Time energy entropic uncertainty relations: an algebraic approach

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Structure of the talk

- Introduction: uncertainty relations
- “Standard” entropic uncertainty relations and their operational motivations
- An operational definition for time-energy entropic uncertainty relations
- Results
- A general method to prove entropic uncertainty relations
- Conclusions

Uncertainty relations

Heisenberg's uncertainty principle[1]

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

Robertson's relation[2]

$$\sigma_V \sigma_W \geq \left| \frac{1}{2i} \langle [\hat{V}, \hat{W}] \rangle \right|$$

[1] E. H. Kennard. *Zur quantenmechanik einfacher bewegungstypen*. Zeitschrift für Physik, 44(4):326– 352, Apr 1927.
[2] H.P. Robertson. *The uncertainty principle*. Phys. rev. 34:163-164, Jul 1929.

Definitions and notation

Relative entropy: state ρ and positive σ

$$D(\rho||\sigma) = \begin{cases} \mathrm{Tr}(\rho(\log\rho - \log\sigma)) & \text{if } \mathrm{supp}\rho \subseteq \mathrm{supp}\sigma \\ \infty & \text{otherwise} \end{cases}$$

ρ_{AB} state on Hilbert space $\mathcal{H}_{AB}$:

Von Neumann entropy of state ρ on system A:

$$S(A)_\rho = -D(\rho_A||\mathbb{1}) = -\mathrm{Tr}(\rho_A \log \rho_A)$$

Conditional entropy:

$$S(A|B)_\rho = -D(\rho_{AB}||\mathbb{1}_A \otimes \rho_B) = S(AB)_\rho - S(B)_\rho$$

Guessing game

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   - A state ρ_A on Alice's Hilbert space.
   - Two observables V and W both acting on Alice's Hilbert space.
2. Alice randomly decides between measuring V or W, and communicates her choice to Bob.
3. Bob guesses the measurement result.

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At step 3, Bob effectively has a probability distribution from which he knows that Alice has sampled, for example if V is measured

$$\mathcal{M}_V(\rho) = \sum_k \langle v_k|\rho|v_k\rangle |v_k\rangle\langle v_k|$$

→ Eigenvectors of V

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Eigenvectors of V

The entropy of this distribution represents how well is Bob able to guess

Maassen-Uffink relation

$$S(A)_{\mathcal{M}_V(\rho)} + S(A)_{\mathcal{M}_W(\rho)} \geq \log \frac{1}{c}$$

$$c = \max_{i,j} |\langle v_i | w_j \rangle|^2$$

Eigenvectors of V and W

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If Bob is good at guessing the result when one observable is measured, he is bad at guessing when the other is measured.

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Bipartite guessing game

Bob can at best randomly guess by choosing the most likely outcome!

Change the game to give Bob a “quantum” help to win the game

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2. Alice randomly decides between measuring V or W, and communicates her choice to Bob.
3. Bob guesses the measurement result using ρ_B.

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$$S(A|B)_{\mathcal{M}_V(\rho)} + S(A|B)_{\mathcal{M}_W(\rho)} \geq \log \frac{1}{c} + S(A|B)_\rho$$

How bad is Bob at guessing if V is measured, given ρ_B

How bad is Bob at guessing if W is measured, given ρ_B

$S(A|B)_\rho$ can be negative for entangled states!

There are cases where the RHS and the LHS vanish simultaneously for noncommuting observables.

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Can Bob guess both outcomes simultaneously? No.

To make sure, define a tripartite game

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2. Alice randomly decides between measuring V or W and communicates her choice to
   - Bob 1 if V is chosen
   - Bob 2 if W is chosen.
3. The player who received the choice information guesses the corresponding result.

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How bad is Bob 2 at guessing if W is measured, given ρ_{B_2}

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The negative term disappeared

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Summary until now

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One adversary, with quantum memory [Berta et al., 2010]:

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Two adversaries, with quantum memory [Berta et al., 2010]:

$$S(A|B_1)_{\mathcal{M}_V(\rho)} + S(A|B_2)_{\mathcal{M}_W(\rho)} \geq \log \frac{1}{c}$$

Time-energy EUR

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It is difficult to find a well defined operator corresponding to time in quantum mechanics[5]

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Time-energy EUR

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Idea[6]:

Measuring the time of a quantum system	$\longrightarrow$	Estimating how much time has passed since some reference time using the system

[5] W. Pauli. *Die allgemeinen Prinzipien der Wellenmechanik*, pages 21–192. Springer Berlin Heidel- berg, Berlin, Heidelberg, 1990.
[6] P. J. Coles, V. Katariya, S. Lloyd, I. Marvian, and M. M. Wilde. *Entropic energy-time uncertainty relation*. Phys. Rev. Lett., 122:100401, Mar 2019.

Time-energy EUR

Guessing game between[6]:

- Performing an energy measurement and guessing the result
- Waiting a random amount of time and guessing the amount

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$$\downarrow$$

$$G \leftrightarrow e^{-iGr}$$

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   - A bipartite state ρ_{AB}
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   - A finite list of real numbers $\{r_k\}_{k=1}^{|R|} \subset \mathbb{R}$ called angles
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2. Alice randomly decides between
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3. Alice communicates her choice to Bob, then
   - If Alice measured G, he must guess the measurement result
   - If Alice performed the rotation, she sends ρ_A to Bob, who must guess r_k

Bipartite guessing game

How bad is Bob at guessing the measurement result:

Bipartite guessing game

How bad is Bob at guessing the measurement result:

$$S(A|B)_\omega, \quad \omega = \sum_{k=1}^{|A|} \langle g_k|\rho_{AB}|g_k\rangle |g_k\rangle\langle g_k|_A$$

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How bad is Bob at guessing the rotation angle:

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How bad is Bob at guessing the rotation angle:

$$S(R|AB)_\kappa$$

Bipartite guessing game

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↑

Classical register
storing the angle

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How bad is Bob at guessing the rotation angle:

$$S(R|AB)_\kappa, \quad \kappa = \sum_{j=1}^{|R|} p_j |r_j\rangle\langle r_j|_R \otimes e^{-iG_A r_j} \rho_{AB} e^{iG_A r_j}$$

↑ Classical register storing the angle

With probability p_j, the register contains r_j and the system is rotated by r_j

Bound for bipartite game

$$S(R|AB)_\kappa + S(A|B)_\omega \geq$$

Bound for bipartite game

$$S(R|AB)_\kappa + S(A|B)_\omega \geq S(R)_\kappa + D(\kappa_A||\omega_A) + S(A|B)_\rho$$

- $S(R)_\kappa$: Entropy of $\{p_k\}_{k=1}^{|R|}$
- $D(\kappa_A||\omega_A)$: Positive term
- $S(A|B)_\rho$: May be negative

Again, Bob can have an advantage using an entangled state.

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   - A probability distribution on the angles $\{p_k\}_{k=1}^{|R|}$.
2. Alice randomly decides between
   - Measuring G.
   - Picking r_k with probability p_k and applying e^{-iGr_k}.
3. Alice announces her choice, then
   - If Alice performed the rotation, she sends ρ_A to Bob 1, who must guess r_k.
   - If Alice measured G, Bob 2 must guess the measurement result.

Bound for tripartite game

$$S(R|AB_1)_\kappa + S(A|B_2)_\omega \geq$$

Bound for tripartite game

$$S(R|AB_1)_\kappa + S(A|B_2)_\omega \geq S(R)_\kappa + D(\kappa_{AB_1}||\omega_{AB_1})$$

Bound for tripartite game

$$S(R|AB_1)_\kappa + S(A|B_2)_\omega \geq S(R)_\kappa + D(\kappa_{AB_1}||\omega_{AB_1})$$
$$+ \max(0, I(A:B_1)_\omega - I(B_1:B_2)_\rho + S(A|B)_\rho)$$

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The negative term disappeared

Summary

Alice either measures an observables or rotates the system using it as a generator, how well can adversaries guess either the measurement result or the angle of rotation?

One adversary:

$$S(R|AB)_\kappa + S(A|B)_\omega \geq S(R)_\kappa + D(\kappa_A||\omega_A) + S(A|B)_\rho$$

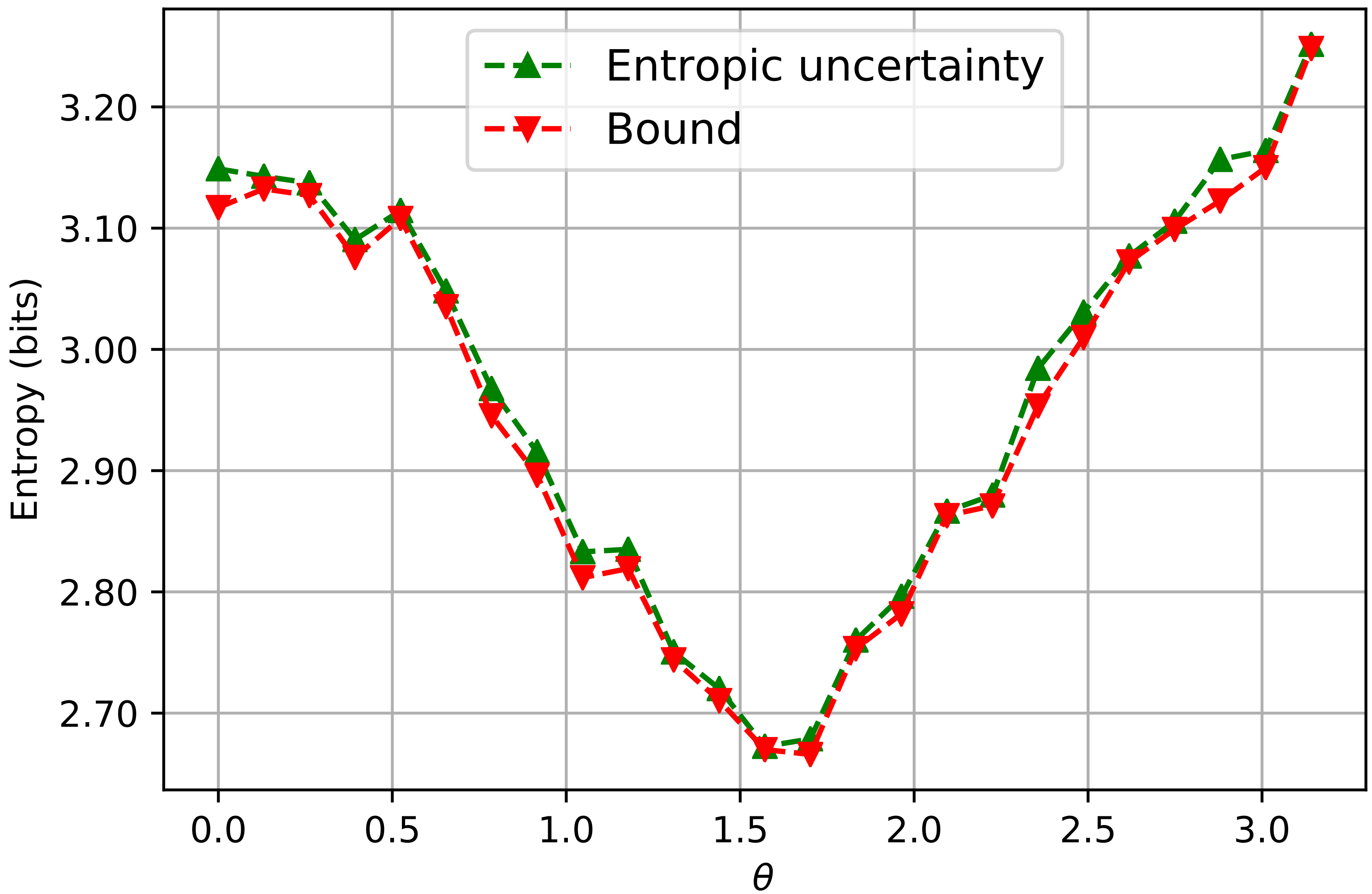
Two adversaries:

$$S(R|AB_1)_\kappa + S(A|B_2)_\omega \geq S(R)_\kappa + D(\kappa_{AB_1}||\omega_{AB_1})$$
$$+ \max(0, I(A:B_1)_\omega - I(B_1:B_2)_\rho + S(A|B)_\rho)$$

Performance of bound for the bipartite game:

$$\rho = \mathcal{N}(|\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi|), |\psi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$$

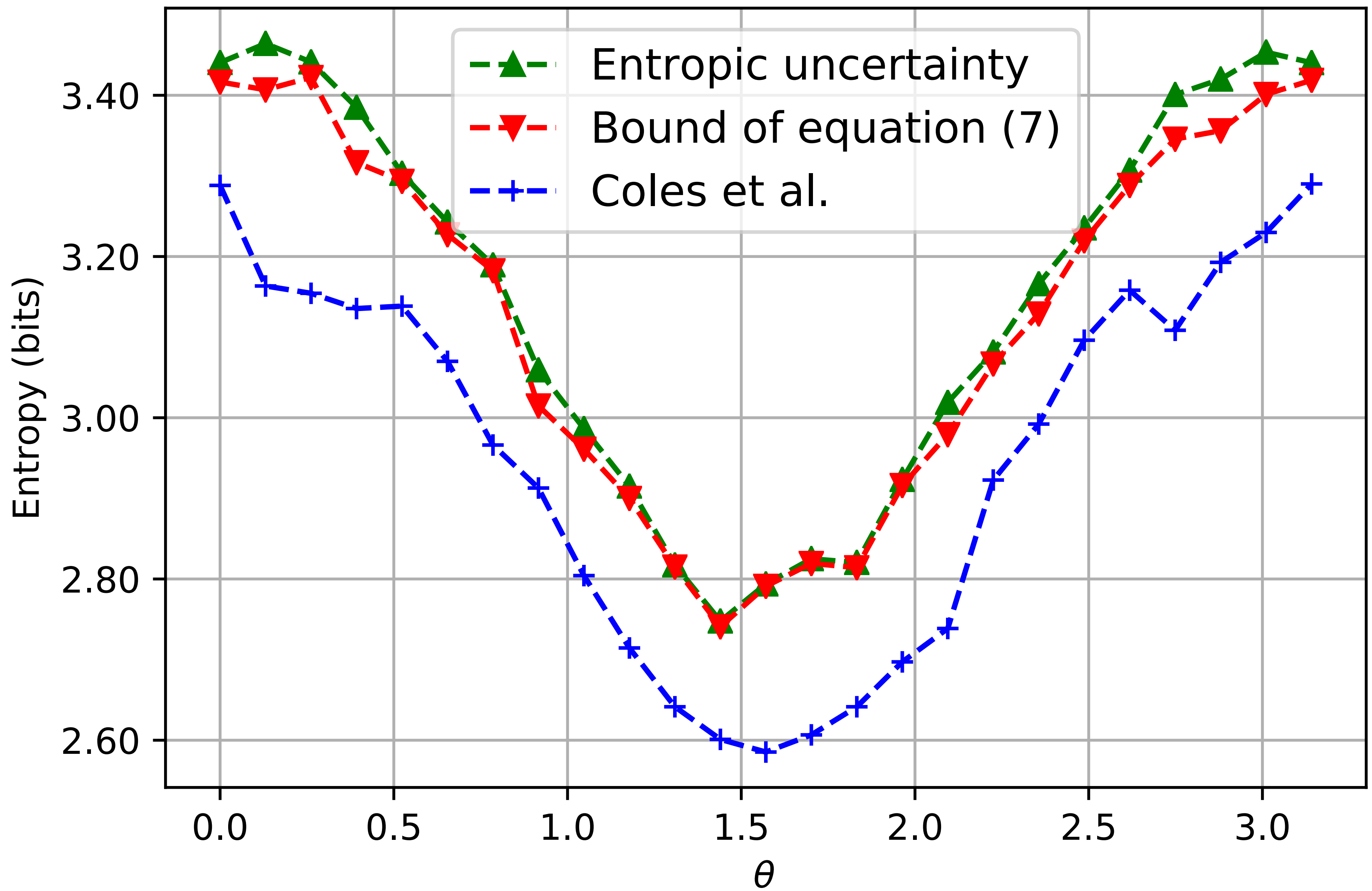
$$S(R|AB)_\kappa + S(A|B)_\omega \geq S(R)_\kappa + D(\kappa_A||\omega_A) + S(A|B)_\rho \,.$$



Comparison of bounds for the tripartite game: B_1 trivial

Coles et al. $S(R|A)_\kappa + S(A|B)_\omega \geq S(R)_\kappa + D(\kappa_A||\omega_A)$

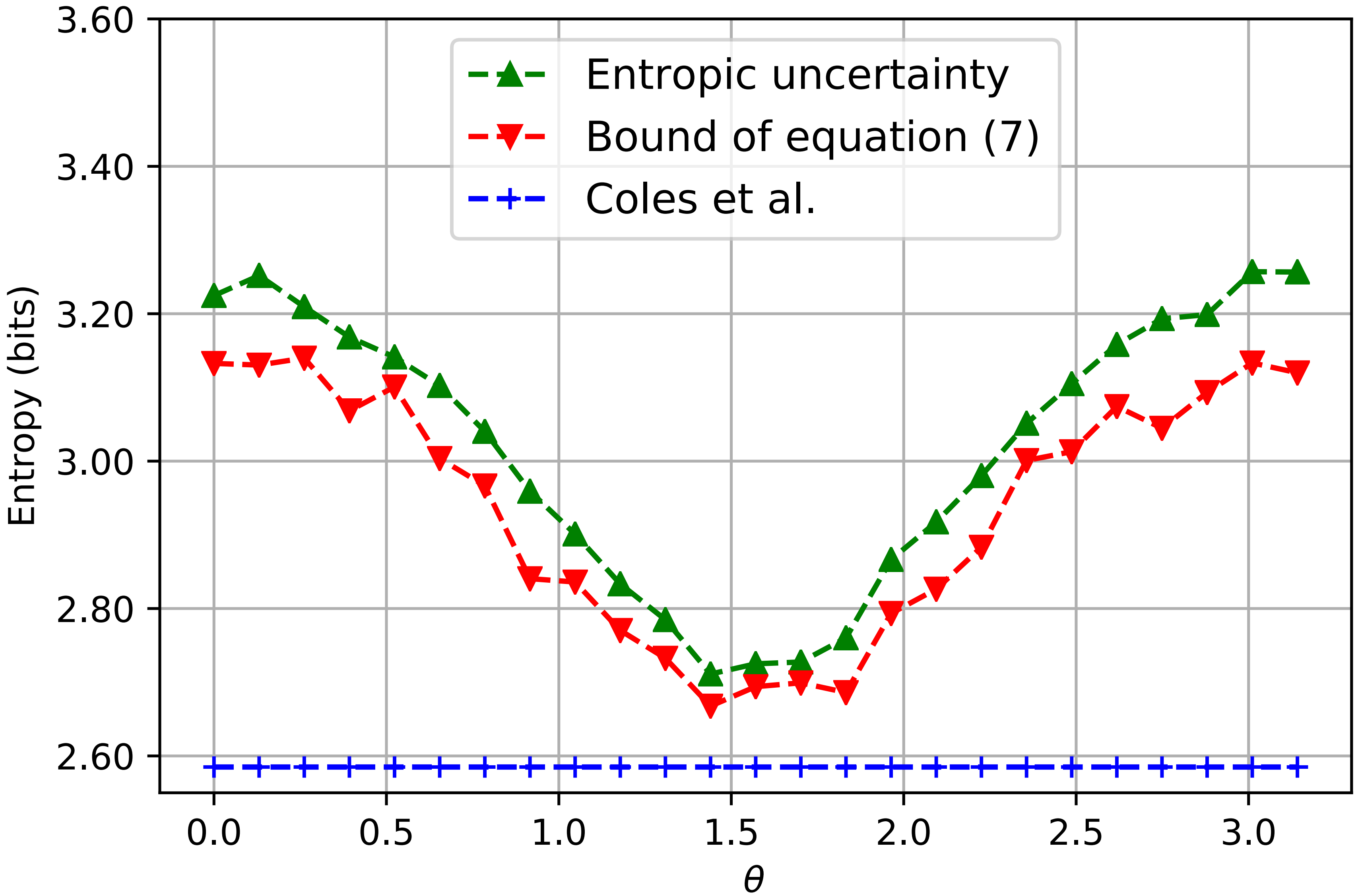
Eq. (7): $S(R|A)_\kappa + S(A|B)_\omega \geq S(R)_\kappa + D(\kappa_A||\omega_A) + \max\{0, S(A|B)_\rho\}.$



Comparison of bounds for the tripartite game: B_1 non trivial

Coles et al. $S(R|AB_1)_\kappa + S(A|B_2)_\omega \geq \log|R|$

Eq. (7): $S(R|AB_1)_\kappa + S(A|B_2)_\omega \geq S(R)_\kappa + D(\kappa_{AB_1}||\omega_{AB_1})$
$+ \max\{0, I(A:B_1)_\omega - I(B_1:B_2)_\rho + S(A|B_1B_2)_\rho\}\,.$

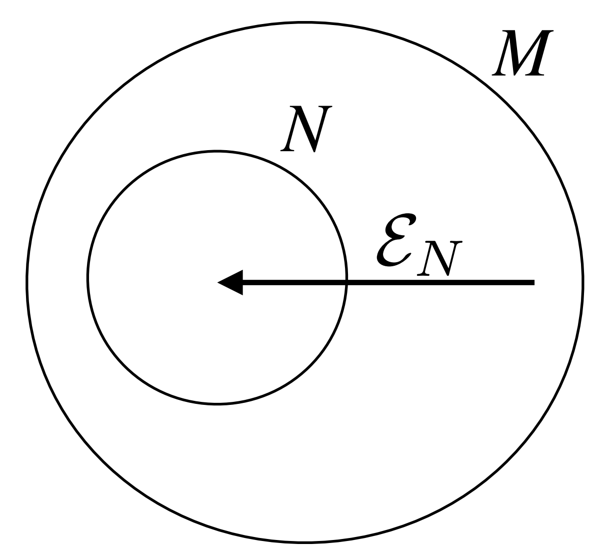


Proving entropic uncertainty relations

Definition: let M be a Von Neumann algebra and let $N \subset M$ be a subalgebra.

The **conditional expectation** onto N is the unique CPTP unital map $\mathcal{E}_N : M \to N$ such that for all $\rho \in M, \sigma \in N$

$$\mathrm{Tr}(\sigma \mathcal{E}_N(\rho)) = \mathrm{Tr}(\sigma\rho)$$

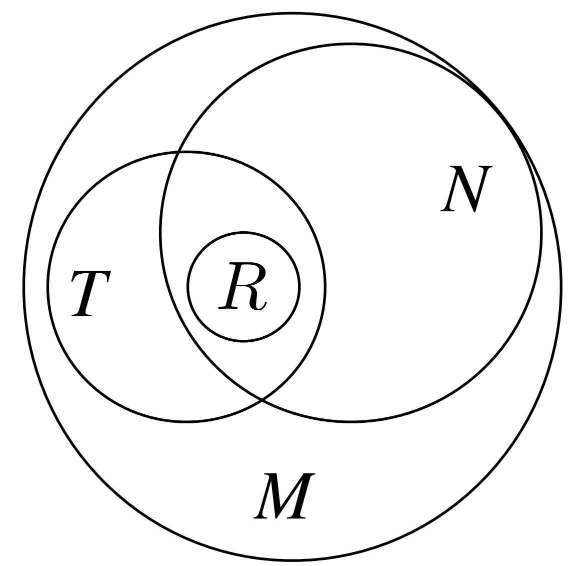


[7] L. Gao, M. Junge, and N. Laracuente. *Strong subadditivity inequality and entropic uncertainty relations*. arXiv:1710.10038v2 [quant-ph], 2017.

Proving entropic uncertainty relations

Definition: a set of four Von Neumann algebras satisfying the inclusions

$$\begin{pmatrix} N & \subset & M \\ \cup & & \cup \\ R & \subset & T \end{pmatrix}$$



is called a **commuting square** if the conditional expectations satisfy

$$\mathcal{E}_N \circ \mathcal{E}_T = \mathcal{E}_T \circ \mathcal{E}_N = \mathcal{E}_R.$$

[7] L. Gao, M. Junge, and N. Laracuente. *Strong subadditivity inequality and entropic uncertainty relations*. arXiv:1710.10038v2 [quant-ph], 2017.

Proving entropic uncertainty relations

Theorem: if

$$\begin{pmatrix} N & \subset & M \\ \cup & & \cup \\ R & \subset & T \end{pmatrix}$$

is a commuting square then the following relation holds

$$S(N)_{\mathcal{E}_N(\rho)} + S(T)_{\mathcal{E}_T(\rho)} \geq S(M)_\rho + S(R)_{\mathcal{E}_R(\rho)}$$

[7] L. Gao, M. Junge, and N. Laracuente. *Strong subadditivity inequality and entropic uncertainty relations*. arXiv:1710.10038v2 [quant-ph], 2017.

Conclusions

- The proof method presented encompasses a large class of entropic uncertainty relations, including “non conventional” ones such as time-energy.
- Even with an operational definition that is quite different from the standard one, the uncertainty relations still follow the standard structure: allowing for the existence of two guesses at once makes the game impossible to win.
- To do: extend the proof to Rényi entropies.

References

[1] E. H. Kennard. *Zur quantenmechanik einfacher bewegungstypen*. Zeitschrift für Physik, 44(4):326– 352, Apr 1927.

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