A geometric framework to compare classical field theories

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Requirements for framework

Framework should be able to answer:

- Are two (classical field) theories equivalent?
- Can one theory be embedded into another one?
- Do two theories share any subtheory?

Preferably, the Framework has a category-theoretical arena.

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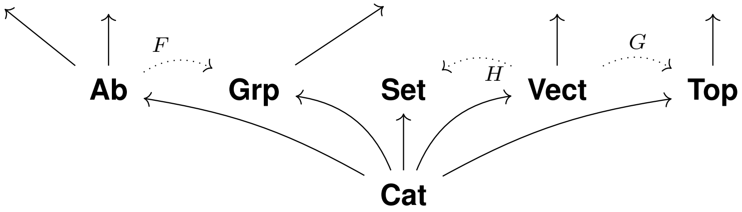
Category theory

- can serve to relate mathematical theories via functors,
- analysis of relations without specification of objects
  $\rightarrow$ comparison of relationship structures,
- very general with relations to all areas of mathematics



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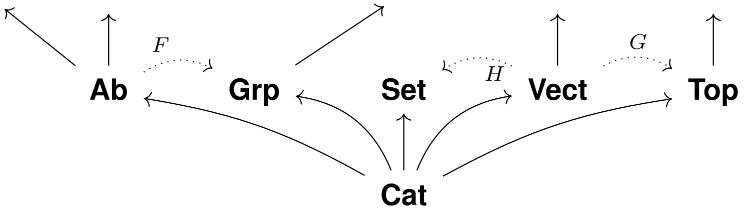

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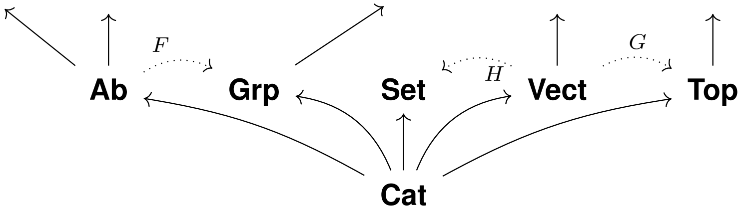

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- Interpretation
- Initial / Boundary Conditions
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- Would like to investigate everything in one category

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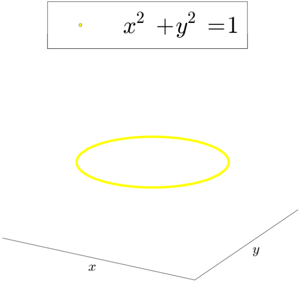
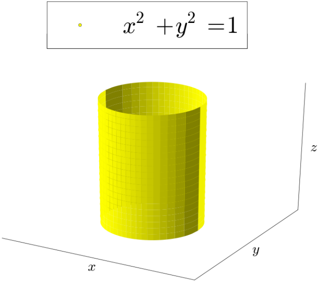
- Equations cut manifolds out of an embedding space.





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Equations cut out spaces

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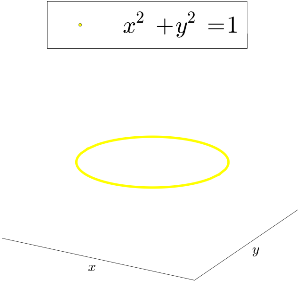
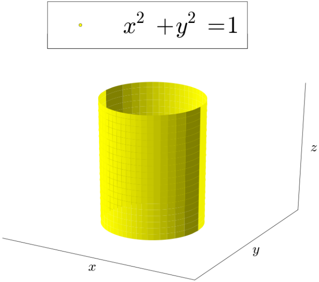
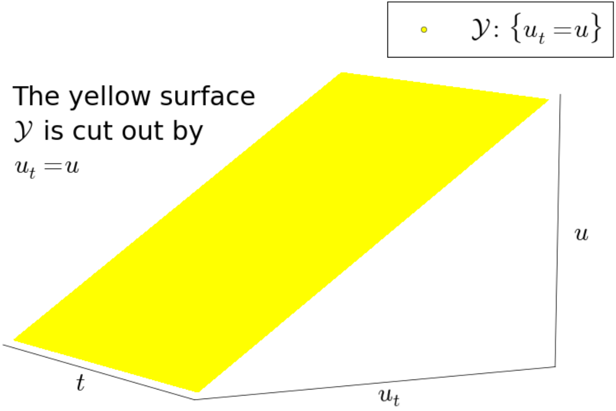




- How do differential equations cut out manifolds?

Simple example

- Consider PDE
$$u_t = u \tag{1}$$
with $u_t = du/dt$ and solution $u(t) = A \cdot \exp(t)$
- Introduce **new coordinate** u_t to plot solution surface $\mathcal{Y}$.



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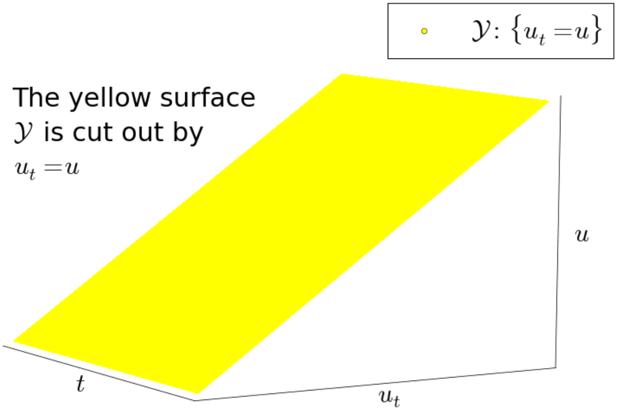

Simple example

- Formally introduce $\pi : E \to M$, $E := \mathbb{R} \times \mathbb{R}$, $M := \mathbb{R}$. $J^1(E) := E \times \mathbb{R}$ is the *Jet Space* with coordinates (t, u, u_t).
- With $F := \mathbb{R} \times \mathbb{R}$, define bundle $\xi : F \to M$. Then our PDE is *the kernel of the operator* $\Phi : J^1(E) \to F$ defined by

$$\Phi(t, u, u_t) := (t, u_t - u) \Rightarrow \ker \Phi = \{\theta \in J^1(E) \mid u_t = u\} \quad (2)$$

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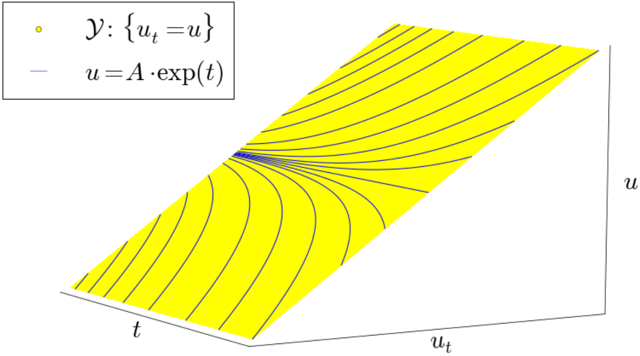
- Solutions $u(t) = A \cdot \exp(t)$ **foliate** solution surface $\mathcal{Y}$.
- But not all sections $S : M \to J^1(E)$ of $\pi^1 : J^1(E) \to M$,

$$S(t) = (t, u(t), u_t(t)) \tag{3}$$

are solutions, even if $u(t) = u_t(t)$.



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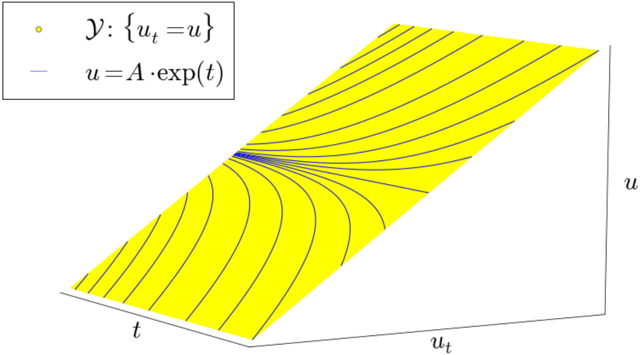

Simple example

- Let $s : M \to E$ be section of $\pi : E \to M$, i.e. $s(t) = (t, u(t))$
- Let $j^1(s) : M \to J^1(E)$ be the *prolongation*

$$j^1(s)(t) := (t, u(t), du/dt(t)) \tag{4}$$

- Those sections $S : M \to J^1(E)$ which are of form $j^1(s)$ and lie in solution surface $\mathcal{Y} = \ker \Phi$ are *solutions* of the PDE.

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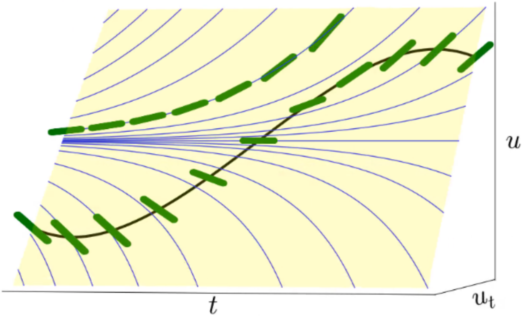
- One can introduce *distribution* $\mathcal{C}$ to single out solutions
- Consists of tangent lines to all sections for which $u_t(t) = du/dt(t)$.



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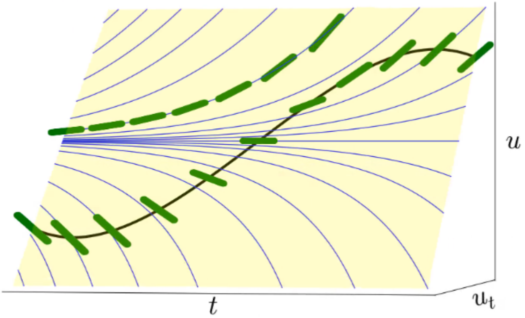

Therefore, $\mathcal{Y}$ and $\mathcal{C}$ suffice to geometrise the PDE.

Simple example

To summarise, we looked at

- Bundle $\pi : E \to M$ with $E := \mathbb{R} \times \mathbb{R}$, $M := \mathbb{R}$. M and E have coordinates (t) and (t, u).
- Jet Space $J^1(E) \simeq \mathbb{R}^3$ with coordinates (t, u, u_t)
- With $F := \mathbb{R} \times \mathbb{R}$, define bundle $\xi : F \to M$. Then our PDE is *the kernel of the operator* $\Phi : J^1(E) \to F$ defined by

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- Prolong $s : M \to E$ to $j^1(s)(t) := (t, u(t), du/dt(t))$. Those sections $S : M \to J^1(E)$ which are of form $j^1(s)$ and lie in $\mathcal{Y} = \ker \Phi$ are *solutions* of PDE.

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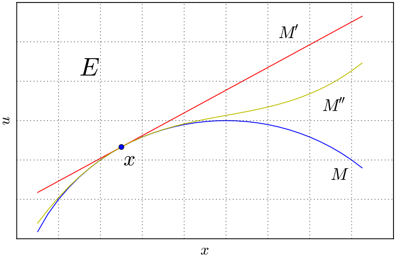
In general

Definition
Let E be an $n+m$-dimensional manifold. Two m-dimensional submanifolds M, M' of E are said to have the same k-th order *Jet* $[M]_a^k$ at $a \in M \cap M' \subset E$ if they are tangent to one another up to order k.



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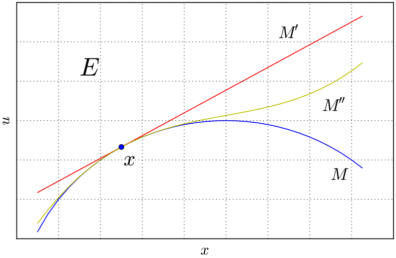

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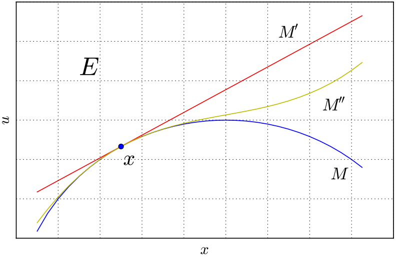

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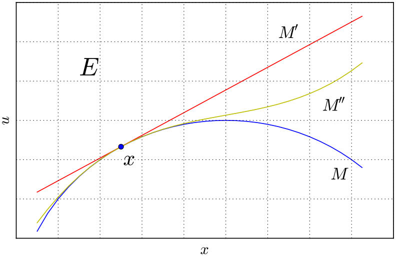

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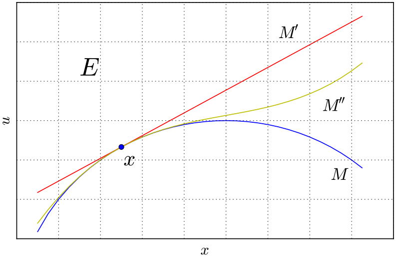

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Definition

If $\pi : E \to M$ and $\xi : F \to M$ are fibered manifolds, then a morphism of fibered manifolds $\Phi : J \subset J^k(E) \to F$ is a *differential operator*.

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Motivating example

1. Magneto-statics:

$$\nabla \times \mathbf{B} = \mathbf{j}, \qquad \nabla \cdot \mathbf{B} = 0. \tag{7}$$

2. Viscous, incompressible Navier-Stokes equations

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \mathbf{u} = -\nabla \left(\frac{p}{\rho}\right) + \nu \Delta \mathbf{u}, \qquad \nabla \cdot \mathbf{u} = 0. \tag{8}$$

If we assume

$$\mathbf{j} = -\nabla \psi,\ 0 = d\mathbf{u}/dt = \partial u/\partial t + (\mathbf{u} \cdot \nabla)\mathbf{u} \tag{9}$$

Use assumptions and vector identities to obtain

$$\begin{gathered} \nabla \times \mathbf{B} = -\nabla \psi, \qquad \nabla \cdot \mathbf{B} = 0 \\ \nabla \times (\nabla \times \mathbf{u}) = -\nabla \phi, \qquad \nabla \cdot (\nabla \times \mathbf{u}) = 0, \qquad \nabla \cdot \mathbf{u} = 0. \end{gathered} \tag{10}$$

where $\phi := p/(\rho\nu)$. Similar under the "correspondence"

$$\mathbf{B} \to \nabla \times \mathbf{u}. \tag{11}$$

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- Let $\pi : E \to M$ and $\xi : F \to M$ be bundles. Compare PDEs

$$\mathcal{E} = \ker \Phi_E \subset J^k(E), \qquad \mathcal{F} = \ker \Phi_F \subset J^l(F) \tag{12}$$

- *Correspondence* is diff operator $\varphi : J \subset J^n(E) \to F$.

$$\begin{array}{ccccccc}
J^k(E) & & J^n(E) & \xrightarrow{\varphi} & F & \xleftarrow[\xi_0^k]{} & J^l(F) \\
\Phi_E \big\downarrow & \overset{\pi^k}{\searrow} & \big\downarrow \pi^n & \overset{\xi}{\swarrow} & & & \big\downarrow \Phi_F \\
E' & \xrightarrow{\pi'} & M & & \xleftarrow{\xi'} & & F'
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Formal definition

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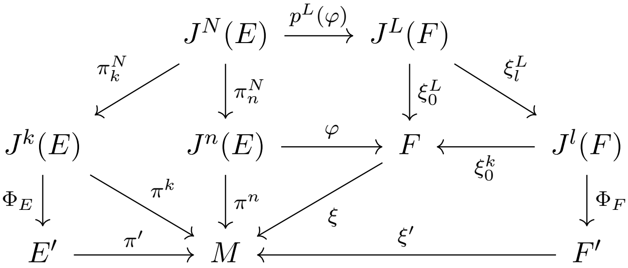
Formal definition

- Define $N := \max(k, n+l)$. Prolong φ to $p^L(\varphi)$, $L = N - n$.



(14)

- Define preimage and *intersection*

$$\boxed{\varphi_L^* \mathcal{F} := p^L(\varphi)^{-1}(\mathcal{F}^{L-l})} = \ker\left(p^{L-l}(\Phi_F) \circ p^L(\varphi)\right)$$
$$\boxed{\mathcal{I} := \mathcal{E}^K \cap \varphi_L^* \mathcal{F}} = \ker\left(p^K(\Phi_E) \vee p^{L-l}(\Phi_F) \circ p^L(\varphi)\right) \quad (15)$$

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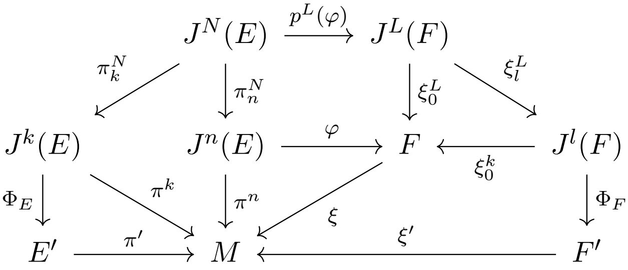

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Solution transfer

Theorem

Suppose we found a solution of an intersection $\mathcal{I} = \mathcal{E}^K \cap \varphi_L^* \mathcal{F}$*, that means a local section* $s : O \subset M \to E$ *such that* $j^N(s)(O) \subset \mathcal{I}$*. Then* s *is also solution of* $\mathcal{E}$ *and*

$$s' := \varphi \circ j^n(s) : O \to F \qquad (16)$$

is a solution of $\mathcal{F}$*.*

The solution is *transferred* from $\mathcal{I}$ to $\mathcal{F}$ via φ.

Contents

Illustrating example

- $\pi : E \to M$, $\xi : F \to M$, $E := \mathbb{R} \times \mathbb{R}$, $M := \mathbb{R}$, $F := E$.
- $\mathcal{E} = \{\theta \in J^2(E) \mid u_{tt} = u\}$, $\mathcal{F} = \{\theta \in J^3(F) \mid u_{ttt} = 4\}$.
- Solutions are: $u_1(t) = A\exp(t) + B\exp(-t)$ and $u_2(t) = 2/3t^3 + A/2t^2 + Bt + C$.
- Define correspondence $\varphi : J^0(E) \to F$ as identity.
- Prolong $\mathcal{E}$ to $\mathcal{E}^1 = \{\theta \in J^3(E) \mid u_{tt} = u,\ u_{ttt} = u_t\}$.
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Illustrating example

- $\mathcal{I} = \{(t, u, 4, u, 4) \mid t, u \in \mathbb{R}\}$.
- Prolongation:

$$\mathcal{I}^1 = \left\{ \theta \in J^4(E) \;\middle|\; \begin{pmatrix} u_{tt} = u, \; u_{ttt} = u_t, \; u_{ttt} = 4 \\ u_{ttt} = u_t, \; u_{tttt} = u_{tt}, \; u_{tttt} = 0 \end{pmatrix} \right\}$$

$\mathcal{I}^1 = \{(t, 0, 4, 0, 4, 0), \mid t \in \mathbb{R}\} \Rightarrow \pi^4_3 : \mathcal{I}^1 \to \mathcal{I}$ not surjective.

- 2nd prolongation yields contradiction:

$$\mathcal{I}^2 = \left\{ \theta \in J^5(E) \;\middle|\; \begin{pmatrix} u_{tt} = u, \; u_{ttt} = u_t, \; u_{ttt} = 4 \\ u_{ttt} = u_t, \; u_{tttt} = u_{tt}, \; u_{tttt} = 0 \\ u_{tttt} = u_{tt}, \; u_{ttttt} = u_{ttt}, \; u_{ttttt} = 0 \end{pmatrix} \right\}$$

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Formal integrability of intersection

Definition

A differential equation R^k is said to be *formally integrable* if

1. $\pi^{k+l+1}_{k+l} : R^{k+l+1} \to R^{k+l}$ is surjective,
2. g^{k+l+1} is a vector bundle

for all $l \in \{0, 1, 2, \cdots\}$.

Theorem

If R^k is a differential equation, then it is formally integrable if

1. *$\pi^{k+1}_{k} : R^{k+1} \to R^{k}$ is surjective,*
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Shared Structure

Definition

Two differential equations *share structure* if they share an intersection $\mathcal{I}$ and this intersection is formally integrable at least for all points on an open subset of $\mathcal{I}$.

Contents

Intersection of Magneto-Statics and Hydrodynamics

Rewrite

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\mathbf{u} = -\nabla\left(\frac{p}{\rho}\right) + \nu\Delta\mathbf{u}, \qquad \nabla\cdot\mathbf{u} = 0.$$

$$\nabla\times\mathbf{B} = \mathbf{I}, \qquad \nabla\cdot\mathbf{B} = 0$$

as kernel of

$$\Phi_E(\theta) := \begin{pmatrix} u_t^i + u^j u^{i,j} + \frac{1}{\rho} p^{\ ,i} - \nu u^{i,jj} \\ u^{i,i} \end{pmatrix} \tag{17}$$

$$\Phi_F(\theta) := \begin{pmatrix} \varepsilon_{ijk} B^{k,j} - I^i \\ B^{i,i} \end{pmatrix} \tag{18}$$

Intersection of Magneto-Statics and Hydrodynamics

- Curl operator as correspondence

$$\varphi(x^i, t, u^i, u^{i,j}, u^i_t) := (x^i, t, \varepsilon_{ijk} u^{k,j}). \tag{19}$$

- For simplicity, set $\nu I^i = -p^{\,,i}/\rho$. Intersection is

$$\mathcal{I}^2 = \ker \begin{pmatrix} \Phi_E \\ \varphi^* \Phi_F \end{pmatrix} = \ker \begin{pmatrix} u^i_t + u^j u^{i,j} - \nu u^{j,ij} \\ u^{i,i} \\ u^{j,ij} - u^{i,jj} - I^i \end{pmatrix}$$

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Intersection of Magneto-Statics and Hydrodynamics

First prolongation

$$\mathcal{I}^3 = \ker \begin{pmatrix} u_t^i + u^j u^{i,j} - \nu u^{j,ij} & \begin{array}{c} u_t^{i,k} + u^{j,k} u^{i,j} + u^j u^{i,jk} - \nu u^{j,ijk} \\ u_{tt}^i + u_t^j u^{i,j} + u^j u_t^{i,j} - \nu u_t^{j,ij} \end{array} \\ u^{i,i} & \begin{array}{c} u^{i,ik} \\ u_t^{i,i} \end{array} \\ u^{j,ij} - u^{i,jj} - I^i & \begin{array}{c} u^{j,ijk} - u^{i,jjk} - I^{i,k} \\ u_t^{j,ij} - u_t^{i,jj} - I_t^i \end{array} \end{pmatrix}$$

$\Rightarrow$ Integrability conditions

$$u^{i,ik} = u^{j,kj} = 0 \;\Rightarrow\; du/dt = 0 \qquad (20)$$

They are minimal physical assumptions needed for consistency.

Intersection of Magneto-Statics and Hydrodynamics

Define new system with those conditions

$$\Rightarrow \mathcal{J}^2 = \ker \begin{pmatrix} & u^{j,j}, \quad u_t^{j,j}, \quad u^{j,ji} & \\ u_t^i + u^j u^{i,j}, & (u_t^i + u^j u^{i,j})^{,k}, & (u_t^i + u^j u^{i,j})_t \\ & u^{i,jj} + I^{i,k} & \end{pmatrix}$$

One can show: This system is formally integrable.

It can be understood as subtheory of the intersected theories.

Contents

Conclusion

One can now answer the questions

1. Are two (classical field) theories equivalent?
2. Can one theory be embedded into another one?
3. Do two theories share any subtheory?

in the category of smooth manifolds.

With the notion of shared structure,

- Analogies of similar systems can be found and analysed.
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- Find method to find the best of all correspondences

- Use homotopy theory to describe the deformation of solution spaces of theories. Such deformations may correspond to approximations of equations and therefore homotopy theory might be a suitable mathematical language to talk about the transition of physical theories.

- Find geometric formulation of functional differential equations $\rightarrow$ application to QFT.

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Reference overview

Category Theory: Eilenberg and Mac Lane, Leinster, Geroch

Jet Spaces: Cartan, Ehresmann, Saunders

Geometric Theory of PDEs: Vinogadov, Krasilshchik, Vitagliano

Formal Integrability: Goldschmidt, Bryant, Pommaret, Seiler