot{\kappa}\frac{1}{4}D_g^\gamma = \frac{1}{2}\dot{\kappa}D_g^\gamma = 2\dot{\kappa}\mathcal{M}_0. \quad (227)$$

The rest of the terms vanish, because they involve P_1 or P_2 and are orthogonal. Collecting the pieces, and proceeding along similar lines for $l>0$, we find the hierarchy for $\mathcal{M}$ [30]

$$\dot{\mathcal{M}}_0 = -\frac{k}{3}V_\gamma \tag{228}$$

$$\dot{\mathcal{M}}_1 = \dot\kappa(V_b - V_\gamma) + k(\Psi - \Phi) + k\left(\mathcal{M}_0 - \frac{2}{5}\mathcal{M}_2\right) \tag{229}$$

$$\dot{\mathcal{M}}_2 = -\dot\kappa\left(\mathcal{M}_2 - \mathcal{C}\right) + k\left(\frac{2}{3}V_\gamma - \frac{3}{7}\mathcal{M}_3\right) \tag{230}$$

$$\dot{\mathcal{M}}_l = -\dot\kappa\,\mathcal{M}_l + k\Bigg(\frac{l}{2l-1}\mathcal{M}_{l-1} \tag{231}$$

$$-\frac{l+1}{2l+3}\mathcal{M}_{l+1}\Bigg), \quad ; l > 2. \tag{232}$$

Again, in similar fashion, this time by integrating over $\mathrm{d}\Omega$ and projecting on spin weighted spherical harmonics ${}_2Y_l^0$, you can find from Equation (218) (try it!)

$$\dot{E}_2 = -\frac{k\sqrt{5}}{7}E_3 - \dot\kappa\left(E_2 + \sqrt{6}\,\mathcal{C}\right) \tag{233}$$

$$\dot{E}_l = k\left(\frac{{}_2\kappa_l}{2l-1}E_{l-1} - \frac{{}_2\kappa_{l+1}}{2l+3}E_{l+1}\right) - \dot\kappa\,E_l \quad ; l > 2. \tag{234}$$

Here, $\mathcal{C} = (M_2 - \sqrt{6}E_2)/10$ as above and ${}_2\kappa_l = \sqrt{l^2-4}$ are combinatorial factors [30] that should not be confused with the differential optical depth $\dot\kappa$.

Massless neutrinos follow the same multipole hierarchy as $\mathcal{M}$, however without polarization and Thomson scattering. In fact, our analysis for photons carries directly over to neutrinos – just omit the scattering. Hence, the perturbed neutrino distribution is

$$\dot{\mathcal{N}}_0 = -\frac{k}{3}V_\nu \tag{235}$$

$$\dot{\mathcal{N}}_1 = k(\Psi - \Phi) + k\left(\mathcal{N}_0 - \frac{2}{5}\mathcal{N}_2\right) \tag{236}$$

$$\dot{\mathcal{N}}_l = k\left(\frac{l}{2l-1}\mathcal{N}_{l-1} - \frac{l+1}{2l+3}\mathcal{N}_{l+1}\right) \quad ; l > 1, \tag{237}$$

where $V_\nu = \mathcal{N}_1$. In contrast to photons, there is no tight coupling to baryons. Thus, moments beyond the dipole may built up from the beginning. However, as $\dot{\mathcal{N}}_l \propto k\mathcal{N}_{l-1}$ for $l > 1$, it follows that $\mathcal{N}_l \propto (k\tau)^{(l-1)}\mathcal{N}_1$ at early times. As $k\tau \ll 1$ for super-horizon modes, higher order moments of $\mathcal{N}$ are suppressed.[7]

[7] This is a bit of circular reasoning. *If* each moment $\mathcal{N}_{l+1}$ is small compared to $\mathcal{N}_{l-1}$, *then* $\dot{\mathcal{N}}_l \propto k\mathcal{N}_{l-1}$. That this leads to the suppression of higher order moments is no wonder, for we have assumed this from the start. Yet, $\mathcal{N}_2$ corresponds to Π_ν and this in turn determines $\Psi - \Phi$ from Einstein's equation. As this difference is not substantial, one concludes that N_2 (and all higher moments) are small initially.

C. The Line of Sight Strategy

Experiments such as WMAP measure the temperature anisotropy *today*. So usually, we are interested in $\mathcal{M}(\tau_0, \mu)$. It turns out that there is a clever way to obtain this that even highlights the different contributions towards the final anisotropy. Let us develop this *Line of Sight* strategy!

Inspecting (220), one notices that the l.h.s can be written as

$$e^{-i\mu k\tau}e^{-\kappa(\tau)}\dot{L}, \tag{238}$$

where

$$L \equiv e^{i\mu k\tau}e^{\kappa(\tau)}\mathcal{M}. \tag{239}$$

Hence, (220) translates into

$$\dot{L} = e^{i\mu k\tau}e^{\kappa(\tau)}\Bigg[i\mu k(\Phi - \Psi) + \dot\kappa\left(\frac{1}{4}D_g^\gamma - i\mu V_b - \frac{1}{2}(3\mu^2 - 1)\mathcal{C}\right)\Bigg], \tag{240}$$

and integrated over conformal time,

$$L(\tau_0) = \int_0^{\tau_0} \mathrm{d}\tau e^{i\mu k\tau}e^{\kappa(\tau)}\Bigg[i\mu k(\Phi - \Psi) + \dot\kappa\left(\frac{1}{4}D_g^\gamma - i\mu V_b - \frac{1}{2}(3\mu^2 - 1)\mathcal{C}\right)\Bigg]. \tag{241}$$

According to Equation (239), the photon perturbation today is given by $\mathcal{M}(\mu, \tau_0) = e^{-i\mu k\tau_0}e^{-\kappa(\tau_0)}L(\tau_0)$, so

$$\mathcal{M}(\mu, \tau_0) = \int_0^{\tau_0} \mathrm{d}\tau\, e^{i\mu k(\tau - \tau_0)}e^{\kappa(\tau) - \kappa(\tau_0)} \times \left[i\mu k(\Phi - \Psi) + \dot\kappa\left(\frac{1}{4}D_g^\gamma - i\mu V_b - \frac{1}{2}(3\mu^2 - 1)\mathcal{C}\right)\right]. \tag{242}$$

The product $g \equiv \dot\kappa\exp(\kappa(\tau) - \kappa(\tau_0))$ plays an important role[8] and is called the visibility function. Its peak defines the epoch of recombination (see also Figure 3). Each term in the above Equation (242) containing factors of μ, can be integrated by parts, in order to get rid of μ. For instance

$$\int_0^{\tau_0} e^{i\mu k(\tau - \tau_0)}i\mu\, gV_b\mathrm{d}\tau \tag{243}$$

$$= \int_0^{\tau_0}\left[e^{i\mu k(\tau - \tau_0)}i\mu k\right]k^{-1}gV_b\mathrm{d}\tau \tag{244}$$

$$= \Big[\Big] - \int_0^{\tau_0} e^{i\mu k(\tau - \tau_0)}k^{-1}\left(g\dot{V}_b + V_b\,\dot{g}\right)\mathrm{d}\tau, \tag{245}$$

[8] Please note that the function κ used in [15] is in fact the function $\kappa(\tau_0, \tau)$ of Equation (2.5) of [33]. It is not the function κ for which $\dot\kappa = an_e\sigma_T = \mathrm{d}\kappa/\mathrm{d}\tau$. Therefore the factor $\exp(\kappa(\tau) - \kappa(\tau_0))$ in this work is equivalent to $\exp(-\kappa(\tau))$ of [15]. However, this usage of κ (meaning $\kappa(\tau_0, \tau)$) obscures the derivation a bit and we therefore choose to display $\exp(\kappa(\tau) - \kappa(\tau_0))$ explicitly.

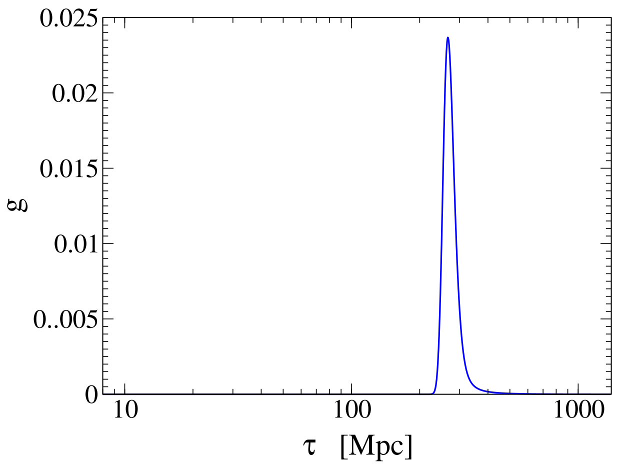

FIG. 3: The visibility $g \equiv \dot{\kappa} \exp(\kappa(\tau) - \kappa(\tau_0))$ as a function of conformal time τ in Mpc. Its peak at about $\tau \approx 300\,\text{Mpc}$ defines the epoch of last scattering. Before the visibility function peaks, photons are very likely to scatter again until the Universe becomes translucent. After the peak, photons do not scatter at a substantial rate. It is thus the balance between frequent scattering and sufficiently low optical depth that will give the largest contribution towards the anisotropy today. And this fact is exactly encoded in g.

where [] stands for the boundary term that here and for all of the terms above can be dropped, as it vanishes for $\tau \to 0$ and only contributes to C_0 for $\tau = \tau_0$. Applying this procedure to all terms involving μ yields

$$\mathcal{M}(\mu, \tau_0) = \int_0^{\tau_0} e^{i\mu k(\tau - \tau_0)} S_T(k, \tau) \mathrm{d}\tau, \tag{246}$$

where the *source* is

$$S_T = -e^{\kappa(\tau)-\kappa(\tau_0)} \left[\dot{\Phi} - \dot{\Psi}\right] + \dot{g} \left[\frac{V_b}{k} + \frac{3}{k^2}\dot{\mathcal{C}}\right] + \ddot{g}\frac{3}{2k^2}\mathcal{C} + g\left[\frac{1}{4}D_g^\gamma + \frac{\dot{V}_b}{k} - (\Phi - \Psi) + \frac{\mathcal{C}}{2} + \frac{3}{2k^2}\ddot{\mathcal{C}}\right]. \tag{247}$$

Let us pause to discuss this result (246, 339) in detail. First, we note that the visibility function g is sharply peaked at the epoch of decoupling (see Figure 3). Hence, $\mathcal{M}(\tau_0)$ gets contributions from D_g^γ and V_b^γ at about this epoch: whatever the density contrast of the photon fluid and the baryon fluid velocity has been at decoupling, it will be imprinted in the temperature anisotropy today. The term from the density contrast D_g^γ is the most important one on scales smaller than the sound horizon. It is the main contributor towards the oscillatory behaviour of the C_l spectrum [34]. Its appearance is plausible, because for a photon gas, $\rho \propto T^4$ and therefore $\delta T/T \propto \frac{1}{4}\delta\rho/\rho$. The V_b^γ-term appears, because a baryon moving in the direction towards the observer will cause a Doppler shift of the emitted photon. For adiabatic initial conditions, this Doppler shift fills the region before the first peak (at $l \approx 220$), which is mainly due to D_g^γ [35]. The first term in the source (involving $\dot{\Phi} - \dot{\Psi}$) accounts for the integrated Sachs-Wolfe (ISW) effect [36]: if the gravitational potential decays, the photons have to climb out of a more shallow potential than they have been in before. Quintessence, for instance can lead to a late ISW effect. The terms involving $\mathcal{C}$ and its derivatives describe polarization effects and are far less important than the D_g^γ term. Finally, the $(\Phi - \Psi)$ term is the (ordinary) Sachs-Wolfe effect. On scales that at decoupling were well outside the horizon, this gives the main contribution.

1. *Line of Sight for Polarization*

To discuss polarization, we need to prepare the ground a bit further. As said, we may decompose

$$(\mathcal{Q}(\boldsymbol{n}) + iU(\boldsymbol{n})) = \sum_{l,m} {}_{-2}a_{lm}\ {}_{-2}Y_l^m(\boldsymbol{n}) \tag{248}$$

and

$$(\mathcal{Q}(\boldsymbol{n}) - iU(\boldsymbol{n})) = \sum_{l,m} {}_{2}a_{lm}\ {}_{2}Y_l^m(\boldsymbol{n}) \tag{249}$$

Using the spin lowering and raising operators, we get spin 0 quantities

$$\bar{\eth}^{\,2}(\mathcal{Q}(\boldsymbol{n}) + iU(\boldsymbol{n})) = \sum_{l,m} \left[\frac{(l+2)!}{(l-2)!}\right]^{1/2} {}_{2}a_{lm}\ Y_l^m(\boldsymbol{n}) \tag{250}$$

and

$$\eth^{\,2}(\mathcal{Q}(\boldsymbol{n}) - iU(\boldsymbol{n})) = \sum_{l,m} \left[\frac{(l+2)!}{(l-2)!}\right]^{1/2} {}_{-2}a_{lm}\ Y_l^m(\boldsymbol{n}). \tag{251}$$

The combinations

$$a_{E,lm} \equiv -({}_2a_{lm} + {}_{-2}a_{lm}) \tag{252}$$

$$a_{B,lm} \equiv i({}_2a_{lm} - {}_{-2}a_{lm}) \tag{253}$$

have definite parity: E remains unchanged (like the electromagnetic E-field), while B changes sign. As U vanishes in our coordinate system, and $\mathcal{Q}$ depends only on μ, the relation

$$\bar{\eth}^{\,2}(\mathcal{Q}(\boldsymbol{n}) + iU(\boldsymbol{n})) = \eth^{\,2}(\mathcal{Q}(\boldsymbol{n}) - iU(\boldsymbol{n})) \tag{254}$$

holds and so

$${}_2a_{lm} = {}_{-2}a_{lm} \tag{255}$$

and hence $a_{E,lm} =_2 a_{lm}$ and $a_{B,lm} = 0$. Hence scalar perturbations will not contribute to B modes.

For polarization, we formally integrate just like for the temperature, but we will not use partial integration to get rid of any μ's appearing. From (218), we get

$$\mathcal{Q} = \frac{3}{4}(1-\mu^2)\int_0^{\tau_0} e^{i\mu k(\tau-\tau_0)}\, g\,\mathcal{C}\mathrm{d}\tau. \tag{256}$$

Acting twice with $\bar{\eth}$ on the integral solution for $\mathcal{Q}$ yields

$$\begin{aligned}\Delta_{\tilde{E}} &= -\frac{3}{4}\int_0^{\tau_0} g\,\mathcal{C}\,\partial_\mu^2\left[(1-\mu^2)^2 e^{i\mu k(\tau-\tau_0)}\right]\mathrm{d}\tau \\ &= \frac{3}{4}\int_0^{\tau_0}\mathrm{d}\tau\, g\,\mathcal{C}\,\left[1+\partial_x^2\right]^2\left(x^2 e^{-ix\mu}\right),\end{aligned} \tag{257}$$

where $x \equiv k(\tau_0 - \tau)$. BTW: verifying the above might keep you busy for half an hour, but you are invited to check it.

D. The Multipole Spectrum

In the previous sections, we obtained an evolution equation for the gauge invariant temperature anisotropy $\mathcal{M}$. Here, we would like to make a connection to observations.

As we have seen, the fluctuations evolve from some initial time on. It is widely assumed that some mechanism (most prominently Inflation) generated these initial fluctuations. While we cannot hope to make predictions about a particular initial perturbation pattern in the Universe, we might hope to be able to describe the initial perturbations statistically. In this approach, we take the particular pattern of hot and cold spots on the CMB sky today as just one realization of the underlying statistics. In other words, we assume that the part of the Universe that we can access with our measurement is a fair sample of the Universe as a whole. For correlations of large angular separation, this is not a particularly good approximation, because there are just not enough regions with such large extend on the sky today that do not overlap and could provide independent measurements to infer the expectation value for the Universe as a whole. This limitation is called *cosmic variance* and seen on the measurement errors on large scale of the CMB. In some sense, these are no measurement errors (the instrument is much better than the errors quoted), but our inability to infer the correct expectation value for the entire Universe.

So we measure the temperature or polarization anisotropy today (τ_0) on earth ($\boldsymbol{x}_0$). Generally speaking a spin-0 field A on the sky can be expanded in terms of

$$A(\boldsymbol{x}_0,\tau_0,\boldsymbol{n}) = \sum_{l,m} a_{lm} Y_l^m(\boldsymbol{n}). \tag{258}$$

In what follows, we will use the relation

$$\int \mathrm{d}\boldsymbol{n} Y_l^m(\boldsymbol{n})\left[Y_{l'}^{m'}(\boldsymbol{n})\right]^* = \delta_{mm'}\delta_{ll'}. \tag{259}$$

Using (259), we can project out a_{lm} in Equation (258), yielding

$$a_{lm} = \int \mathrm{d}\boldsymbol{n}[Y_l^m(\boldsymbol{n})]^* A(\boldsymbol{x}_0,\tau_0,\boldsymbol{n}). \tag{260}$$

As the *particular* pattern on the sky cannot be predicted, we rather predict (and measure) *correlations* between different directions on the sky

$$\begin{aligned}&\langle A(\boldsymbol{x}_0,\tau_0,\boldsymbol{n})B(\boldsymbol{x}_0,\tau_0,\boldsymbol{n}')^*\rangle = \\ &\left\langle\left[\sum_{l,m} a_{lm}Y_l^m(\boldsymbol{n})\right]\left[\sum_{l',m'} b_{l'm'}Y_{l'}^{m'}(\boldsymbol{n}')\right]^*\right\rangle.\end{aligned} \tag{261}$$

The expectation $\langle\rangle$ on the r.h.s acts only on the a_{lm}'s. Hence

$$\begin{aligned}&\langle A(\boldsymbol{x}_0,\tau_0,\boldsymbol{n})B(\boldsymbol{x}_0,\tau_0,\boldsymbol{n}')^*\rangle = \\ &\sum_{l,l',m,m'}\langle a_{lm}b_{l'm'}^*\rangle\left[Y_l^m(\boldsymbol{n})\right]\left[Y_{l'}^{m'}(\boldsymbol{n}')\right]^*.\end{aligned} \tag{262}$$

If the angle $\boldsymbol{n}\cdot\boldsymbol{n}'$ is statistically independent of orientation, i.e. it does not matter in which direction we measure an angular correlation of a given angle, we may write

$$\langle a_{lm}b_{l'm'}^*\rangle = \delta_{ll'}\delta_{mm'}C_l, \tag{263}$$

or equivalently in this case

$$\frac{1}{2l+1}\sum_{m=-l}^{l}\langle a_{lm}b_{lm}^*\rangle = C_l. \tag{264}$$

It is intuitively clear why this relation must hold in that case: the m's select orientation for a given multipole in the Y_l^m's. A net correlation between different orientations would mean that some orientations are more important than others. In addition, a correlation between different l's would mean that there is a net cross-talk between fluctuations on different scales. If all processes are well described by linear perturbation theory and the initial fluctuations didn't cross talk then they won't mix coherently later on. So a $\delta_{ll'}$ is in order. Please keep in mind that this may not be true and must always be justified by observations. For instance non-linearities during inflation would couple modes. Measuring the degree to which (263) holds therefore provides important clues about the physics of inflation. Plugging (263) into (262) (and using the relation (D2)) yields

$$\begin{aligned}\langle A(\boldsymbol{n})B(\boldsymbol{n}')^*\rangle &= \sum_l C_l\sum_{m=-l}^{l} Y_l^m(\boldsymbol{n})\left[Y_l^m(\boldsymbol{n}')\right]^* \\ &= \frac{2l+1}{4\pi}\sum_l C_l P_l(\boldsymbol{n}\cdot\boldsymbol{n}'),\end{aligned} \tag{265}$$

where we have suppressed the arguments $(\boldsymbol{x}_0,\tau_0)$ of A and B. Hence, the C_l's turn out to be the coefficients of a Legendre polynomial expansion. It is these C_l's that encode the two point function (provided (263) holds) and we would like to find an expression for C_l in terms of A and B. To this end, we note that using the Fourierdecomposition, Equation (260) becomes

$$a_{lm} = \int \mathrm{d}\boldsymbol{n}[Y_l^m(\boldsymbol{n})]^* \int \frac{\mathrm{d}^3k}{(2\pi)^3} A(\boldsymbol{k},\tau_0,\boldsymbol{n}) \exp(i\boldsymbol{k}\boldsymbol{x}). \quad (266)$$

As the evolution of A does not depend on the direction of $\boldsymbol{k}$, we may write

$$A(\boldsymbol{k},\tau_0,\boldsymbol{n}) = \psi^{ini}(\boldsymbol{k})A(k,\tau_0,\boldsymbol{n}), \quad (267)$$

where $\psi^{ini}(\boldsymbol{k})$ is the initial perturbation and the initial correlation is usually cast in the *ansatz*

$$\left\langle \psi^{ini}(\boldsymbol{k})\psi^{ini}(\boldsymbol{k}')^*\right\rangle = (2\pi)^3 P_\psi(k)\delta_D(\boldsymbol{k}-\boldsymbol{k}'), \quad (268)$$

where $P_\psi(k)$ is the initial power spectrum and $\delta_D(\boldsymbol{k}-\boldsymbol{k}')$ is the Dirac delta function.[9] Please note that in CMBEASY,

$$P_\psi(k) = \frac{2\pi^2}{k^3} P_\chi(k) \quad (269)$$

, where P_χ is the power spectrum of the initial curvature perturbation

$$P_\chi(k) = A_s \left(\frac{k}{k_\star}\right)^{n_s-1}, \quad (270)$$

and A_s is the intial scalar amplitude, quoted at a reference scale $k_\star$ (one chooses $k_\star = 0.05$Mpc) and the spectrum is a feature-less power law with scalar spectral n_s.

Using (266) and (267) to write the correlation, we get

$$\langle a_{lm}b_{lm}^*\rangle = \int \mathrm{d}\boldsymbol{n}\,\mathrm{d}\boldsymbol{n}'[Y_l^m(\boldsymbol{n})]^*\,Y_l^m(\boldsymbol{n}') \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3} \left\langle \psi^{ini}(\boldsymbol{k})\psi^{ini}(\boldsymbol{k}')\right\rangle A(k,\tau_0,\boldsymbol{n})B(k',\tau_0,\boldsymbol{n}')^* e^{i(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{x}}. \quad (271)$$

The ansatz (268), simplifies this to

$$\langle a_{lm}b_{lm}^*\rangle = \int \mathrm{d}\boldsymbol{n}\,\mathrm{d}\boldsymbol{n}'[Y_l^m(\boldsymbol{n})]^*\,Y_l^m(\boldsymbol{n}') \int \frac{\mathrm{d}^3k}{(2\pi)^3} P_\psi(k)A(k,\tau_0,\boldsymbol{n})B(k,\tau_0,\boldsymbol{n})^*, \quad (272)$$

and provided that you can express A (and B) in the form of sources

$$A(k,\tau_0,\boldsymbol{n}) = \int \mathrm{d}\tau \exp(i\,k\,\mu[\tau-\tau_0])S_A(k,\tau), \quad (273)$$

this becomes

$$\begin{aligned} C_l &= \frac{1}{2l+1}\sum_m \langle a_{lm}b_{lm}^*\rangle \\ &= \frac{1}{2l+1}\int \frac{\mathrm{d}^3k}{(2\pi)^3}P_\psi(k)\sum_m \\ &\times \left(\int \mathrm{d}\tau\mathrm{d}\boldsymbol{n}[Y_l^m(\boldsymbol{n})]^*\exp(i\,k\,\mu[\tau-\tau_0])S_A(k,\tau)\right) \\ &\times \left(\int \mathrm{d}\tau'\mathrm{d}\boldsymbol{n}' Y_l^m(\boldsymbol{n}')\exp(-i\,k\,\mu[\tau'-\tau_0])S_B(k,\tau')\right). \end{aligned} \quad (274)$$

In Appendix D, we show that in our $\boldsymbol{k}||z$ system

$$\int \mathrm{d}\boldsymbol{n}[Y_l^m(\boldsymbol{n})]^* e^{-i\mu x} = \sqrt{4\pi(2l+1)}(-i)^l j_l(x)\delta_{0m}, \quad (275)$$

and by complex conjungation

$$\int \mathrm{d}\boldsymbol{n} Y_l^m(\boldsymbol{n}) e^{i\mu x} = \sqrt{4\pi(2l+1)}i^l j_l(x)\delta_{0m}. \quad (276)$$

So

$$\begin{aligned} C_l &= \frac{1}{2l+1}\int \frac{\mathrm{d}^3k}{(2\pi)^3}P_\psi(k) \\ &\times \left(\int \mathrm{d}\tau\sqrt{4\pi(2l+1)}(-i)^l j_l(k[\tau_0-\tau])S_A(k,\tau)\right) \\ &\times \left(\int \mathrm{d}\tau'\sqrt{4\pi(2l+1)}i^l j_l(k[\tau_0-\tau])S_B(k,\tau')\right) \\ &= (4\pi)\int \frac{\mathrm{d}^3k}{(2\pi)^3}P_\psi(k)\left\{\int \mathrm{d}\tau j_l(k[\tau_0-\tau])S_A(k,\tau)\right\} \\ &\times \left\{\int \mathrm{d}\tau' j_l(k[\tau_0-\tau'])S_B(k,\tau')\right\} \end{aligned} \quad (277)$$

In case that $A = B = \Delta$, we therefore get

$$C_l^{TT} = 4\pi\int \frac{\mathrm{d}^3k}{(2\pi)^3}P_\psi(k)\left|\int \mathrm{d}\tau j_l(k[\tau_0-\tau])S_T(k,\tau)\right|^2, \quad (278)$$

In other words, the C_l's are obtained by integrating over Fourier modes k the initial power spectrum times the square of the result of folding the sources with spherical Bessel functions.[10] BTW: by projecting out the moments Δ_l of the line-of-sight integral and using equations (275), you can verify that

[9] Please note that for a real quantity R (such as the density contrast), $R(-\boldsymbol{k}) = R^*(\boldsymbol{k})$.

[10] If you compare (**??**) to the corresponding expression in Ma & Bertschinger [23], please keep in mind that $P_\psi^{Ma} = (2\pi)^3 P_\psi$ and hence we seem to be off by a total factor of $(2\pi)^{-6}$. However, our definition of the Fourier decomposition leads to $M_l^{Ma} = M_l/(2\pi)^3$ and hence all factors of (2π) cancel.

$$\Delta_l = (2l+1)\int d\tau j_l(k[\tau-\tau_0])S_T(k,\tau). \qquad (279)$$

For polarization, we want to quote

$$C_l^{EE} = \frac{1}{2l+1}\sum_m \langle a^*_{E,lm} a_{E,lm}\rangle \qquad (280)$$

and the temperature-polarization cross correlation

$$C_l^{TE} = \frac{1}{2l+1}\sum_m \langle a^*_{T,lm} a_{E,lm}\rangle. \qquad (281)$$

For EE, we get from Equation (274) and (257) and using again $x \equiv k(\tau_0-\tau)$,

$$\begin{aligned} C_l^{EE} &= \frac{1}{2l+1}\sum_m \langle a_{E,lm} a^*_{E,lm}\rangle \\ &= \frac{1}{2l+1}\frac{(l-2)!}{(l+2)!}\int \frac{d^3k}{(2\pi)^3} P_\psi(k)\sum_m \\ &\times \left|\frac{3}{4}d\boldsymbol{n}Y_l^m(\boldsymbol{n})\int_0^{\tau_0} d\tau\, g\,\mathcal{C}\,\left[1+\partial_x^2\right]^2\left(x^2 e^{ix\mu}\right)\right|^2, \end{aligned} \qquad (282)$$

where the factor $\frac{(l-2)!}{(l+2)!}$ comes from acting twice with $\eth$ (see Equation (257)). Again, we can use (275) to get rid of the angular part

$$C_l^{EE} = (4\pi)^2\frac{(l+2)!}{(l-2)!}\int \frac{d^3k}{(2\pi)^3}P_\psi(k) \left(\frac{3}{4}\int_0^{\tau_0} d\tau\, g\,\mathcal{C}\,\left[1+\partial_x^2\right]^2\left[x^2 j_l(x)\right]\right)^2 \qquad (283)$$

The final step (which leads to a surprisingly simple result) involves the defining relation of spherical bessel functions, namely

$$j_l'' + \frac{2}{x}j_l' + \left[1-\frac{l(l+1)}{x^2}\right]j_l = 0. \qquad (284)$$

Using this relation (do it!)[11] (283) simplifies to

$$C_l^{EE} = (4\pi)^2\frac{(l+2)!}{(l-2)!}\int \frac{d^3k}{(2\pi)^3}P_\psi(k) \times\left[\frac{3}{4}\int_0^{\tau_0} d\tau\, g\,\mathcal{C}\frac{j_l(x)}{x^2}\right]^2, \qquad (285)$$

a surprisingly elegant expression.

Again, as in the temperature case, we could write this in terms of [12]

$$\Delta_l^E = (2l+1)\sqrt{\frac{(l-2)!}{(l+2)!}}\int_0^{\tau_0} d\tau S_E(k,\tau) j_l(x), \qquad (286)$$

where

$$S_E(k,\tau) = \frac{3g\mathcal{C}}{4x^2}. \qquad (287)$$

Repeating for convenience, we found

$$\begin{aligned} \Delta_l^T &= (2l+1)\int d\tau j_l(k[\tau-\tau_0])S_T(k,\tau) \qquad (288)\\ \Delta_l^E &= (2l+1)\sqrt{\frac{(l-2)!}{(l+2)!}}\int_0^{\tau_0} d\tau S_E(k,\tau) j_l(x) \qquad (289) \end{aligned}$$

with the sources

$$\begin{aligned} S_T = &-e^{\kappa(\tau)-\kappa(\tau_0)}\left[\dot\Phi-\dot\Psi\right] + \dot g\left[\frac{V_b}{k}+\frac{3}{k^2}\dot{\mathcal{C}}\right] + \ddot g\frac{3}{2k^2}\mathcal{C} \\ &+ g\left[\frac{1}{4}D_g^\gamma + \frac{\dot V_b}{k} - (\Phi-\Psi) + \frac{\mathcal{C}}{2} + \frac{3}{2k^2}\ddot{\mathcal{C}}\right]. \end{aligned} \qquad (290)$$

and

$$S_E(k,\tau) = \frac{3g\mathcal{C}}{4x^2} \qquad (291)$$

in terms of which the $C_l's$ are given by[13]

$$\begin{aligned} C_l^{TT} &= \frac{4\pi}{(2l+1)^2}\int\frac{d^3k}{(2\pi)^3}P_\psi(k)\left[\Delta_l^T(k)\right]^2 \qquad (292)\\ C_l^{EE} &= \frac{4\pi}{(2l+1)^2}\int\frac{d^3k}{(2\pi)^3}P_\psi(k)\left[\Delta_l^E(k)\right]^2 \qquad (293)\\ C_l^{TE} &= \frac{4\pi}{(2l+1)^2}\int\frac{d^3k}{(2\pi)^3}P_\psi(k)\Delta_l^E(k)\Delta_l^T(k) \qquad (294) \end{aligned}$$

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[11] The trick is to use the recursion relation whenever possible after acting with ∂_x once: using the relation, simplify (i.e. get rid of j_l'' and avoid getting even higher derivatives of j_l.). Then act with the next ∂_x etc. until you have completed the $(1+\partial_x^2)(1+\partial_x^2)$ acting on $x^2 j_l(x)$.

[12] Please note that we differ by a factor of $(2l+1)$ from the original definition. This is due to the slightly more recent convention for the expansion coefficients in terms of spherical harmonics which we used.

[13] Again, we differ by a factor of $(2l+1)^{-2}$ which cancels the factor of $(2l+1)$ squared from the Δ's from the earlier literature, but are in line with the more recent one.

1. Putting it all together

As far as the CMB is concerned what one really wants is the correlation functions, commonly quoted using the coefficients C_l. The slow way would be to get the C_l's directly from the (vast) multipole hierarchy of the photon distribution via (292-294) and the multipole hierarchy up to $l \approx 3000$. In contrast, the line of sight integration gets the Δ_l's (in our case the gauge-invariant $\mathcal{M}_l$'s) by folding the source term S with the spherical Bessel functions j_l. While the Bessel functions oscillate rapidly in this convolution, the source term is most of the time rather slowly changing. It thus suffices to calculate the sources at few (cleverly chosen) points and interpolate between. In order to determine the sources, one needs to know (among other things) D_g^γ and $\mathcal{C}$. Therefore, one still needs to solve a multipole hierarchy for $\mathcal{M}$ and E. However, for sufficient precision, only a few multipoles are needed: they build up rather slowly starting from initially shear-free conditions ($M_l = 0,\ l > 1$) due to the tight coupling to baryons. In order to suppress truncation effects, the multipole beyond the highest one in the hierarchy is approximated by the recursion relation of Bessel functions (see Appendix E).

VI. INITIAL CONDITIONS

To integrate the perturbation equations in k-Space, we need initial conditions for all density perturbations, velocities and in case of the neutrinos also the neutrino shear Π_ν. Let us briefly write down the equations. For photons (without the shear and higher moments, for those wait until later)

$$\dot{\delta}_\gamma = -\frac{4}{3}kv_\gamma + 4\dot{\phi} \tag{295}$$

$$\dot{v}_\gamma = k\frac{1}{4}\delta_\gamma + k\psi + \tau_c^{-1}(v_b - v_\gamma),, \tag{296}$$

(297)

For baryons

$$\dot{\delta}_b = -kv_b + 3\dot{\phi} \tag{298}$$

$$\dot{v}_b = -\frac{\dot{a}}{a}v_b + c_s^2 k\delta_b + R\tau_c^{-1}(v_\gamma - v_b) + k\psi, \tag{299}$$

where a piece for momentum conservation in Thomson scattering has been added. And the other species look similar. These equations are supplemented by Einstein's equations from which we get ψ and ϕ which we need in the above equations.

One usually starts in the radiation dominated regime, where one can neglect all components but radiation and therefore from the Friedmann equation

$$\frac{\dot{a}}{a} = \tau^{-1}. \tag{300}$$

Using this again in the Friedmann equation

$$3\mathrm{M}_\mathrm{P}^2 a^{-2}\left(\frac{\dot{a}}{a}\right)^2 = 3\mathrm{M}_\mathrm{P}^2 a^{-2}\tau^{-2} = \bar{\rho}, \tag{301}$$

and hence

$$\mathrm{M}_\mathrm{P}^{-2} a^2\bar{\rho} = 3\tau^{-2}, \tag{302}$$

which is exactly what we need to get the potentials Ψ and Φ from the fluid perturbations δ, v and π using the Einstein equations.

The standard strategy to find the relations between δ, v and π at early times is to power expand all perturbations in powers of $x = k\tau \ll 1$ and solve the evolution equations plus the perturbed Einstein equations order by order by comparing coefficients. That's not too difficult, just remember that $\Omega_\gamma = const$, and $\Omega_\nu = const$ during radiation domination which simplifies the task.

Much more elegantly, you can verify that using scaled variables for the velocities and the shear

$$\tilde{v} \equiv \frac{v}{x} \tag{303}$$

and

$$\tilde{\Pi} \equiv \frac{\Pi}{x^2}, \tag{304}$$

one can write the system of differential equations as a matrix equation for $\boldsymbol{U} \equiv (\delta_b, \delta_\gamma, \delta_c, \ldots \tilde{v}_b, \ldots, \tilde{\Pi}_\nu)$, namely

$$\frac{\mathrm{d}}{\mathrm{d}\ln x}\boldsymbol{U} = A(x)\boldsymbol{U}, \tag{305}$$

where $A(x)$ is only mildly x dependent. Let's do this for the velocity equation of the photons

$$\frac{\mathrm{d}}{\mathrm{d}x}\tilde{v}_\gamma = \frac{\mathrm{d}}{\mathrm{d}x}\frac{v_\gamma}{x} = -\frac{v_\gamma}{x^2} + x^{-1}\frac{\mathrm{d}}{\mathrm{d}x}v_\gamma \tag{306}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}v_\gamma = \frac{\mathrm{d}\tau}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\tau}v_\gamma = \frac{1}{4}\delta_\gamma + \psi \tag{307}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tilde{v}_\gamma = -\frac{\tilde{v}_\gamma}{x} + x^{-1}(\frac{1}{4}\delta_\gamma + \psi) \tag{308}$$

$$x\frac{\mathrm{d}}{\mathrm{d}x}\tilde{v}_\gamma = \frac{\mathrm{d}}{\mathrm{d}\ln x}\tilde{v}_\gamma = -\tilde{v}_\gamma + \frac{1}{4}\delta_\gamma + \psi \tag{309}$$

In much the same manner, one can re-write all the perturbation equations. In general, for $x \to 0$, the right hand sides are time independent to first order. A mild x dependence is hidden in ψ which depends on Ω_c etc. But those quantities are small at early times. The nice thing about a matrix formulation is that you can easily count solutions, here called modes, because the general solution to the differential equation is

$$\boldsymbol{U}(x) = \sum_i c_i\left(\frac{x}{x_{0\,i}}^{\lambda}\right)\boldsymbol{U}_i, \tag{310}$$

where λ_i is the eigenvalue to the eigenvector $\boldsymbol{U}_i$. Only the modes with largest λ_i will be relevant after some evolution. Those are the ones that are usually specified.

But back to our slightly x dependend A: It is clever to Taylor expand

$$A(x) = A_0 + A_1 x + \dots \tag{311}$$

Making the same split for $\boldsymbol{U}$,

$$\boldsymbol{U} = \boldsymbol{U}_0 + x\boldsymbol{U}_1, \tag{312}$$

one gets from the constant pieces A_0 and $\boldsymbol{U}_0$ from (305)

$$A_0 \boldsymbol{U}_0 = 0, \tag{313}$$

which is relativly simple to solve. The next to leading order corrections (which are rather unimportant) are then regiven by simple matrix multiplication

$$\boldsymbol{U_1} = -(A_0 - \mathbf{1})^{-1} A_1 \boldsymbol{U}_0 \tag{314}$$

No matter how you get the solution to the differential equation system, you will find that

$$\tilde{v}_\gamma = \frac{1}{4}\delta + \psi \tag{315}$$

and several more such relations. As these relations come from the equations of motion, they need to be fulfilled by all modes.

As far as the density contrasts δ are concerned, the equations of motion only demand that to leading order $\delta = const$. A specific mode is now usually selected by choosing the δ's. The way to classify this is by using

$$S_{a:b} \equiv \frac{\delta_a}{1+w_a} - \frac{\delta_b}{1+w_b}, \tag{316}$$

which is the entropy perturbation between the fluids a and b. The adiabatic mode is the one where all S vanish. I.e. $\delta_\gamma = \delta_n u$, $\delta_b = \delta_c = \frac{3}{4}\delta_\gamma$. In addition to this single adiabatic mode, there is one so called isocurvature mode per fluid (slight subtlty: this is only true for a fluid with $\Gamma = 0$). The iso-cdm mode, for example is given by the requirement that

$$S_{\gamma:cdm} \neq 0, \tag{317}$$

but all other species are adiabatic and that the total curvature perturbation χ vanishes. As at early times $\chi \propto \Psi$, this simply means that the gravitational potential should vanish for such a mode.

A general initial condition is then simply a linear superposition of such modes. Usually the only relevant one is the adiabatic.

VII. TIGHT COUPLING

At early times, the photon and baryon fluids are strongly coupled via Thomson scattering. The mean free path between collisions of a photon $\tau_c^{-1} \equiv a n_e \sigma_T$ is given in terms of the number density of free electrons n_e, the scale factor of the Universe a and Thomson cross section σ_T. During early times, Hydrogen and Helium are fully ionized, hence $n_e \propto a^{-3}$ and $\tau_c \propto a^2$. During Helium and Hydrogen recombination, this scaling argument does not hold (see Figure 4). To avoid these periods we resort to the correct value of $\dot{\tau}_c$ computed beforehand instead of using $\dot{\tau}_c = 2\frac{\dot{a}}{a}\tau_c$ for redshifts $z < 10^4$. The effect of assuming that the scaling holds would however be considerably less than 1% on the final CMB spectrum.

To discuss the tight coupling regime, let us recapitulate the evolution equations for baryons and photons. We do this in terms of their density perturbation δ and bulk velocity v. For photons, we additionally consider the shear σ_γ and higher multipole moments $\mathcal{M}_l$ of the intensity as well as polarization multipoles E_l. Our variables are related to the ones of [23] by substituting $v \to k^{-1}\theta$. In longitudinal gauge, baryons evolve according to

$$\dot{\delta}_b = -k v_b + 3\dot{\phi} \tag{318}$$

$$\dot{v}_b = -\frac{\dot{a}}{a} v_b + c_s^2 k \delta_b + R\tau_c^{-1}(v_\gamma - v_b) + k\psi, \tag{319}$$

where $R \equiv (4/3)\rho_\gamma/\rho_b$, the speed of sound of the baryons is denoted by c_s^2 and ϕ and ψ are metric perturbations. By definition, $R \propto a^{-1}$ (provided no baryons are converted to other forms of energy) and at the time of interest, $c_s^2 \propto T_b = T_\gamma \propto a^{-1}$ (for more detail see e.g. [23]). Photons evolve according to the hierarchy

$$\dot{\delta}_\gamma = -\frac{4}{3} k v_\gamma + 4\dot{\phi} \tag{320}$$

$$\dot{v}_\gamma = k\left(\frac{1}{4}\delta_\gamma - \sigma_\gamma\right) + k\psi + \tau_c^{-1}(v_b - v_\gamma), \tag{321}$$

$$\frac{5}{2}\dot{\sigma}_\gamma = \dot{\mathcal{M}}_2 = -\tau_c^{-1}\left(\frac{9}{10}\mathcal{M}_2 + \frac{\sqrt{6}}{10}E_2\right) + k\left(\frac{2}{3}v_\gamma - \frac{3}{7}\mathcal{M}_3\right) \tag{322}$$

$$\dot{\mathcal{M}}_l = k\left(\frac{l}{2l-1}\mathcal{M}_{l-1} - \frac{l+1}{2l+3}\mathcal{M}_{l+1}\right) - \tau_c^{-1}\mathcal{M}_l \tag{323}$$

where the E-type polarization obeys

$$\dot{E}_2 = -k\frac{\sqrt{5}}{7}E_3 - \tau_c^{-1}\left(\frac{4}{10}E_2 + \frac{\sqrt{6}}{10}M_2\right) \tag{324}$$

$$\dot{E}_l = -\tau_c^{-1}E_l + k\left(\frac{\sqrt{l^2-4}}{2l-1}E_{l-1} - \frac{\sqrt{l^2+2l-3}}{2l+3}E_{l+1}\right) \tag{325}$$

The overwhelmingly large value of τ_c^{-1} precludes a straight forward numerical integration at early times: tiny errors in the propagation of v_b and v_γ lead to strong restoring forces. This severely limits the maximum step size of the integrator and hence the speed of integration. Ever since Peebles and Yu [34] first calculated the

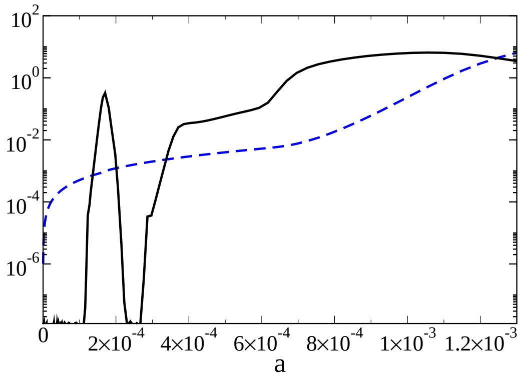

FIG. 4: Relative deviation of $\dot{\tau}_c$ from the naive scaling relation: $[\dot{\tau}_c - 2(\dot{a}/a)\tau_c]/[2(\dot{a}/a)\tau_c]$ (solid line). We also depict the product $\tau_c \frac{\dot{a}}{a}$ (dashed line) vs. the scale factor a, which compares the mean free path to the expansion rate of the Universe. In the cosmological model used, matter radiation equality is at $a_{equ} = 3 \times 10^{-4}$ and last scattering defined by the peak of the visibility function is at $a_{ls} = 9 \times 10^{-4}$. The deviation around $a = 2 \times 10^{-4}$ is from Helium recombination and is practically negligible, because the visibility is still small during that period. At later times, however the deviation is due to the onset of Hydrogen recombination and takes on substantial values before last scattering.

CMB fluctuations, one resorts to the so called tight coupling approximation. This approximation eliminates all terms of order τ_c^{-1} from the evolution equations assuming[14] tight coupling at initial times. Our discussion will closely lean on that of [23], taking a slightly different route. In contrast to [23], however, we will keep *all* terms in the derivation. Like [23], we start by solving (321) for $(v_b - v_\gamma)$ and write $\dot{v}_\gamma = \dot{v}_b + (\dot{v}_\gamma - \dot{v}_b)$ to get Equation (71) of [23]

$$(v_b - v_\gamma) = \tau_c \left[\dot{v}_b + (\dot{v}_\gamma - \dot{v}_b) - k \left(\frac{1}{4}\delta_\gamma - \sigma_\gamma + \psi \right) \right]. \tag{326}$$

Substituting Equation (319) for $\dot{v}_b$ into this Equation (326), one gets Equation (72) of [23]

$$\frac{(1+R)}{\tau_c}(v_b - v_\gamma) = -\frac{\dot{a}}{a}v_b + (\dot{v}_\gamma - \dot{v}_b) + k\left(c_s^2\delta_b - \frac{1}{4}\delta_\gamma + \sigma_\gamma \right). \tag{327}$$

[14] There is no restoring force left, as we will see. Any error in the approximation is therefore amplified over time. One could, in principle retain a fraction of the restoring force to eliminate small numerical errors. However, this is not necessary in practice and we therefore will not discuss this possibility further.

Deriving the LHS of this Equation (327) yields

$$\begin{aligned} \dot{\text{lhs}} &= \frac{(1+R)}{\tau_c}(\dot{v}_b - \dot{v}_\gamma) \\ &\quad -(v_b - v_\gamma)\left[\frac{\dot{a}}{a}\frac{R}{\tau_c} - \frac{1+R}{\tau_c}\frac{\dot{\tau}_c}{\tau_c} \right] \qquad (328) \\ &\stackrel{*}{=} \frac{(1+R)}{\tau_c}(\dot{v}_b - \dot{v}_\gamma) - \frac{2+3R}{\tau_c}\frac{\dot{a}}{a}(v_b - v_\gamma), \qquad (329) \end{aligned}$$

where the last line holds provided the assumed scaling of τ_c is correct (see also Figure 4). All in all, deriving Equation (327) with respect to conformal time yields

$$\begin{aligned} &\frac{(1+R)}{\tau_c}(\dot{v}_b - \dot{v}_\gamma) - \left[\frac{\dot{a}}{a}\frac{R}{\tau_c} - \frac{1+R}{\tau_c}\frac{\dot{\tau}_c}{\tau_c} \right](v_b - v_\gamma) \\ &\quad = (\ddot{v}_\gamma - \ddot{v}_b) - \frac{\ddot{a}}{a}v_b + \left(\frac{\dot{a}}{a}\right)^2 v_b - \frac{\dot{a}}{a}\dot{v}_b \\ &\quad + k\left(\dot{c}_s^2\delta_b + c_s^2\dot{\delta}_b - \frac{1}{4}\dot{\delta}_\gamma + \dot{\sigma}_\gamma \right) \qquad (330) \end{aligned}$$

Multiplying Equation (319) by $\frac{\dot{a}}{a}$ to substitute $\frac{\dot{a}}{a}\dot{v}_b$ in (330), we get

$$\begin{aligned} &\frac{(1+R)}{\tau_c}(\dot{v}_b - \dot{v}_\gamma) = \left[\frac{\dot{a}}{a}\frac{R}{\tau_c} - \frac{1+R}{\tau_c}\frac{\dot{\tau}_c}{\tau_c} \right](v_b - v_\gamma) \\ &\quad + (\ddot{v}_\gamma - \ddot{v}_b) - \frac{\ddot{a}}{a}v_b + 2\left(\frac{\dot{a}}{a}\right)^2 v_b - 2\frac{\dot{a}}{a}c_s^2 k\delta_b \\ &\quad + k\left(c_s^2\dot{\delta}_b - \frac{1}{4}\dot{\delta}_\gamma + \dot{\sigma}_\gamma \right) + \frac{\dot{a}}{a}k\psi \qquad (331) \end{aligned}$$

where we have used $\dot{c}_s^2 = -\frac{\dot{a}}{a}c_s^2$. We could stop here, however it is numerically better conditioned to write $2\left(\frac{\dot{a}}{a}\right)^2 v_b = 2\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}v_b\right)$ where $\frac{\dot{a}}{a}v_b$ is obtained from solving Equation (327) for $\frac{\dot{a}}{a}v_b$. This expression for $\left(\frac{\dot{a}}{a}\right)^2 v_b$ is then plugged into Equation (331) to yield the final result for the slip (denoted by $\dot{\mathcal{V}}$)

$$\begin{aligned} \dot{\mathcal{V}} \equiv (\dot{v}_b - \dot{v}_\gamma) &= \Bigg\{ \left[\frac{\dot{\tau}_c}{\tau_c} - \frac{2}{1+R} \right]\frac{\dot{a}}{a}(v_b - v_\gamma) \\ &+ \frac{\tau_c}{1+R}\Bigg[-\frac{\ddot{a}}{a}v_b + (\ddot{v}_\gamma - \ddot{v}_b) + k\left(\frac{1}{2}\delta_\gamma - 2\sigma_\gamma + \psi \right) \\ &+ k\left(c_s^2\dot{\delta}_b - \frac{1}{4}\dot{\delta}_\gamma + \dot{\sigma}_\gamma \right) \Bigg] \Bigg\} \Bigg/ \left\{ 1 + 2\frac{\dot{a}}{a}\frac{\tau_c}{1+R} \right\}. \qquad (332) \end{aligned}$$

or alternatively, at times when the scaling of τ_c holds,

$$\begin{aligned} \dot{\mathcal{V}} \equiv (\dot{v}_b - \dot{v}_\gamma) &= \Bigg\{ \frac{2R}{1+R}\frac{\dot{a}}{a}(v_b - v_\gamma) \\ &+ \frac{\tau_c}{1+R}\Bigg[-\frac{\ddot{a}}{a}v_b + (\ddot{v}_\gamma - \ddot{v}_b) + k\left(\frac{1}{2}\delta_\gamma - 2\sigma_\gamma + \psi \right) \\ &+ k\left(c_s^2\dot{\delta}_b - \frac{1}{4}\dot{\delta}_\gamma + \dot{\sigma}_\gamma \right) \Bigg] \Bigg\} \Bigg/ \left\{ 1 + 2\frac{\dot{a}}{a}\frac{\tau_c}{1+R} \right\}. \qquad (333) \end{aligned}$$

This Equation (332) (or more obviously (333)) is essentially Equation (74) of [23] up to some corrections. Having kept all terms, we note that our Equation (332) is *exact*. To obtain Equations of motion for v_b and v_γ during tight coupling, we plug our result for $(\dot{v}_b - \dot{v}_\gamma)$, Equation (332) into the RHS of Equation (326) and this in turn into the RHS of Equations (319) and (321). This yields

$$\begin{aligned} \dot{v}_b &= \frac{1}{1+R}\left(kc_s^2\delta_b - \frac{\dot{a}}{a}v_b\right) + k\psi \\ &\quad \frac{R}{1+R}\left[k\left(\frac{1}{4}\delta_\gamma - \sigma_\gamma\right) + \dot{\mathcal{V}}\right] \\ \dot{v}_\gamma &= \frac{R}{1+R}k\left(\frac{1}{4}\delta_\gamma - \sigma_\gamma\right) + k\psi \\ &\quad + \frac{1}{1+R}\left(kc_s^2\delta_b - \frac{\dot{a}}{a}v_b - \dot{\mathcal{V}}\right) \end{aligned} \tag{334}$$

Up to now, we have made no approximations. Conceptually, we would like to separate the question of tight coupling for the velocities v_γ and v_b from any approximations of the shear σ_γ which we make below. As far as the tight coupling of the velocities and hence the slip $\dot{\mathcal{V}}$ is concerned, our approximation is to drop the term $(\ddot{v}_\gamma - \ddot{v}_b)$. We reserve the expression 'tight coupling' for the validity of our assumption that $(\ddot{v}_\gamma - \ddot{v}_b)$ can be neglected in the slip $\dot{\mathcal{V}}$. As a criterion, we use $k\tau_c < \frac{2}{10}$ for the photon fluid. When this threshold is passed, we use Equation (321) to evolve the photon velocity. Likewise, for the baryons, we use $\max(k, \frac{\dot{a}}{a})\tau_c/R < \frac{4}{100}$. Again, when this limit is exceeded, we switch to Equation (319). In any case, we switch off the approximation $\Delta\tau = 30$Mpc before the first evaluation of the CMB anisotropy sources (see below). For a $\Lambda-$CDM model, this is at $\tau \approx 200$Mpc.

To obtain high accuracy *during* tight coupling, it is crucial to determine σ_γ. Not so much for the slip (332), but more so for the Equations of motion (334): the shear reflects the power that is drained away from the velocity in the multipole expansion. This leads to an additional damping for photons. For the shear, we distinguish two regimes: an early one, where we use a high-order analytic approximation and a later one in which the full multipole equations of motion are used.

Since $\tau_c \ll 1$ at early times, one gets from multiplying (323) by τ_c that $\mathcal{M}_l \approx (k\tau_c)\,\mathcal{M}_{l-1}l/(2l-1)$. Hence, higher multipoles are suppressed by powers of $k\tau_c$. Approximating this situation by $\dot{\mathcal{M}}_3 = \dot{E}_3 = \mathcal{M}_4 = E_4 = 0$ in Equations (323) and (325), we get

$$\begin{aligned} \mathcal{M}_3 &= \frac{3}{5}(k\tau_c)\,\mathcal{M}_2 \\ E_3 &= \frac{1}{\sqrt{5}}(k\tau_c)\,E_2. \end{aligned} \tag{335}$$

Likewise, we obtain a leading order estimate of the quadrupoles by temporarily setting $\dot{\mathcal{M}}_2 = \dot{E}_2 = 0$,

$$\frac{5}{2}\sigma_\gamma^{l.o.} = \mathcal{M}_2^{l.o.} = \frac{8}{9}(k\tau_c)\,v_\gamma \tag{336}$$

$$E_2^{l.o.} = -\frac{\sqrt{6}}{4}\mathcal{M}_2^{l.o.}. \tag{337}$$

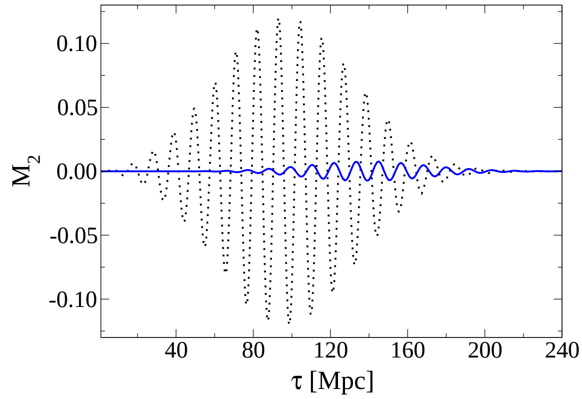

FIG. 5: The quadrupole $\mathcal{M}_2$ obtained by a full numerical evolution for a mode of $k = 1\text{Mpc}^{-1}$ (dotted line). The solid (blue) line depicts the deviation of our analytic result, Equation (338) from the numerical value. For this mode, we normally switch to the full numerical evolution at $\tau = 65$Mpc when the analytic estimate still holds very well.

Inserting Equations (335) into the quadrupole Equations (322) and (324) and using $\dot{\mathcal{M}}_2 = \dot{\mathcal{M}}_2^{l.o.}$ and $\dot{E}_2 = \dot{E}_2^{l.o.}$ as an estimate for the derivative, we get the desired expression for the shear

$$\frac{5}{2}\sigma_\gamma = \mathcal{M}_2 = \frac{8}{9}k\tau_c v_\gamma\left[1 - \frac{29}{70}(k\tau_c)^2\right] - \frac{11}{6}\tau_c\dot{\mathcal{M}}_2^{l.o.}, \tag{338}$$

which is precise to order τ_c and $(k\tau_c)^2$ (see also Figure 5). The inclusion of the octupole reduces the power of $\mathcal{M}_2$ as expected.

In practice, we use $\mathcal{M}_2^{l.o.}$ to calculate the slip $\dot{\mathcal{V}}^{l.o}$ to leading order. This in turn is used to calculate $\dot{v}_\gamma^{l.o}$. From $\dot{v}_\gamma^{l.o}$, we get $\dot{\mathcal{M}}_2^{l.o.}$ which in turn is needed to obtain the accurate value of $\mathcal{M}_2$ according to Equation (338). The difference $\Delta\mathcal{M}_2 \equiv \mathcal{M}_2 - \mathcal{M}_2^{l.o.}$ is then used to promote $\dot{\mathcal{V}}^{l.o} \rightarrow \dot{\mathcal{V}}$ as well as $\dot{v}_\gamma^{l.o.} \rightarrow \dot{v}_\gamma$. Finally, having $\mathcal{M}_2$ and $\dot{\mathcal{V}}$ at hand, we get $\dot{v}_b$ from Equation (334).

When this approximation breaks down (sometimes long before tight coupling ends), we switch to the full multipole evolution equations. Tight coupling is applicable for $k\tau_c \ll 1$. Equation (338) on one hand goes to higher order in $k\tau_c$, namely, as $\mathcal{M}_2^{l.o.}$ is already of order $(k\tau_c)$, our results incorporates quantities up to $(k\tau_c)^3$. In terms of τ_c alone, however, Equation (338) is accurate to order $\tau_c\,(k\tau_c)$ only. Hence, when τ_c reaches $\sim 10^{-1}$Mpc, our analytic expression is not sufficiently accurate anymore. This signals the breakdown of our assumption that $\dot{\mathcal{M}}_3 = \dot{E}_3 = 0$ (and likewise for higher multipoles). Luckily, it is not critical to evolve the full multipole equations even when τ_c^{-1} is still substantial. This is in strong contrast to the coupled velocity equations which are far more difficult to evolve at times when the analytic quadrupole formulae breaks down. In essence, distinguishing between tight coupling and the treatment of the quadrupole evolution is the key to success here.

VIII. A CURE FOR RAPID OSCILLATIONS

While the gain in speed from the method described in the last section is impressive, high k-modes would still require long integration times. To see this, one must consider the evolution of the photon and neutrino multipole hierarchies.[15] Our discussion is aimed at small scale modes which are supposed to be well inside the horizon, i.e. $k\tau \gg 1$.

Before last scattering, $(k\tau_c) \ll 1$ and $\mathcal{M}_l \propto (k\tau_c)^{l-1}$ for $l > 1$ and so the influence of higher multipoles on δ_γ and v_γ may be neglected to first order. In the small scale limit that we are interested in, δ_γ and v_γ are oscillating according to $\delta_\gamma \sim \cos(c_s^\gamma k\tau)$ and $v_b \sim \sin(c_s^\gamma k\tau)$. As the speed of sound of the photon-baryon fluid is $c_s^\gamma \approx \sqrt{1/3}$, we encounter oscillations with period $\Delta\tau \approx (2\pi)/(kc_s^\gamma) \approx 11/k$. Estimating the time of last scattering with $\tau_{ls} \approx 280$Mpc, we see that a mode will perform $\tau_{ls}/\Delta\tau \approx 25k$Mpc oscillations until last scattering. Yet, there are many more oscillations *after* last scattering which we turn to now. After last scattering, τ_c^{-1} is negligible and the multipole hierarchy of photons effectively turns into recursion relations for spherical Bessel functions. The same is true for neutrino multipoles which roughly evolve like spherical Bessel functions from the start. Spherical Bessel functions have a leading order behavior similar to $j_l(k\tau) \propto (k\tau)^{-3/2}\sin(k\tau)$ for $k\tau \gg 1$ and $k\tau > l$. The period is then given by $\Delta\tau = (2\pi)/k$. The time passed from last scattering to today, is $\tau_0 - \tau_{ls} \approx \tau_0 \approx 14000$Mpc for current cosmological models. So we encounter $\sim \tau_0/\Delta_\tau \approx k\tau_0/(2\pi)$ $=2200 \times k$Mpc oscillations. Numerically, each oscillation necessitates $\sim 20\ldots40$ evaluations of the full set of evolution equations. We therefore estimate a total of $\sim 6\times10^4 \times k$Mpc evaluations induced by the oscillatory nature of the solution. So a mode $k = 5\text{Mpc}^{-1}$ needs $\sim 3\times10^5$ evaluations – a substantial number.

Since the introduction of the line-of-sight algorithm, what one really needs for the CMB and LSS are the low multipoles up to the quadrupoles. In fact, the sources for temperature and polarization anisotropies are given by

$$S_T = e^{\kappa(\tau)-\kappa(\tau_0)}\left[\dot\phi+\dot\psi\right] + \dot g\left[\frac{v_b}{k}+\frac{3}{k^2}\dot{\mathcal{C}}\right] + \ddot g\frac{3}{2k^2}\mathcal{C} + g\left[\frac{1}{4}\delta_\gamma + \frac{\dot v_b}{k} + (\phi+\psi) + \frac{\mathcal{C}}{2} + \frac{3}{2k^2}\ddot{\mathcal{C}}\right] \quad (339)$$

and

$$S_E = \frac{3\,g}{2}\mathcal{C}\,\left(k\left[\tau_0-\tau\right]\right)^{-2} \quad (340)$$

Here, $g \equiv \dot\kappa\exp(\kappa(\tau)-\kappa(\tau_0))$ is the visibility with $\dot\kappa \equiv \tau_c^{-1}$ the differential optical depth and $\mathcal{C} \equiv (\mathcal{M}_2 - \sqrt{6}E_2)/10$ contains the quadrupole information. The role of higher multipole moments is therefore reduced to draining power away from δ_γ, v_γ and $\mathcal{M}_2$ and E_2 (and likewise for neutrinos). As the oscillations are damped and tend to average out, it suffices to truncate the multipole hierarchy at low $l \sim 8\ldots25$ in the line-of-sight approach. This is one of the main reasons for its superior speed. Truncating the hierarchy, though leads to unwanted reflection of power from the highest multipole l_{max}. As one can see in Figure 6, the power reflected back spoils the mono frequency of the oscillations. At best, the further high frequency evolution of the multipoles is wrong but negligible, because the oscillations are small and average out. This is indeed the case in the CMBFAST/CAMB/CMBEASY truncation.

We will now show that the overwhelming contribution from δ_γ and $\mathcal{C}$ (and its derivatives) of some small scale mode $k > 10^{-1}\text{Mpc}^{-1}$ towards CMB fluctuations comes from times *before* re-ionization. To do this, let us find an

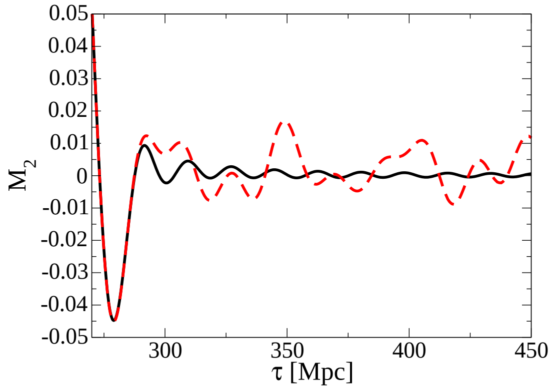

FIG. 6: The quadrupole as a function of conformal time τ for a mode of $k/h = 0.5\text{Mpc}^{-1}$ and $h = 0.7$. The multipole expansion for photons and neutrinos has been truncated at $l_{max} = 200$ (solid line) and $l_{max} = 8$ (dashed line) respectively. In the case of $l_{max} = 8$, power reflected back from the highest multipole l_{max} renders the further evolution of the quadrupole unphysical. Indeed the magnitude of the physical oscillations are much smaller than the reflected ones. For $l_{max} = 200$, reflection effects dominate the evolution from $\tau \sim 1300$Mpc on. In both cases, the effect of shear of realistic particles on the potentials ϕ and ψ is negligible by the time the truncation effects set in.

[15] We include the monopole δ_γ and dipole v_γ here.

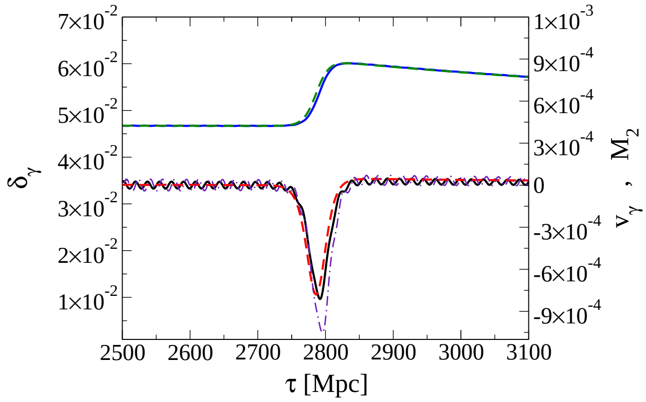

FIG. 7: Photon density contrast δ_γ (upper solid [blue] line), v_γ (lower solid [black] line) and quadrupole $\mathcal{M}_2 \equiv (5/2)\sigma_\gamma$ (dashed dotted [indigo] line) as a function of conformal time τ before and after re-ionization at $\tau \approx 2800$Mpc. The [green] upper dashed line is the analytic estimate for δ_γ, Equation (344) and the lower [red] dashed line is the analytic estimate for v_γ, Equation (345). The analytic estimate of δ_γ falls almost on top of the correct numerical result. Please note the different scales for δ_γ and v_γ and $\mathcal{M}_2$ respectively. The quadrupole is roughly of the same order as v_γ. The mode shown is for $k/h = 5\text{Mpc}^{-1}$ where $h = 0.7$ and the optical depth to the last scattering surface is $\tau_{opt} = 0.3$. Please note that we truncated the multipole hierarchy at sufficiently high $l_{max} = 2500$. With insufficient l_{max}, rapid unphysical oscillations of considerably higher amplitude would be present.

analytic approximation to the photon evolution after decoupling and in particular during re-ionization. Without re-ionization, and neglecting $\mathcal{M}_2$ as well as using $\phi \approx \psi$ and $\ddot{\phi} \approx 0$, the equation of motions (320) and (321) can be cast in the form

$$\ddot{\delta}_\gamma = -\frac{4}{3}k^2\left(\frac{1}{4}\delta_g + \psi\right), \tag{341}$$

which has the particular solution

$$\delta_\gamma = -4\psi. \tag{342}$$

As the oscillations of v_γ and higher multipoles are damped roughly $\propto (k\tau)^{-3/2}$, we see that to good approximation, $\delta_\gamma = -4\psi$ after decoupling (and before re-ionization) and all higher moments vanish.

During re-ionization, τ_c^{-1} reaches moderate levels again. As v_b has grown substantial during matter domination, the photon velocity v_γ starts to evolve towards v_b. Any increase in magnitude of v_γ, is however swiftly balanced by a growth of δ_γ according to Equation (320). So roughly speaking, during re-ionization, we may approximate

$$0 \approx \dot{v}_\gamma \approx \tau_c^{-1}v_b + k\left[\psi + \frac{1}{4}\delta_\gamma\right], \tag{343}$$

where we omit the tiny term $\tau_c^{-1}v_\gamma$ and (a bit more worrisome) $\mathcal{M}_2$. Hence, during re-ionization, the particular solution to the equation of motion is

$$\delta_\gamma \approx -4\psi - 4\frac{v_b}{k\tau_c}. \tag{344}$$

This approximation holds well (see Figure 7) and oscillations on top of it are again damped and tend to average out. Deriving the above (344), one gets

$$v_\gamma \approx \frac{3}{k}\left(2\dot{\psi} - \frac{\dot{v}_b}{k\tau_c} + \frac{v_b}{k\tau_c}\frac{\dot{\tau}_c}{\tau_c}\right). \tag{345}$$

Please note that during the onset of re-ionization, $\dot{\tau}_c = 2\frac{\dot{a}}{a}\tau_c$ does not hold and it depends on the details of the re-ionization history to what peak magnitude v_γ will reach. Both CMBFAST and CMBEASY implement a swift switch from neutral to re-ionized and it is likely that both serve as upper bounds on any realistic contribution of higher k modes towards the CMB anisotropies at late time. In other words: as the effects are negligible for the currently implemented re-ionization history, they will be even more so for the real one. Going back on track, we give an estimate for the amplitude of $\mathcal{M}_2$: assuming $\dot{\mathcal{M}}_l \approx 0$ and $\tau_c^{-1}\mathcal{M}_l \approx 0$, one gets from the equations of motion (323) that neighboring multipoles $\mathcal{M}_l$ are of roughly the same amplitude. So the amplitude of $\mathcal{M}_2$ and hence that of the shear σ_γ is related to that v_γ, i.e. we find the bound

$$\max(|\sigma_\gamma|) \sim \max(|v_\gamma|), \tag{346}$$

where it is understood that the maximum is taken of full oscillations. After radiation domination, the metric potential ψ is given by

$$\psi \sim \frac{a^2\rho_c\delta_c}{2\text{M}_\text{P}^2k^2}, \tag{347}$$

where M_P is the reduced Plank mass, ρ_c is the energy density of cold dark matter and δ_c is its relative density perturbation. For modes that enter the horizon during radiation domination, δ_c is roughly independent of scale (we omit the overall dependence on the initial power spectrum in this argument). Hence, $\psi \propto k^{-2}$ during matter domination and we see that $\psi \to 0$ and so $\delta_\gamma \to 0$ according to Equation (344). Provided that $\dot{\tau}_c/\tau_c$ remains reasonable, v_γ and hence $\mathcal{M}_2$ and E_2 will remain negligible as well during re-ionization and afterwards.

For the LSS evolution, neglecting the shear is a good approximation because Einstein's Equation gives

$$\frac{12}{5}a^2\left[\bar{p}_\gamma\mathcal{M}_2 + \bar{p}_\nu\mathcal{N}_2\right] = \text{M}_\text{P}^2k^2(\phi - \psi), \tag{348}$$

where $\mathcal{N}_2$ is the neutrino quadrupole. As $\bar{p}_{\gamma,\nu} \propto a^{-4}$, the difference of the metric potentials vanishes for small scale modes, i.e. at least

$$(\phi - \psi) \propto (ka)^{-2}, \tag{349}$$

where we have neglected the decay of the quadrupoles $\mathcal{M}_2$ and $\mathcal{N}_2$ which give an additional suppression (see also Figure 8).

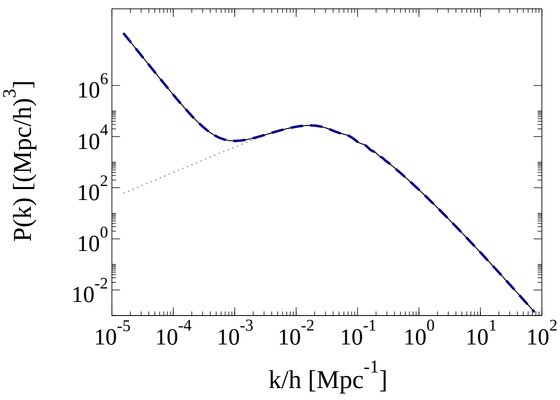

FIG. 8: Cold dark matter power spectrum using the old gauge invariant implentation (dashed line) and the new strategy in gauge invariant variables (thin solid line). The density contrast shown is the gauge invariant combination $D_g^{cdm} \equiv \delta^{longit.} - 3\phi$. The mean deviation between the curves is $\approx 0.02\%$. To guide the eye, we also depict the synchronous gauge power spectrum [thin gray dotted line]. The difference at large scales is due to gauge ambiguities. Again, we used $h = 0.7$.

As the effect of δ_γ and $\mathcal{M}_2$ and E_2 at late times for small scale modes can be neglected (or very well approximated in the case of δ_γ), we see that there is really no need to propagate relativistic species at later times. The key to our final speed up is therefore to avoid integrating these oscillations after they have become irrelevant. We do this by multiplying the RHS of equations (320 - 325) as well as the corresponding multipole evolution equations for relativistic neutrinos by a damping factor Γ. Defining $x \equiv k\tau$, we employ $\Gamma = \{1 - \tanh([x - x_c]/w)\}/2$ with the cross over $x_c = \max(1000, k\tau_{dilute})$, where $a(\tau_{dilute}) = 5a_{equ}$ and a_{equ} is the scale factor at matter-radiation equality. This later criterion ensures that the contribution of relativistic species to the perturbed energy densities is negligible: from equality on, $\delta_c \propto a$, whereas δ_γ decays and $\rho_c/\rho_{rel} \propto a^{-1}$ so at least

$$\delta_c\rho_c : \delta_\gamma\rho_\gamma \propto a^{-2}, \tag{350}$$

and similar arguments hold for neutrinos. Hence, from τ_{dilute} on, one can safely ignore this contribution. The former criterion $x_c < 1000$ ensures that oscillations have damped away sufficiently. The cross-over width w is rather uncritical. We used $w = 50$ to make the transition smooth. Typically, $\tau_{dilute} \approx 400$Mpc and one therefore has to follow only a fraction of τ_{dilute}/τ_0 oscillations as compared to the standard strategy. This corresponds to a gain in efficiency by a factor $\tau_0/\tau_{dilute} \approx 30$.

To compute the sources S_T and S_E, we use the expressions

$$\delta_\gamma = \Gamma\delta_\gamma^{numeric.} - 4(1-\Gamma)\left[\psi + \frac{v_b}{k\tau_c}\right] \tag{351}$$
$$\mathcal{C} = \Gamma\mathcal{C}^{numeric.}, \tag{352}$$
$$\dot{\mathcal{C}} = \Gamma\dot{\mathcal{C}}^{numeric.}, \tag{353}$$
$$\ddot{\mathcal{C}} = \Gamma\ddot{\mathcal{C}}^{numeric.}, \tag{354}$$

which interpolate between the numerical value before Γ-damping and the analytic approximations, Equation (344) and $\mathcal{C} \equiv 0$. Setting $\mathcal{C} \equiv 0$ is an approximation to the small value of the quadrupoles averaged over several oscillations.

For general dark energy models with rest frame speed of sound $c_s^2 > 0$ of the dark energy fluid, the dark energy perturbations well inside the horizon oscillate with high frequency. In this case, one needs to suppress the damped oscillations of the dark energy fluid perturbations much like those of photons to achieve faster integration.

Appendix A: From Rest to FRW frame

In longitudinal gauge, the transformation matrices relating FRW to Minkowski rest frame (denoted by ˜) are:

$$\left(\frac{\partial\tilde{x}^0}{\partial x^0}\right) = a(1 + \Psi\, Q) \tag{A1}$$
$$\left(\frac{\partial\tilde{x}^i}{\partial x^0}\right) = -a\, v\, Q^i \tag{A2}$$
$$\left(\frac{\partial\tilde{x}^0}{\partial x^i}\right) = -a\, v\, Q_i \tag{A3}$$
$$\left(\frac{\partial\tilde{x}^i}{\partial x^j}\right) = a(1 + \Phi\, Q)\delta^i_j. \tag{A4}$$

and

$$\left(\frac{\partial x^0}{\partial\tilde{x}^0}\right) = a^{-1}(1 - \Psi\, Q) \tag{A5}$$
$$\left(\frac{\partial x^i}{\partial\tilde{x}^0}\right) = a^{-1}\, v\, Q^i \tag{A6}$$
$$\left(\frac{\partial x^0}{\partial\tilde{x}^i}\right) = a^{-1}\, v\, Q_i \tag{A7}$$
$$\left(\frac{\partial x^i}{\partial\tilde{x}^j}\right) = a^{-1}(1 - \Phi\, Q)\delta^i_j. \tag{A8}$$

To give an example, the energy momentum tensor transforms as

$$\tilde{T}^\mu_{\ \nu} = T^\alpha_{\ \beta}\left(\frac{\partial\tilde{x}^\beta}{\partial x^\nu}\right)\left(\frac{\partial x^\mu}{\partial\tilde{x}^\alpha}\right) \tag{A9}$$

Using these transformations, we get

$$\tilde{p}^0 = a(1+\Psi)p^0 - a\, v\, Q_i n^i \tag{A10}$$
$$= a\, p\, (1 + \Phi - vQ_i n^i) \tag{A11}$$
$$= a\, p\, (1 + \Phi + iv\mu Q). \tag{A12}$$

This little term above will later give us the Doppler shift.

Exercise 17: Obtain an expression for $\Delta\mathcal{P}$ from (A12). Hint: use μ as the direction cosine for $\boldsymbol{n}$ and λ as the direction cosine for $\boldsymbol{n'}$. The velocity to transform from FRW to the Minkowski rest frame is just the velocity of the electrons v_b. In addition, $\tilde{p}^0 = \tilde{p}'^0$, because Thomson scattering is elastic in the electron rest frame.

The solution to this exercise is given in the main text.

Appendix B: Gauge invariant temperature perturbation

Unfortunately, the temperature perturbation is not gauge invariant. In principle, this is no problem, for the data is always quoted in terms of multipole moments $l = 2, \ldots$ which are gauge independent.[16] Yet, it takes only a little more work to find a gauge invariant temperature anisotropy so let's do it! The distribution function is a scalar and hence does not transform under *coordinate transformations*. To complete the *gauge transformation*, however, we still have to go back from the new argument $\tilde{x}$ to the old one using a Taylor expansion. There is a small complication, though: the argument p^μ transforms under the coordinate transformation $x^\mu \to \tilde{x}^\mu = x^\mu + \epsilon^\mu$ according to

$$\tilde{p}^i = p^i + \epsilon^i_{,\mu} p^\mu \tag{B1}$$

and we have to go back from $\tilde{p}$ to the old momentum p just like we do for the spatial coordinates:

$$\begin{aligned}
\tilde{f}(x^\mu, p^i) &= f(x^\mu, p^i) - \epsilon^\mu \partial_\mu f - \frac{\partial f}{\partial p^i}\delta p^i && \text{(B2)}\\
&= f(x^\mu, p^i) - \epsilon^\mu \partial_\mu f - \frac{\partial f}{\partial p^i}\epsilon^i_{,\nu} p^\nu && \text{(B3)}\\
&= f(x^\mu, p^i) - \frac{\partial f}{\partial \mathcal{P}}\frac{\partial \mathcal{P}}{\partial a}\,\dot{a}\,\epsilon^0 && \text{(B4)}\\
&\quad - \bar{f}' a^2 n_i \left[\dot{L} Q^i p + L Q^i_{,j} n^j p\right] && \text{(B5)}\\
&= f(x^\mu, p^i) - 2\frac{\dot{a}}{a}\bar{f}'\mathcal{P} T Q && \text{(B6)}\\
&\quad - \bar{f}'\mathcal{P}\left[n_i Q^i \dot{L} + L n^i n^j Q_{i,j}\right]. && \text{(B7)}\\
& && \text{(B8)}
\end{aligned}$$

From (B2), we can then read off the transformation law for F:

$$\tilde{F} = F - \bar{f}'\mathcal{P}\left[2\frac{\dot{a}}{a}TQ + n_i Q^i \dot{L} + L n^i n^j Q_{i,j}\right]. \tag{B9}$$

Using $F = -\mathcal{P}\bar{f}'\Delta$ and the definition for the direction cosine μ, we see that Δ transforms as

$$\begin{aligned}
\tilde{\Delta} &= \Delta + 2\frac{\dot{a}}{a}TQ + n_i Q^i \dot{L} + L n^i n^j Q_{i,j} && \text{(B10)}\\
&= \Delta + 2\frac{\dot{a}}{a}TQ - i\mu Q\dot{L} + k\mu^2 LQ. && \text{(B11)}
\end{aligned}$$

We easily construct a gauge invariant temperature perturbation

$$\mathcal{M} \equiv \Delta + 2\left(H_L + \frac{1}{3}H_T\right) + i\mu k^{-1}\dot{H}_T - \mu^2 H_T, \tag{B12}$$

which reduces to

$$\mathcal{M}(\tau, \boldsymbol{x}, \boldsymbol{n}) = \Delta(\tau, \boldsymbol{x}, \boldsymbol{n}) + 2\Phi(\tau) Q(\boldsymbol{x}) \tag{B13}$$

in longitudinal gauge.

Appendix C: The energy momentum tensor from the distribution

The connection between the distribution language and the fluid language is given by means of [27]

$$T^\mu_{\ \nu} = \int \sqrt{-g}\,\frac{p^\mu p_\nu}{|p_0|} f(p, x)\,\mathrm{d}^3 p, \tag{C1}$$

where for the scalar case and in longitudinal gauge, $\sqrt{g}$, p_0 and p_i are given by equations (124- 126). Let us take a closer look at $T^0_{\ 0}$. To warm up (and because we will need the result for the perturbed case), we consider the background only, neglecting all perturbation variables in the formulae for p^0 and σ_γ. So

$$\begin{aligned}
-\bar{\rho} = \bar{T}^0_{\ 0} &= \int \sqrt{-g}\,\frac{\bar{p}^0 \bar{p}_0}{|\bar{p}_0|} f(p, x)\,\mathrm{d}^3 p && \text{(C2)}\\
&= -a^4 \int \bar{p}^0 \bar{f}(\mathcal{P})\,\mathrm{d}^3 p && \text{(C3)}\\
&= -a^4 \int p \bar{f}(\mathcal{P})\,\mathrm{d}^3 p && \text{(C4)}\\
&= -a^4 \int p^3 \bar{f}(\mathcal{P})\,\mathrm{d}p\mathrm{d}\Omega && \text{(C5)}\\
&= -4\pi\, a^4 \int p^3 \bar{f}(\mathcal{P})\,\mathrm{d}p && \text{(C6)}\\
&= -4\pi\, a^{-4} \int \mathcal{P}^3 \bar{f}(\mathcal{P})\,\mathrm{d}\mathcal{P}. && \text{(C7)}
\end{aligned}$$

---

[16] The monopole and dipole are gauge dependent but not of interest to us right now.

The total (background and perturbed part) is

$$
\begin{aligned}
T^0_{\ 0} &= \int \sigma_\gamma \frac{p^0 p_0}{|p_0|} f(p,x)\, \mathrm{d}^3p && \text{(C8)}\\
&= -\int \sigma_\gamma p^0 \, f \, \mathrm{d}^3 p && \text{(C9)}\\
&= -a^4 \int (1+\Psi+3\Phi) p^0 \, f \, \mathrm{d}^3 p && \text{(C10)}\\
&= -a^4 \int (1+\Psi+3\Phi) p (1-\Psi+\Phi) f \, \mathrm{d}^3 p && \text{(C11)}\\
&= -a^4 \int (1+4\Phi) p \, f \, p^2 \mathrm{d}p \, \mathrm{d}\Omega && \text{(C12)}\\
&= -a^{-4} \int \mathcal{P}^3 \, f (1+4\Phi)\, \mathrm{d}\mathcal{P}\, \mathrm{d}\Omega. && \text{(C13)}
\end{aligned}
$$

From this, we can readily read off the perturbation

$$
\begin{aligned}
-\delta T^0_{\ 0} &= a^{-4} \int \left[4\Phi\, \bar{f}(\mathcal{P}) + F(\tau, \boldsymbol{x}, \mathcal{P}, \boldsymbol{n})\right] \mathcal{P}^3 \mathrm{d}\mathcal{P}\, \mathrm{d}\Omega\\
&= a^{-4} \int \left[4\Phi\, \bar{f}(\mathcal{P}) - \mathcal{P} \bar{f}' \Delta(\tau, \boldsymbol{x}, \boldsymbol{n})\right] \mathcal{P}^3 \mathrm{d}\mathcal{P}\, \mathrm{d}\Omega\\
&= a^{-4} \int \left[4\Phi\, \mathcal{P}^3 \bar{f}(\mathcal{P}) - \mathcal{P}^4 \bar{f}' \Delta(\tau, \boldsymbol{x}, \boldsymbol{n})\right] \mathrm{d}\mathcal{P}\, \mathrm{d}\Omega\\
&= a^{-4} \int \mathcal{P}^3 \bar{f} \left[4\Phi + 4\Delta(\tau, \boldsymbol{x}, \boldsymbol{n})\right] \mathrm{d}\mathcal{P}\, \mathrm{d}\Omega\\
&= a^{-4} \int \mathcal{P}^3 \bar{f} \left[4\Phi + 4\mathcal{M}(\tau, \boldsymbol{x}, \boldsymbol{n}) - 8\Phi\right] \mathrm{d}\mathcal{P}\, \mathrm{d}\Omega\\
&= a^{-4} \int \mathcal{P}^3 \bar{f}(\mathcal{P})\, 4\, \left[\mathcal{M}(\tau, \boldsymbol{x}, \boldsymbol{n}) - \Phi(\tau) Q\right] \mathrm{d}\mathcal{P}\, \mathrm{d}\Omega\\
&= a^{-4} \int \mathcal{P}^3 \bar{f}(\mathcal{P}) \mathrm{d}\mathcal{P}\, 4 \int \mathcal{M}(\tau, \boldsymbol{x}, \boldsymbol{n}) - \Phi(\tau) Q \, \mathrm{d}\Omega,
\end{aligned}
$$

where we have integrated by parts in line 3 to get rid of $\bar{f}'$. Using our background result $\bar{T}^0_{\ 0}$, leads to

$$
\begin{aligned}
-\delta T^0_{\ 0} &= \frac{\bar{\rho}}{4\,\pi}\, 4 \int \mathcal{M}(\tau, \boldsymbol{x}, \boldsymbol{n}) - \Phi \, \mathrm{d}\Omega\\
&= \frac{\bar{\rho}}{\pi} \left[-4\pi\, \Phi + \int \mathcal{M}(\tau, \boldsymbol{x}, \boldsymbol{n})\right] \mathrm{d}\Omega\\
&= -4\bar{\rho}\,\Phi + \frac{\bar{\rho}}{\pi} \int \mathcal{M}(\tau, \boldsymbol{x}, \boldsymbol{n})\, \mathrm{d}\Omega
\end{aligned}
\qquad \text{(C14)}
$$

Finally, we know that $\delta\rho = -\delta T^0_0$ and hence

$$
\begin{aligned}
\delta &= \frac{\delta\rho}{\bar{\rho}} && \text{(C15)}\\
&= -\frac{\delta T^0_0}{\bar{\rho}} && \text{(C16)}\\
&= -4\Phi + \frac{1}{\pi} \int \mathcal{M}(\tau, \boldsymbol{x}, \boldsymbol{n})\, \mathrm{d}\Omega, && \text{(C17)}
\end{aligned}
$$

where as always in this chapter, δ is in longitudinal gauge. As $w = 1/3$ for a radiation fluid, we can read off a simple expression for the gauge invariant density perturbation D_g^γ. Hence, the calculation above and similar calculations for $T^i_{\ 0}$ and $T^i_{\ j}$ yield

$$
\begin{aligned}
D_g^\gamma &= \frac{1}{\pi} \int \mathcal{M} \mathrm{d}\Omega && \text{(C18)}\\
V_\gamma &= \frac{3i}{4\pi} \int \mu \mathcal{M} \mathrm{d}\Omega && \text{(C19)}\\
\Pi_\gamma &= -\frac{3}{\pi} \int \frac{1}{2}\left(3\mu^2 - 1\right) \mathcal{M} \mathrm{d}\Omega. && \text{(C20)}
\end{aligned}
$$

Appendix D: Useful Expressions for Y_l^m's, j_l's and others

Spherical harmonics are orthogonal:

$$
\int \mathrm{d}\boldsymbol{n} Y_l^m(\boldsymbol{n}) \left[Y_{l'}^{m'}(\boldsymbol{n})\right]^* = \delta_{mm'} \delta_{ll'}. \qquad \text{(D1)}
$$

A Legendre Polynomial can be decomposed according to

$$
P_l(\boldsymbol{n} \cdot \boldsymbol{n}') = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_l^m(\boldsymbol{n}) \left[Y_l^m(\boldsymbol{n}')\right]^*. \qquad \text{(D2)}
$$

A plane wave can be composed

$$
e^{-i\mu x} = \sum_l (-i)^l (2l+1) j_l(x) P_l(\mu) \qquad \text{(D3)}
$$

Using these last two decompositions and orthogonality, we can show that

$$
\begin{aligned}
&\int \mathrm{d}\boldsymbol{n} \left[Y_l^m(\boldsymbol{n})\right]^* e^{-i\boldsymbol{\mu x}} && \text{(D4)}\\
&= \int \mathrm{d}\boldsymbol{n} \left[Y_l^m(\boldsymbol{n})\right]^* \sum_{l'} (-i)^{l'} (2l'+1) j_{l'}(x) P_{l'}(\mu) && \text{(D5)}\\
&= \int \mathrm{d}\boldsymbol{n} \left[Y_l^m(\boldsymbol{n})\right]^* \sum_{l'} (-i)^{l'} (2l'+1) j_{l'}(x) && \text{(D6)}\\
&\qquad \times \frac{4\pi}{2l'+1} \sum_{m'=-l'}^{l'} Y_{l'}^{m'}(\boldsymbol{n}) \left[Y_{l'}^{m'}(\hat{\boldsymbol{k}})\right]^* && \text{(D7)}\\
&= 4\pi \sum_{l'} (-i)^{l'} j_{l'}(x) \sum_{m'=-l'}^{l'} \left[Y_{l'}^{m'}(\hat{\boldsymbol{k}})\right]^* \int \mathrm{d}\boldsymbol{n} \left[Y_l^m(\boldsymbol{n})\right]^* Y_{l'}^{m'}(\boldsymbol{n})\\
&= 4\pi (-i)^l j_l(x) \left[Y_l^m(\hat{\boldsymbol{k}})\right]^*. && \text{(D8)}
\end{aligned}
$$

If $z||\boldsymbol{k}$, then $Y_l^m(\hat{\boldsymbol{k}}) = 0$ for all $m \neq 0$ and as $Y_l^0(\hat{z}) = \sqrt{\frac{2l+1}{4\pi}} P_l(1) = \sqrt{\frac{2l+1}{4\pi}}$, this yields

$$
\int \mathrm{d}\boldsymbol{n} \left[Y_l^m(\boldsymbol{n})\right]^* e^{-i\boldsymbol{\mu x}} = \sqrt{4\pi(2l+1)} (-i)^l j_l(x) \qquad \text{(D9)}
$$

Symbol	Meaning
$\mathcal{M}_l$	Photon multipole, $l = 0 \dots 7$
E_l	Photon polarization multipole, $l = 2 \dots 7$
$\mathcal{N}_l$	Neutrino multipole, $l = 0 \dots 9$
$D_g^{\rm b}$	Density perturbation for baryons
$D_g^{\rm c}$	Density perturbation for cold dark matter
$V_{\rm b}$	Velocity of baryons
$V_{\rm c}$	Velocity of cold dark matter
D_g^γ	Density perturbation for photons (from $\mathcal{M}_0$)
D_g^ν	Density perturbation for massless neutrinos (from $\mathcal{N}_0$)
V_γ	Velocity of photons (from $\mathcal{M}_1$)
V_ν	Velocity of neutrinos (from $\mathcal{N}_1$)
Π_γ	Photon shear (from $\mathcal{M}_2$)
Π_ν	Neutrino shear (from $\mathcal{N}_2$)

TABLE I: Perturbations propagated through conformal time (upper half). Quantities that are not propagated themselves, but derived algebraically from quantities propagated are grouped in the lower half.

Appendix E: Full set of Perturbation Equations

In this appendix, we summarize the formulae needed to evolve the gauge-invariant perturbation variables listed in Table I.

1. Scalar fluctuations

a. *The Metric Potentials Φ and Ψ*

In order to solve Einstein's equations, we first use Equation (106) in the form

$$-\Psi = \Phi + \mathrm{M_P^{-2}}\, k^{-2} a^2 \bar{p}\,\bar{\Pi}, \tag{E1}$$

where $\bar{p}\bar{\Pi} \equiv \bar{p}_\gamma \Pi_\gamma + \bar{p}_\nu \Pi_\nu$. In addition, for each species i,

$$a^2 \bar{\rho}_i D_i = a^2 \bar{\rho}_i D_g^i - 3\bar{\rho}_i a^2 (1 + w_i)\Phi. \tag{E2}$$

Therefore, Equation (104) yields

$$\Phi = \frac{\sum a^2 \bar{\rho}_i D_g^i}{2\mathrm{M_P^2} k^2 + \sum 3\bar{\rho}_i a^2 (1 + w_i)} \tag{E3}$$

where the summation runs over all species. This fixes Φ, because the right hand side of Equation (E3) contains only known variables. The gravitational potential Ψ follows then immediately from Equation (E1).

b. *Cold dark matter*

For cold dark matter, we set $w = c_s^2 = 0$, from Equations (107) and (108) one then gets:

$$\dot{D}_g^{\rm c} = -kV_{\rm c} \tag{E4}$$

$$\dot{V}_{\rm c} = -\frac{\dot{a}}{a} V_{\rm c} + k\Psi \tag{E5}$$

c. *Baryons and Photons*

As baryons have a non - negligible sound speed at early times, one needs to incorporate it in the baryon evolution. However following [23], we neglect the $3c_s^2 V_{\rm b}$ term in the evolution of the velocity perturbation.[17] Hence, from Equations (107) and (108) and including the momentum transfer between photons and baryons due to Compton scattering [13], we have

$$\dot{D}_g^{\rm b} = -k\, V_{\rm b} - 3\, c_s^2 \frac{\dot{a}}{a} D_g^{\rm b} \tag{E6}$$

$$\dot{V}_{\rm b} = -\frac{\dot{a}}{a} V_{\rm b} + k\, c_s^2 D_g^{\rm b} + k(\Psi - 3c_s^2 \Phi) + \dot{\kappa}\, R\, (V_\gamma - V_{\rm b}), \tag{E7}$$

where $R \equiv 4\rho_\gamma/(3\rho_b)$ is large in the early universe. The photons evolve according to Equations (229-232). Formulated in terms of velocity and density perturbations it reads

$$\dot{V}_\gamma = \dot{\kappa}(V_b - V_\gamma) + k(\Psi - \Phi) + k\left(\frac{1}{4} D_g^\gamma - \frac{1}{6}\Pi_\gamma\right) \tag{E8}$$

During the early stages of evolution, the differential optical depth $\dot{\kappa}$ as well as R are large. Hence the coupling between baryons and photons is very strong. This poses difficulties for the numerical integration. The aim is thus to find equations for the tight coupling regime, which do not involve terms proportional to $\dot{\kappa}$. To this end, we follow the procedure described in [23] and derive an expression for the slip between photons and baryons, starting from Equations (E7) and (E8). The result is[18]:

---

[17] That our approach is indeed equivalent to the one leading from Equation (29) of [23] to Equation (66) of [23], is seen by using our Equation (**??**).

[18] A shortcut to reach it would be to simply transform Equation (74) of [23] to gauge invariant variables.

$$\dot V_{\rm b} - \dot V_\gamma = \frac{2R}{1+R}(V_{\rm b}-V_\gamma) + [\dot\kappa(1+R)]^{-1} \times \Big\{ -\frac{\ddot a}{a}V_{\rm b} - \frac{\dot a}{a}k\left(\frac{1}{2}D_g^\gamma + \Psi - 2\Phi\right) + k\left(c_s^2 \dot D_g^{\rm b} - \frac{k}{4}\dot D_g^\gamma + \dot\Phi(1-3c_s^2)\right)\Big\} \quad \text{(E9)}$$

Depending on whether the tight coupling approximation is justified or not, we therefore use the following equations for baryons and photons:

(I) Tight coupling: From $R\times$(E8)+(E7) and neglecting Π_γ, one gets

$$\dot V_{\rm b} = (1+R)^{-1}\Big\{ -\frac{\dot a}{a}V_{\rm b} - k\Phi(R+3c_s^2) + \frac{k}{4}RD_g^\gamma + kc_s^2 D_g^{\rm b} + R\left(\dot V_{\rm b} - \dot V_\gamma\right)\Big\} + k\,\Psi, \quad \text{(E10)}$$

where Equation (E9) is used to evaluate the r.h.s of the above Equation (E10). The evolution of the photon velocity follows then by solving (E10) for $\dot V_\gamma$,

$$\dot V_\gamma = R^{-1}\Big\{ -\dot V_{\rm b} - \frac{\dot a}{a}V_{\rm b} + kc_s^2 D_g^{\rm b} - 3kc_s^2\Phi + k(R+1)\Psi\Big\} + k\left(\frac{1}{4}D_g^\gamma - \frac{1}{6}\Pi_\gamma\right), \quad \text{(E11)}$$

where we have re-instated Π_γ. The densities evolve according to Equation (E6) and (228). All higher moments of $\mathcal{M}$ as well as the polarization terms E_l are set to zero.

(II) No Tight coupling:
Outside the tight coupling regime, baryons evolve according to Equations (E6) and (E7). Photons evolve according to Equations (228 - 234). As far as the photon velocity is concerned, one may alternatively evolve V_γ according to Equation (E11). The multipole expansion for the photons is truncated at some $l < 10$ for sufficient precision. In order to avoid truncation effects best as possible, one uses a recursion relation for spherical Bessel functions similar to the one in [23]:

$$\dot{\mathcal{M}}_{l_{\max}} = \frac{2l_{\max}+1}{2l_{\max}-1}\,k\,\mathcal{M}_{l_{\max}-1} - \mathcal{M}_{l_{\max}}\left(\frac{l_{\max}+1}{\tau} + \dot\kappa\right) \quad \text{(E12)}$$

The polarization E is propagated using Equations (233,234), and the recursion relation

$$\dot E_{l_{\max}} = \frac{2l_{\max}+1}{2l_{\max}-1}\,kE_{l_{\max}-1} - E_{l_{\max}}\left(\frac{l_{\max}+1}{\tau} + \dot\kappa\right), \quad \text{(E13)}$$

for truncation.

d. Massless Neutrinos

Massless neutrinos evolve according to Equations (235-237). The hierarchy is truncated using

$$\dot N_{l_{\max}} = \frac{2l_{\max}+1}{2l_{\max}-1}\,kN_{l_{\max}-1} - N_{l_{\max}}\frac{l_{\max}+1}{\tau}. \quad \text{(E14)}$$

2. Tensor fluctuations

For completeness, we quote the results of [32]. The anisotropies are

$$\Delta^{\rm tens.}_{\rm T;l} = \sqrt{\frac{(l+2)!}{(l-2)!}}\int_0^{\tau_0} {\rm d}\tau S^{\rm tens.}_{\rm T}(k,\tau)\frac{j_l(x)}{x^2}, \quad \text{(E15)}$$

$$\Delta^{\rm tens.}_{\rm E,B;l} = \int_0^{\tau_0} {\rm d}\tau S^{\rm tens.}_{\rm E,B}(k,\tau) j_l(x). \quad \text{(E16)}$$

where $x \equiv k[\tau_0 - \tau]$ and the sources are

$$S^{\rm tens.}_{\rm T}(k,\tau) = -\dot h\exp(-\kappa) + g\psi, \quad \text{(E17)}$$

$$S^{\rm tens.}_{\rm E}(k,\tau) = g\left\{\psi - \frac{\ddot\psi}{k^2} + \frac{2\psi}{x^2} - \frac{\dot\psi}{kx}\right\} - \dot g\left\{\frac{2\dot\psi}{k^2} + \frac{4\psi}{kx}\right\} - 2\ddot g\frac{\psi}{k^2}, \quad \text{(E18)}$$

$$S^{\rm tens.}_{\rm B}(k,\tau) = g\left\{\frac{4\psi}{x} + \frac{2\dot\psi}{k}\right\} + 2\dot g\frac{\psi}{k}. \quad \text{(E19)}$$

and ψ is given by

$$\psi = \frac{1}{10}\tilde\Delta_0^T + \frac{1}{7}\tilde\Delta_2^T + \frac{3}{70}\tilde\Delta_4^T - \frac{3}{5}\tilde\Delta_0^P + \frac{6}{7}\tilde\Delta_2^P - \frac{3}{70}\tilde\Delta_4^P \quad \text{(E20)}$$

and the hierarchy for the tensor temperature and polarization multipoles is

$$\dot{\tilde\Delta}_0^T = -k\tilde\Delta_1^T - \dot\kappa\left[\tilde\Delta_0^T - \psi\right] - \dot h \quad \text{(E21)}$$

$$\dot{\tilde\Delta}_0^P = -k\tilde\Delta_2^T - \dot\kappa\left[\tilde\Delta_1^T + \psi\right] \quad \text{(E22)}$$

$$\dot{\tilde\Delta}_l^{T,P} = \frac{k}{2l+1}\left[l\tilde\Delta_{l-1}^{T,P} - (l+1)\tilde\Delta_{l+1}^{T,P}\right] - \dot\kappa\tilde\Delta_l^{T,P} \quad ; l \geq 1 \quad \text{(E23)}$$

Appendix F: Conventions, Symbols and Conversion Factors

⋆ We take the metric with signature $(-,+,+,+)$.
⋆ Greek indices run from $0\ldots3$ and are raised and lowered by the metric $g_{\mu\nu}$.

Symbol	Meaning
t	time
τ	conformal time
a	scale factor, normalized $a(\text{today}) = 1$
$\dot{y}$	conformal time derivative $\frac{\mathrm{d}}{\mathrm{d}\tau} y$
V'	derivative with respect to the field φ
y_0	Quantity y today
$\mathrm{M_P}$	Reduced Planck mass $\mathrm{M_P} = (8\pi G)^{-1/2}$.
H	Hubble parameter $H = (\mathrm{d}a/\mathrm{d}t)/a$
$\bar{\rho}_y$	Background energy density of y
$\bar{p}_y$	Background pressure of y
w_y	Equation of state $w = \bar{p}/\bar{\rho}$ of species y
c_s	Sound speed (of some species) see (88)
Ω^y	Fraction of energy $\bar{\rho}_y/\rho_{\text{total}}$.
Ω^y_0	$\Omega^y(\text{today})$.
D^y_g	Density perturbation of species y
V_y	Velocity perturbation of species y.
Π_y	Shear of species y.
$\mathcal{M}_l$	Photon multipole
E_l	Photon polarization multipole
$\mathcal{N}_l$	Neutrino multipole
Q	Scalar perturbation basis function
Q_i	Scalar basis function for vector fields
Q_{ij}	Scalar basis function for tensor fields
n_e	Number density of free electrons
σ_T	Thomson scattering cross-section
$\dot{\kappa}$	Differential optical depth $\dot{\kappa} = a n_e \sigma_T$.
n_s	spectral index of initial fluctuations

TABLE II: Frequently used symbols.

⋆ Latin indices run from $1 \dots 3$ and are raised and lowered by δ_{ij}.
⋆ The partial derivative of a tensor $\partial_\mu T$ is abbreviated by $T_{,\mu}$.
⋆ The co-variant derivative of a tensor $\nabla_\mu T$ is abbreviated by $T_{;\mu}$.
⋆ Perturbations in a certain gauge are denoted by lower case letters: $\delta,\ v,\ \chi \dots$
⋆ Gauge-invariant variables are denoted by capital letters: $D_g,\ \Pi,\ X \dots$
⋆ Three-vectors are denoted by bold letters and their scalar product is given by $\boldsymbol{a} \cdot \boldsymbol{b} = a^i\, b^j\, \delta_{ij}$.
⋆ The Christoffel symbols are

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\sigma} \left(g_{\sigma\beta,\alpha} + g_{\alpha\sigma,\beta} - g_{\alpha\beta,\sigma} \right),$$

⋆ while the Riemann tensor is

$$R^\mu{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\sigma_{\nu\beta}\Gamma^\mu_{\sigma\alpha} - \Gamma^\sigma_{\nu\alpha}\Gamma^\mu_{\sigma\beta}.$$

One Mpc is ...	One Mpc^{-1} is ...
$1.5637 \times 10^{38}\ \mathrm{Gev}^{-1}$	6.3952×10^{-39} Gev
3.0856×10^{22} m	$3.2408 \times 10^{-23}\ \mathrm{m}^{-1}$
1.0292×10^{14} s	$9.7163 \times 10^{-15}\ \mathrm{s}^{-1}$
3.264×10^{6} years	
In terms of Mpc is ...	
$\mathrm{M_P} = 3.7685 \times 10^{56}\ \mathrm{Mpc}^{-1}$	
$H = 3.335 \times 10^{-4}\, h\ \mathrm{Mpc}^{-1}$	
One Mpc^{-4} is ...	
$1.673 \times 10^{-153} \mathrm{Gev}^4$	

TABLE III: All quantities in CMBEASY are in Mpc. For convenience, we summarize conversion factors to other units.

1. Christoffel symbols

The Christoffel symbols for the Robertson Walker metric (4) in the unperturbed case are $\Gamma^0_{00} = \frac{\dot{a}}{a}$, $\Gamma^0_{ij} = \frac{\dot{a}}{a}\delta_{ij}$, $\Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j$, $\Gamma^i_{jl} = \Gamma^0_{i0} = 0$. Using the scalar longitudinal gauge metric (71), the first order perturbations become

$$\delta\Gamma^0_{00} = AQ \tag{F1}$$

$$\delta\Gamma^0_{i0} = -kAQ_i - B\frac{\dot{a}}{a}Q_i \tag{F2}$$

$$\begin{aligned} \delta\Gamma^0_{ij} &= \delta_{ij} Q \left(\frac{1}{3}\dot{H}_T + \dot{H}_L + 2\frac{\dot{a}}{a}[H_L - A] \right) \\ &\quad + Qk^{-2}k_i k_j \left(kB - 2\frac{\dot{a}}{a}H_T - \dot{H}_T \right) \end{aligned} \tag{F3}$$

$$\delta\Gamma^i_{00} = -\left(kA + \frac{\dot{a}}{a}B + \dot{B} \right) Q^i \tag{F4}$$

$$\delta\Gamma^i_{j0} = \dot{H}_T Q^i_j + \dot{H}_L Q \delta^i_j \tag{F5}$$

$$\begin{aligned} \delta\Gamma^i_{jl} &= k \left(H_L + \frac{1}{3}H_T \right) \left(Q^i \delta_{jl} - Q_j \delta^i_l - Q_l \delta^i_j \right) \\ &\quad + k^{-1} k_j k_l H_T Q^i + \frac{\dot{a}}{a}\delta_{jl} B Q^i \end{aligned} \tag{F6}$$

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