QFT I - Problem Set 1 - Solution

14th December 2005

1 Self Interactions

We have given a Hamiltonian with nearest neighbor interaction,

$$H = \sum_j \left\{ \frac{D}{2} Q_j Q_j + \frac{1}{2M} P_j P_j \right\}. \quad (1)$$

Now we add a so-called self-interaction term, i. e. an interaction at the same lattice site:

$$H_{self} = \sum_j \lambda Q_j Q_j Q_j Q_j = \sum_j \frac{\lambda}{4DM} (a_j + a_j^\dagger)^4 \quad (2)$$

Now we want to express H_{self} in terms of annihilation and creation operators in Fourier space. Therefore, we express Q_j in terms of annihilation and creation operators in position space and make the Fourier transformation.

$$\begin{aligned} H_{self} &= \sum_j \lambda Q_j Q_j Q_j Q_j = \sum_j \frac{\lambda}{4DM} (a_j + a_j^\dagger)^4 & (3) \\ &= \frac{\lambda}{4DMN^2} \sum_{j,q_1,q_2,q_3,q_4} \left(e^{iaq_1 j} a_{q_1} + e^{-iaq_1 j} a_{q_1}^\dagger\right) \left(e^{iaq_2 j} a_{q_2} + e^{-iaq_2 j} a_{q_2}^\dagger\right) \ldots \left(e^{iaq_4 j} a_{q_4} + e^{-iaq_4 j} a_{q_4}^\dagger\right) & (4) \\ &= \frac{\lambda}{4DMN^2} \sum_{j,q_1,q_2,q_3,q_4} \left(e^{iaq_1 j} a_{q_1} + e^{iaq_1 j} a_{-q_1}^\dagger\right) .. \left(e^{iaq_4 j} a_{q_4} + e^{iaq_4 j} a_{-q_4}^\dagger\right) & (5) \\ &= \frac{\lambda}{4DMN^2} \sum_{j,q_1,q_2,q_3,q_4} e^{iaj(q_1+q_2+q_3+q_4)} \left(a_{q_1} + a_{-q_1}^\dagger\right) \ldots \left(q_{q_4} + a_{-q_4}^\dagger\right) & (6) \\ &= \frac{\lambda}{4DMN} \sum_{q_1,q_2,q_3,q_4} \delta_{0,q_1+q_2+q_3+q_4} \left(a_{q_1} + a_{-q_1}^\dagger\right) \ldots \left(q_{q_4} + a_{-q_4}^\dagger\right). & (7) \end{aligned}$$

In the last line we have used the identity $\frac{1}{N} \sum_j e^{iajq} = \delta_{0,q}$ for our periodic lattice. This Kronecker-Delta corresponds to conservation of momentum.

2 One-Phonon State

2.1 Uncoupled phonon state

For uncoupled phonons on our one-dimensional grid, we want to compute the mean square displacement for a one-phonon state, i. e. $\langle j|Q_j^2|j\rangle$. First we express Q_j in terms of annihilation and creation operators. Therefore

$$\langle j|Q_j^2|j\rangle = \frac{1}{2(DM)^{1/2}} \langle j|a_j^2 + a_j a_j^\dagger + a_j^\dagger a_j + a_j^{\dagger 2}|j\rangle. \quad (8)$$

Now a_j^2 acting on $|j\rangle$ gives zero ($a_j^2|j\rangle = a_j|0\rangle = 0$), as well as $a_j^{\dagger 2}$ acting on $\langle j|$. Now we remember $\hat{n}_q = a_q^\dagger a_q$ and use the commutator $[a_q, a_q^\dagger] = 1$. Then we get

$$\langle j|Q_j^2|j\rangle = \frac{1}{2(DM)^{1/2}} \langle j|(1 + a_j^\dagger a_j) + a_q^\dagger a_q|j\rangle = \frac{3}{2(DM)^{1/2}} \quad (9)$$

2.2 Phonon state with nearest-neighbor interaction

In this case we also compute

$$\langle j|Q_j^2|j\rangle = \frac{1}{2(DM)^{1/2}}\langle j|(a_j + a_j^\dagger)^2|j\rangle \tag{10}$$

$$= \frac{1}{2(DM)^{1/2}N^2}\sum_{p_1,p_2,p_3,p_4} e^{i(p1+p2+p3-p4)j}\langle 0|A_{p1}(a_{p_2}+a^\dagger_{-p_2})(a_{p_3}+a^\dagger_{-p_3})A^\dagger_{p_4}|0\rangle, \tag{11}$$

where we have used the definition of Q_j and the Fourier expansion of the creation and annihilation operators. Now the relation between a_q and A_q ist given by

$$a_q = \alpha A_q + \beta A^\dagger_{-q} \tag{12}$$

$$\Rightarrow a^\dagger_{-q} = \alpha A^\dagger_{-q} + \beta A_q \tag{13}$$

$$\Rightarrow a_q + q^\dagger_{-q} = (\alpha+\beta)(A_q + A^\dagger_{-q}). \tag{14}$$

Inserting this into (11) and defining $\alpha_2 = \alpha_{p_2}$ and accordingly for β and p_3, terms with an unequal number of creation and annihilation operators drop out and we get

$$\begin{aligned}
\langle j|Q_j^2|j\rangle &= \frac{1}{2(DM)^{1/2}N^2}\sum_{p_1,p_2,p_3,p_4} e^{i(p1+p2+p3-p4)j}(\alpha_2+\beta_2)(\alpha_3+\beta_3)\langle 0|A_{p_1}(A_{p_2}A^\dagger_{-p_3} + A^\dagger_{-p_2}A_{p_3})A_{p_4}|0\rangle \\
&= \frac{1}{2(DM)^{1/2}N^2}\sum_{p_1,p_2,p_3,p_4} e^{i(p1+p2+p3-p4)j}(\alpha_2+\beta_2)(\alpha_3+\beta_3)\langle 0|A_{p_1}(\delta_{p_2,-p_3}A^\dagger_{p_4} + A^\dagger_{-p_3}A_{p_2}A^\dagger_{p_4} + A^\dagger_{-p_2}\delta_{p_3,p_4})|0\rangle \\
&= \frac{1}{2(DM)^{1/2}N^2}\sum_{p_1,p_2,p_3,p_4} e^{i(p1+p2+p3-p4)j}(\alpha_2+\beta_2)(\alpha_3+\beta_3)\langle 0|\delta_{p_2,-p_3}\delta_{p_1,p_4} + \delta_{p_1,-p_3}\delta_{p_2,p_4} + \delta_{p_1,-p_2}\delta_{p_3,p_4} \quad (15) \\
&= \frac{1}{2(DM)^{1/2}N^2}\left(\sum_{p_1,p_2}(\alpha_2+\beta_2)^2 + \sum_{p_1,p_2}(\alpha_1+\beta_1)(\alpha_2+\beta_2) + \sum_{p_1,p_3}(\alpha_1+\beta_1)(\alpha_3+\beta_3)\right) \\
&= \frac{1}{2(DM)^{1/2}N^2}\sum_{p_1,p_2}\left((\alpha_2+\beta_2)^2 + 2(\alpha_1+\beta_1)(\alpha_2+\beta_2)\right). \quad (16)
\end{aligned}$$