# QFT II - Problem Set 10

## (48) Local gauge transformations

Suppose you have a theory of fermions

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi + im\bar{\psi}\psi.$$

a) Convince yourself that $\mathcal{L}$ is invariant under a transformation $\psi \to e^{i\theta}\psi$, where $\theta$ is a constant.

b) Our Universe is really really big. Suppose you pick a phase $\theta$. In another corner of our Universe, Alf (to you he looks quite alien but he is a really good field theorist) picks a different phase $\theta_{alf}$. In an intergalactic meeting on field theory, you meet Alf and you start talking about the phases you picked. Being on the same meeting in the same place, you work with the same theory but a different phase. For days you argue with Alf and you are both convinced that the other is wrong in picking his phase. Wouldn't it be nicer if everyone in the Universe could pick his own phase? So take $\theta = \theta(x)$ as a function of coordinates. How does $\mathcal{L}$ transform now?

c) As you have seen in (b), the derivative spoilt the invariance under $\psi \to e^{i\theta(x)}\psi$. That is actually quite sensible, because the derivative in direction $n^\mu$ is given by the limiting process

$$n^\mu\partial_\mu\psi = \lim_{\epsilon\to 0}\frac{\psi(x+\epsilon n) - \psi(x)}{\epsilon},$$

and the phase transformation acts differently on the space time points $x$ and $x + \epsilon n$. To compensate for the different phase transformations, we can introduce a covariant derivative

$$n^\mu D_\mu\psi = \lim_{\epsilon\to 0}\frac{\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)}{\epsilon}, \tag{1}$$

where a function $U(y, x)$ multiplies $\psi(x)$.

i) What value must $U(x, x)$ take on in order for $D_\mu$ to reduce to $\partial_\mu$ in the case of constant $\theta$. In other words, what is $\lim_{\epsilon\to 0} U(x + \epsilon n, x)$?

ii) If $U$ transforms under $\psi \to e^{i\theta(x)}\psi$ as

$$U(y, x) \to e^{i\theta(y)}U(y, x)e^{-i\theta(x)}, \tag{2}$$

how does $n^\mu D_\mu\psi$ transform?

d) As $U$ compensates for a difference in phases and $U(x, x) = 1$, $U$ must itself be a phase, i.e. $U(x, y) = \exp(i\Phi(x, y))$. For small separations, you can thus expand

$$U(x + \epsilon n, x) = 1 + ie\epsilon n^\mu A_\mu(x) + \mathcal{O}(\epsilon^2), \tag{3}$$

where $A_\mu(x)$ is called a connection and $e$ is a constant. From the definition (1), what is $D_\mu\psi$ in terms of $\partial_\mu$ and $A_\mu$?

e) By using (2) and (3), how does $A_\mu$ transform?

## (49) The Wegner-Wilson Loop

Consider the Wilson line from $y$ to $x$ along some arbitrary path $P$

$$U_P(y, x) = P\left[\exp\left\{ig\int_x^y A_\mu(z)\mathrm{d}z^\mu\right\}\right],$$

or a little more explicitly parameterizing the path as $z^\mu(t)$ with $z(0) = x$ and $z(1) = y$:

$$U_P(y,x) = P\left[\exp\left\{ig\int_0^1 A_\mu(z(t))\frac{\mathrm{d}z^\mu(t)}{\mathrm{d}t}\mathrm{d}t\right\}\right].$$

In the above $P[]$ stands for *path ordering*. In the Abelian case, this is not necessary. However, in the non-Abelian case it means that in a power series expansion of the exponential above, all matrices are ordered from left to right according to their argument $z$ as $y \to x$. So if e.g. $0 < t_1 < t_2 < 1$, then $P[A(x(t_1))A(x(t_2))] = A(x(t_2))A(x(t_1))$ and $P[A(x(t_2))A(x(t_1))] = A(x(t_2))A(x(t_1))$.

a) In the **Abelian** case in which $A_\mu$ transforms as

$$A_\mu \to A_\mu + \frac{1}{g}\partial_\mu\theta,$$

how does $U_P(y,x)$ transform? In order to solve this a bit more intuitively (and because we will need to proceed similarily later anyhow):

i) Divide the path in discrete steps $z_i$, such that $x, z_1, z_2, \ldots z_n, y$.

ii) Split the integral $\int_x^y$ into the little steps $\int_x^y = \int_x^{z_1} + \int_{z_1}^{z_2} \ldots$.

iii) Expand $\theta(z_i)$ in a Taylor series around $z_{i-1}$ in terms of the difference $z_i - z_{i-1}$ to see what $(\partial_\mu\theta)\mathrm{d}z^\mu$ is (if you don't know that anyhow!).

b) Consider now the **non-Abelian case**. In this case $A_\mu$ is a matrix

$$A_\mu \equiv \frac{\lambda^a}{2}A_\mu^a,$$

where $\lambda_a$ are matrices and $A_\mu$ transforms under a gauge transformation $\psi \to e^{i\theta(x)}\psi$ as

$$A_\mu \to A_\mu + i[\theta, A_\mu] + \frac{1}{g}\partial_\mu\theta. \tag{4}$$

i) Again, like in (a), split the path in intermediate steps $z_i$ and hence the exponential into a product of exponentials, that you path order.

ii) So far, you have found that

$$U_P(y,x) = e^{ig\int_{z_n}^y A_\mu(z)\mathrm{d}z^\mu} e^{ig\int_{z_{n-1}}^{z_n} A_\mu(z)\mathrm{d}z^\mu} \ldots e^{ig\int_x^{z_1} A_\mu(z)\mathrm{d}z^\mu}.$$

Suppose you make the steps $z_{i+1} - z_i$ smaller and smaller:

1.) Evaluate the integrals in the limit that the step size is tiny, i.e $\int_a^b f(x)dx \approx (b-a)f(b)$

2.) Expand the exponentials.

iii) Show that under the *infinitesimal* gauge transformation (4), the transformation of $1 + igA_\mu\mathrm{d}z^\mu$ is to lowest order in $\theta$ equivalent to

$$1 + igA_\mu\mathrm{d}z^\mu \to e^{i\theta(z_i)}[1 + igA_\mu\mathrm{d}z^\mu]e^{-i\theta(z_{i-1})},$$

where $\mathrm{d}z^\mu = z_i^\mu - z_{i-1}^\mu$.

iv) Finally, combining (ii) and (iii), how does $U_P(y,x)$ transform under a gauge transformation?

c) How does $\bar\psi(y)U_P(y,x)\psi(x)$ transform?

d) Is $U_P(x,x)$ gauge invariant?

e) Compute the transformation property of the trace of $U_P(x,x)$, i.e. Tr $U_P(x,x)$.