Real-time physics from Dyson-Schwinger Equations via spectral renormalisation

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Outline

I. Methodology

- Motivation & technical introduction (DSE, Källen-Lehmann representation)
- **Spectral renormalisation** (dimensional & BPHZ)

II. Physics

- **Scalar ϕ^4-theory** in d=2+1
- **Yang-Mills theory** in d=3+1

Real-time QFT

- **Non-perturbative** approaches to QFT based on **Euclidean** formulation
  - Functional methods (FRG, DSE), lattice theory
- Resolving **dynamics**: real-time quantities needed
  - e.g. non-equilibrium phenomena, bound states, confinement
- Map from $\mathbb{R}^4$ to Minkowski space is non-trivial

Algebraic access to momentum structure?

no → **Numerical reconstruction**
e.g. Bayesian reconstruction, Padé

yes → **Analytic continuation**
via Wick rotation

Dyson-Schwinger Equations

- **Quantum equations of motion** of the theories Greens functions

$$\frac{\delta\Gamma[\phi]}{\delta\phi} = \frac{\delta S}{\delta\varphi}\left[\varphi = G\cdot\frac{\delta}{\delta\phi} + \phi\right]$$

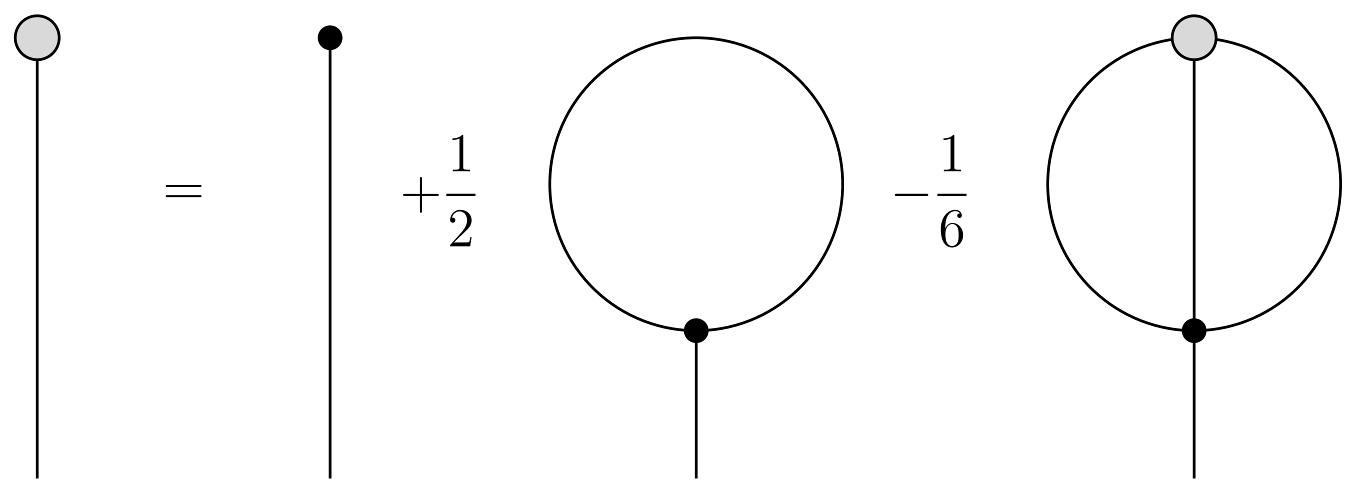
Master DSE

- Encode **shift symmetry** of path integral measure

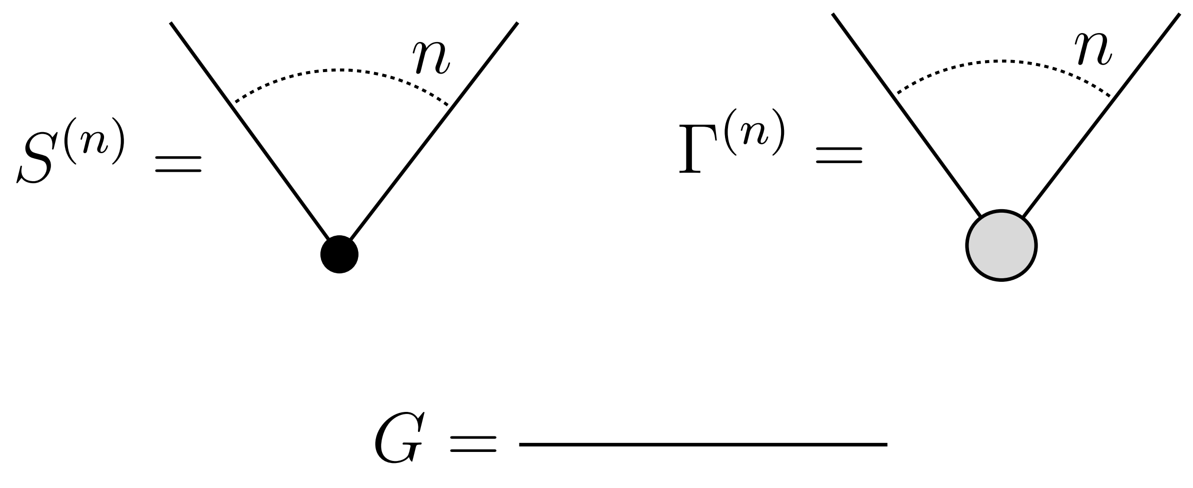
$$\int D\phi\; e^{-S[\phi]} \quad \text{invariant under} \quad \phi(x) \to \phi(x) + \Lambda(x)$$

- Generate higher correlations by acting with $\left(\frac{\delta}{\delta\phi}\right)^n$ on master DSE

Example: Scalar theory



Master DSE with



Källén-Lehmann spectral representation

- Källén-Lehmann **spectral representation** of propagator (vacuum)

$$G(p) = \int_0^\infty \frac{d\lambda}{\pi} \frac{\lambda\,\rho(\lambda)}{p^2 + \lambda^2}$$

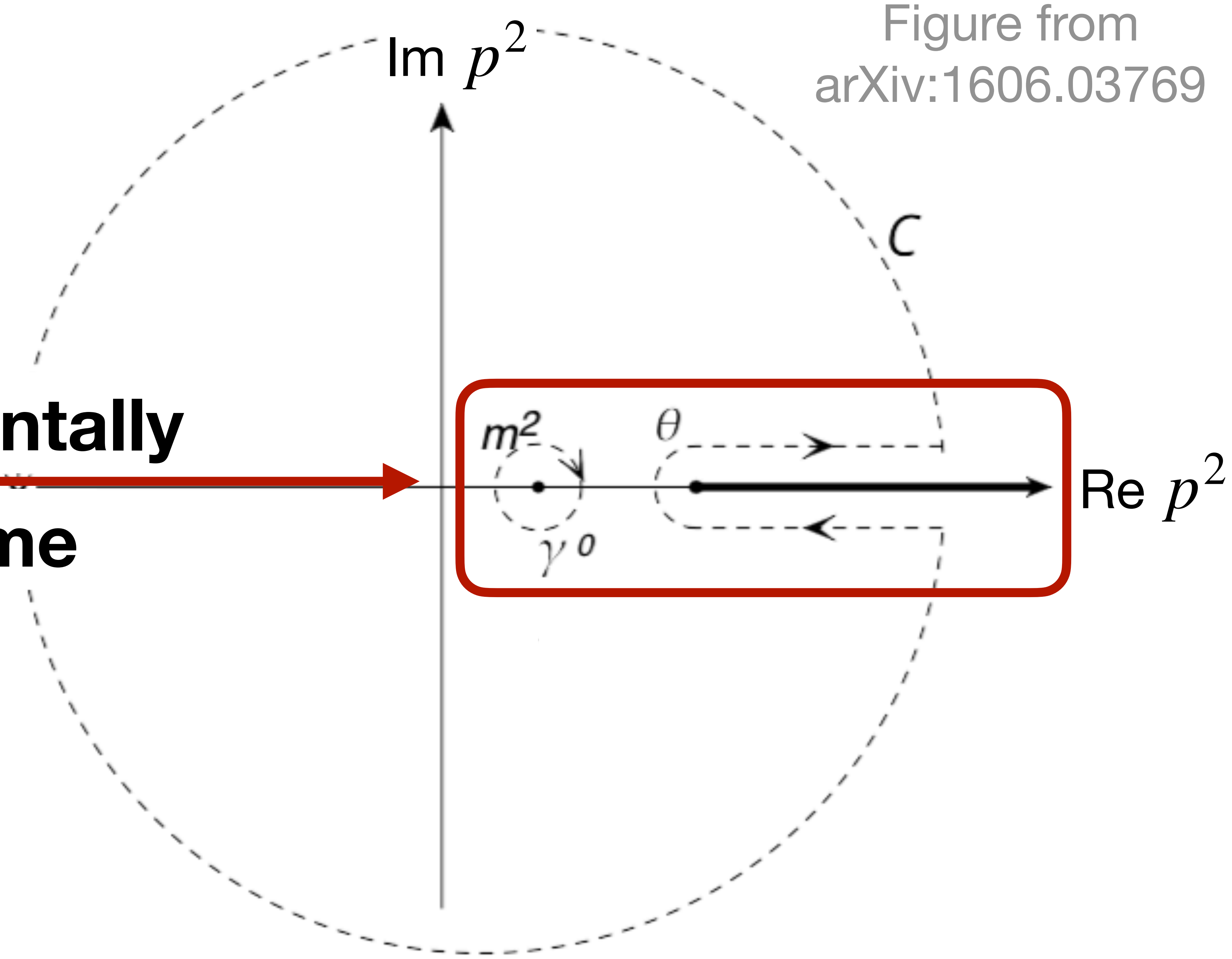
- **Spectral function**

$$\rho(\omega) = \lim_{\varepsilon \to 0^+} 2\ \mathrm{Im}\ G\big(-i(\omega + i\varepsilon)\big)$$

fundamentally

real-time

Figure from arXiv:1606.03769



Propagator in the complex p^2-plane

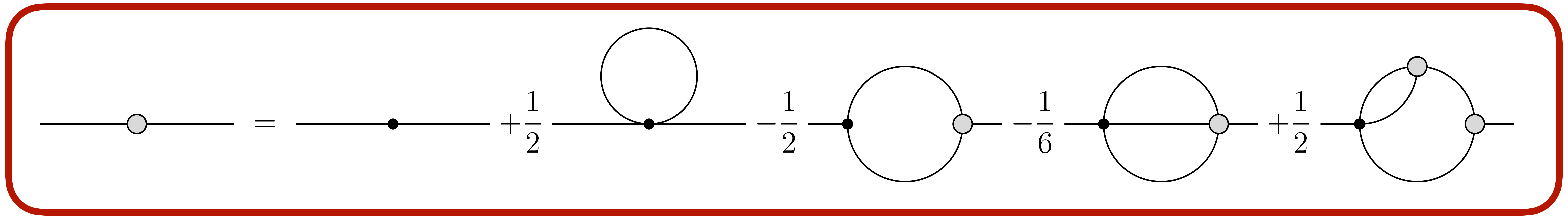
- Spectral representations for higher correlation functions?

  → should exist for fundamental fields from axiomatic perspective

The spectral DSE: Scalar ϕ^4-theory

Propagator DSE of scalar ϕ^4-theory in the broken phase $\left(<\phi> \equiv \phi_c \neq 0 \right)$

$$\Gamma^{(2)}(p) = p^2 + m_\phi^2 + \sum_j D_j(p)$$



Set up:

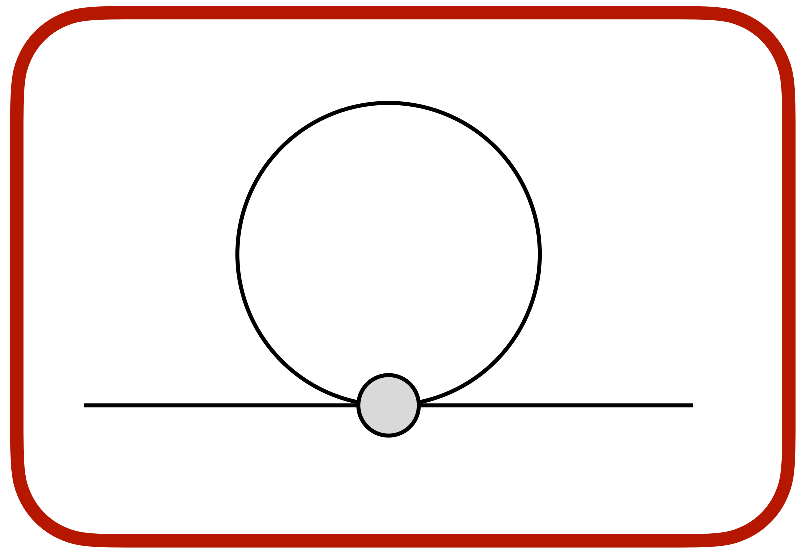
- Plug in spectral representation for propagators
- Assume vertices with canonical momentum scaling or spectral representation (classical here)

$$\longrightarrow \quad D_j(p) = g_j \prod_i^{N_j} \left(\int_{\lambda_i} \lambda_i \rho(\lambda_i) \right) I_j(p, \vec{\lambda}) \quad \text{with} \quad I_j(p, \vec{\lambda}) = \prod_k^{N_j^{\text{loops}}} \int_{q_k} \prod_i^{N_j} \frac{1}{\lambda_i^2 + l_i(\vec{q}, p)^2}$$

II. spectral integral **I. perturbative momentum integral**

Spectral renormalisation: tadpole example

Renormalisation: analytic structure allows for **dimensional regularisation**



- Example: tadpole diagram in d=3

$$D_{\text{tad}}(p) = g_{\text{tad}}(p) \int_0^\infty \frac{d\lambda}{\pi} \, \lambda \, \rho(\lambda) \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\lambda^2 + q^2}$$

dim. reg.

$$= \; g_{\text{tad}}(p) \int_0^\infty \frac{d\lambda}{\pi} \, \rho(\lambda) \, \mu^{2\varepsilon} \, \lambda^{2-2\varepsilon}$$

- **Problem:** analytic spectral function ρ needed in order to single out spectral divergence from $\varepsilon \to 0$

spectral dimensional renormalisation

spectral BPHZ renormalisation

Spectral dimensional renormalisation

UV asymptotics of ρ given by **perturbation theory** propagator, e.g.

$$\rho_{UV}(\lambda, k) = \frac{1}{\lambda^2 + k^2} \longrightarrow \text{split } \rho(\lambda) = \rho_{IR}(\lambda, k) + \rho_{UV}(\lambda, k)$$

$$[G] = -2 \text{ and } \rho \sim \text{Im } G$$
$$\longrightarrow [\rho] = -2$$

Infrared part ρ_{IR} can only carry sub-leading UV behaviour $\longrightarrow \rho_{IR} \sim \frac{1}{\lambda^4}$

$$\longrightarrow D_{\text{tad}}(p) \sim \int_0^\infty \frac{d\lambda}{\pi} \rho(\lambda)\, \mu^{2\varepsilon} \lambda^{2-2\varepsilon} = \int_0^\infty \frac{d\lambda}{\pi} \rho_{IR}(\lambda, k)\, \lambda^2 - \frac{\pi\, k}{2\cos(\frac{\pi\varepsilon}{2})} \left(\frac{\mu^2}{k^2}\right)^\varepsilon$$

finite $\qquad \to -\frac{\pi}{2} k$

Spectral dimensional renormalisation

Renormalised, finite result based entirely on **dimensional regularisation**

$\longrightarrow$ manifestly **gauge-invariant**, respecting **all symmetries** of the theory

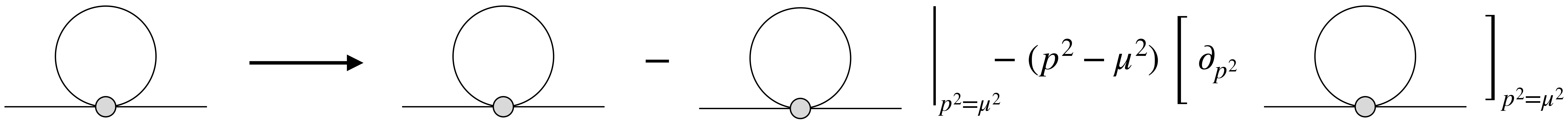
Spectral BPHZ renormalisation

Analytic integration of UV part not always feasible (or possible) ⟶ BPHZ approach

Pull $\lim_{\varepsilon \to 0}$ through integral:

$$D_{\text{tad}}(p) \sim \int_0^\infty \frac{d\lambda}{\pi}\, \rho(\lambda)\, \lambda^2$$ ⟶ **linearly divergent!**

Expand integrand around $p^2 = \mu^2$ and subtract 0th (and 1st) order terms through counterterms:



⟶ $$D_{\text{tad}}(p) \sim \int_0^\infty \frac{d\lambda}{\pi}\, \rho(\lambda)$$

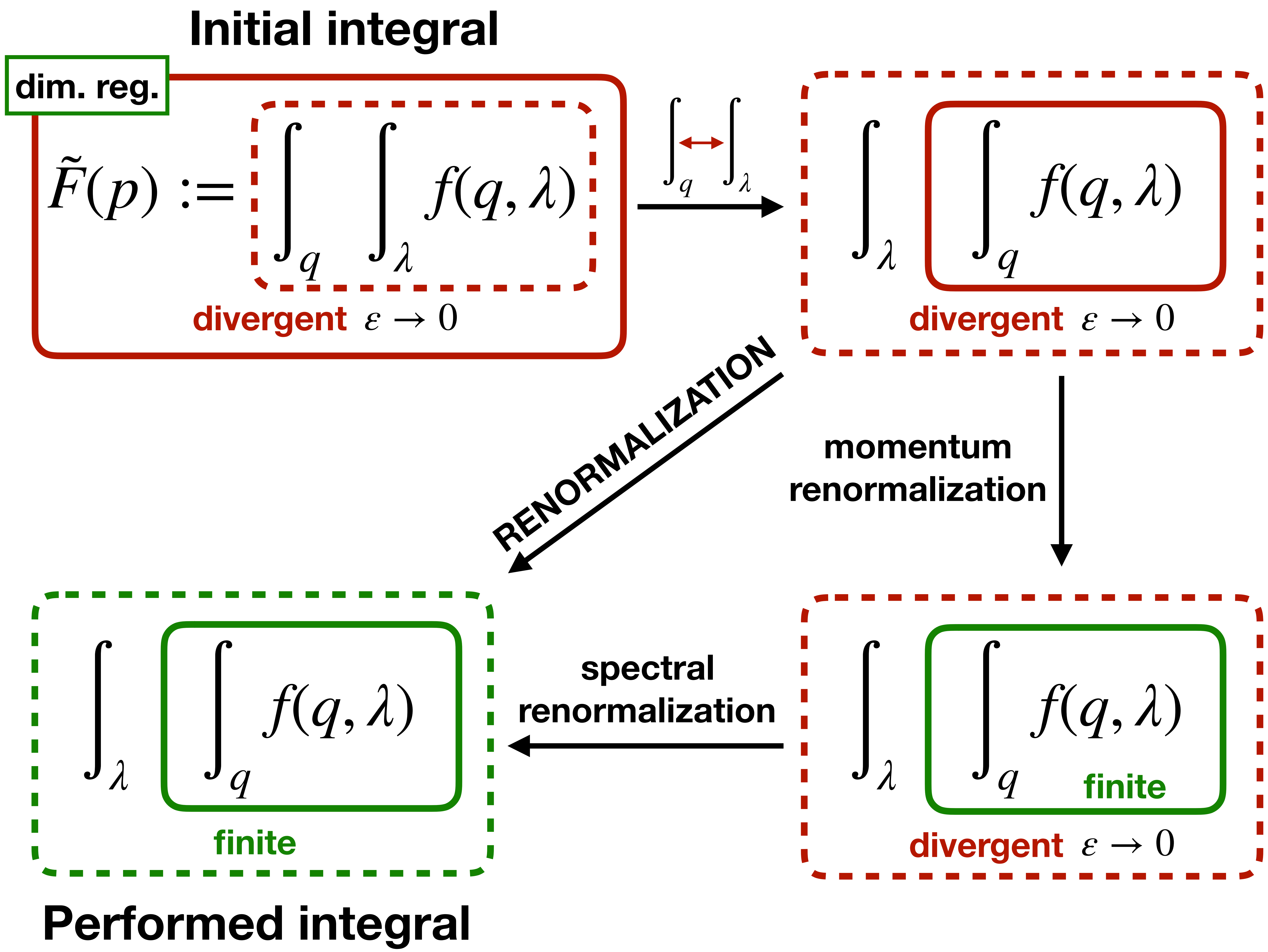
finite

Spectral BPHZ renormalisation

Cancels (sub-)leading UV behaviour in λ

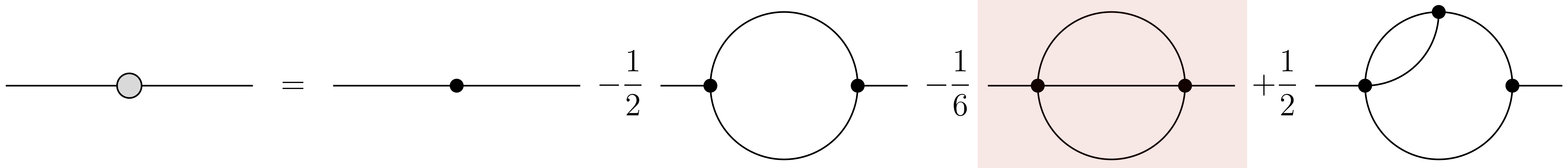
⟶ renders spectral integrals finite

Spectral renormalisation: schematic overview



ϕ^4-theory in d=2+1 with classical vertices

I. **Renormalise** spectral propagator DSE via **spectral BPHZ (on-shell) renormalisation**



Propagator DSE with classical vertices

II. **Analytic continuation** to real momentum axis by **Wick rotation**, $p \to \lim_{\varepsilon\to 0} -\mathrm{i}(\omega + \mathrm{i}\varepsilon)$

$$\Gamma^{(2)}(\omega) = -\omega^2 + m_\phi^2 + \sum_{\{j\}} g_j \prod_i^{N_j} \left(\int_{\lambda_i} \lambda_i \rho(\lambda_i) \right) I_j(\omega, \vec{\lambda})$$

III. **Solve** DSE by **iteration:**

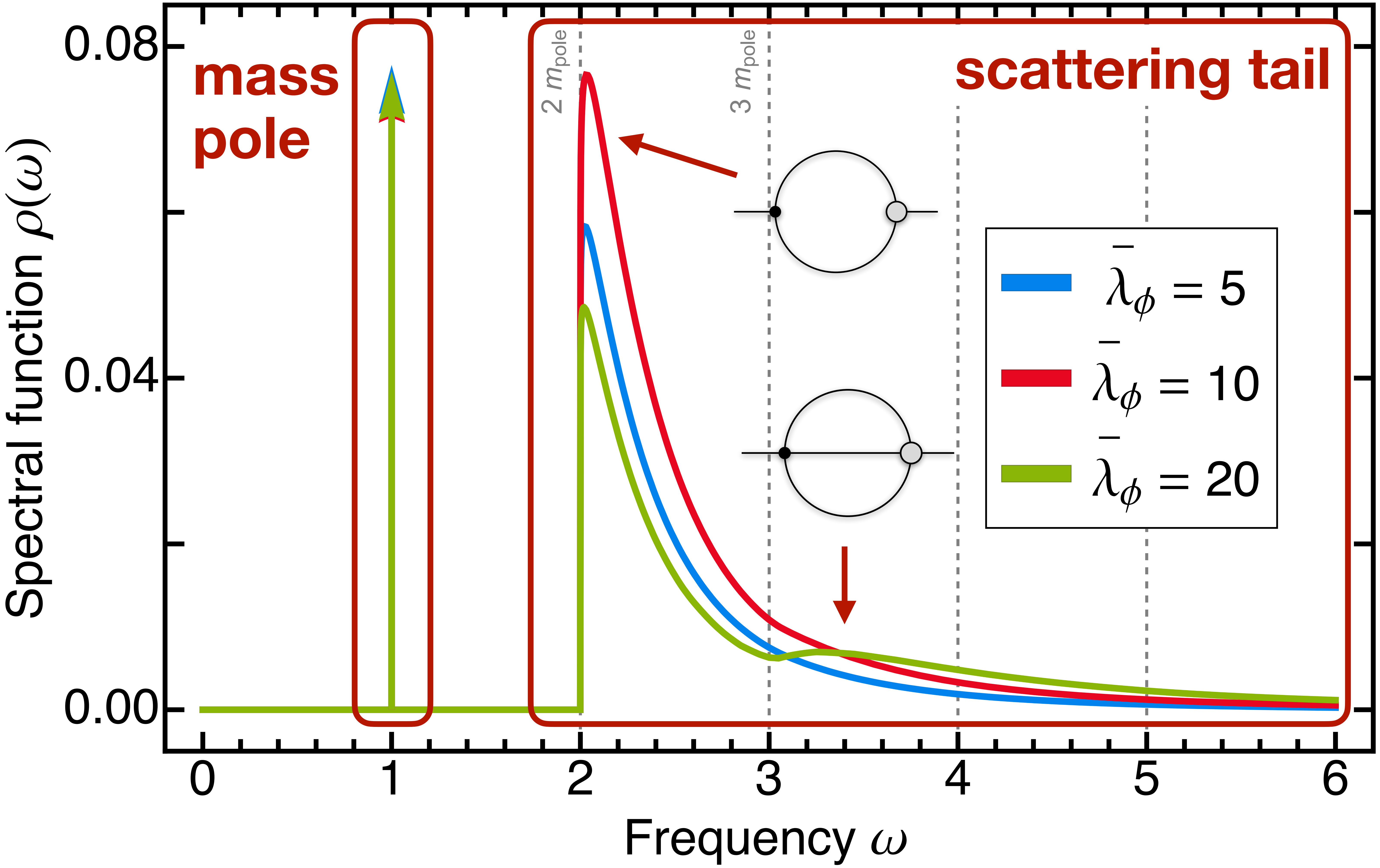
1. Make initial guess ρ_0 →

2. Calculate $\Gamma^{(2)}$ via DSE

3. Compute ρ from propagator

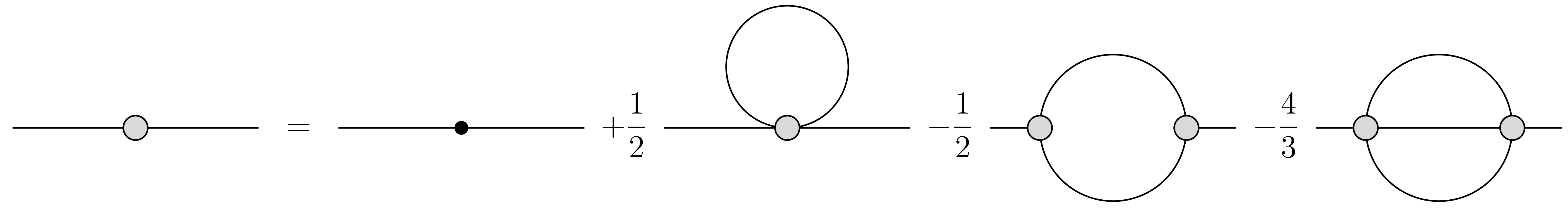
Iterate until convergence

ϕ^4-theory in d=2+1 with classical vertices

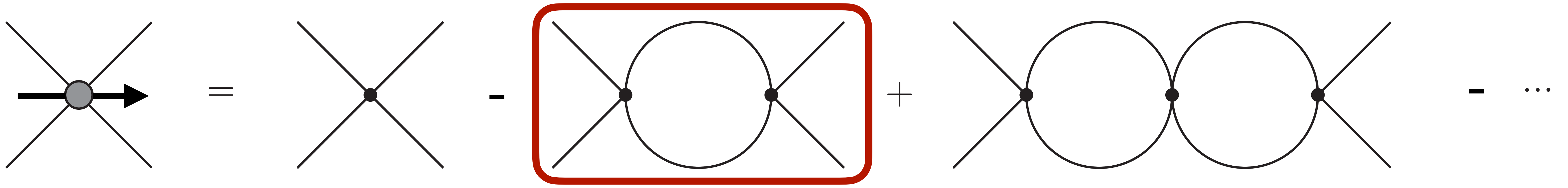


ϕ^4-theory in d=2+1 in the skeleton expansion

Skeleton expansion: dress all vertices



- **Four-point-function** from bubble resummation of s-channel expansion



$\lambda_\phi^2\,\Pi(p)$

$$\Gamma^{(4)}(p) = \frac{\lambda_\phi}{1+\lambda_\phi\,\Pi(p)}$$

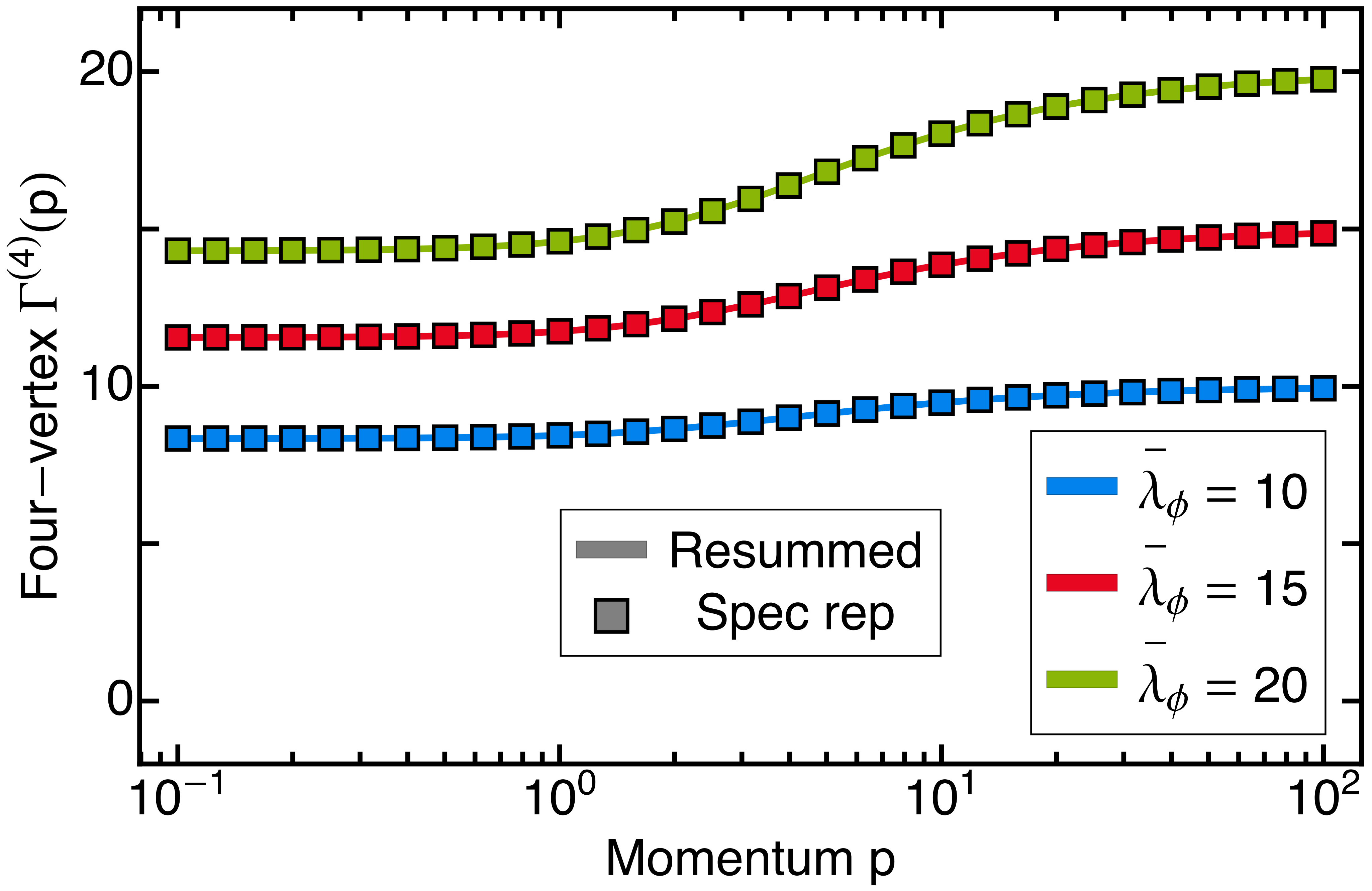
→ **coupled system** of $\Gamma^{(4)}$ and $\Gamma^{(2)}$

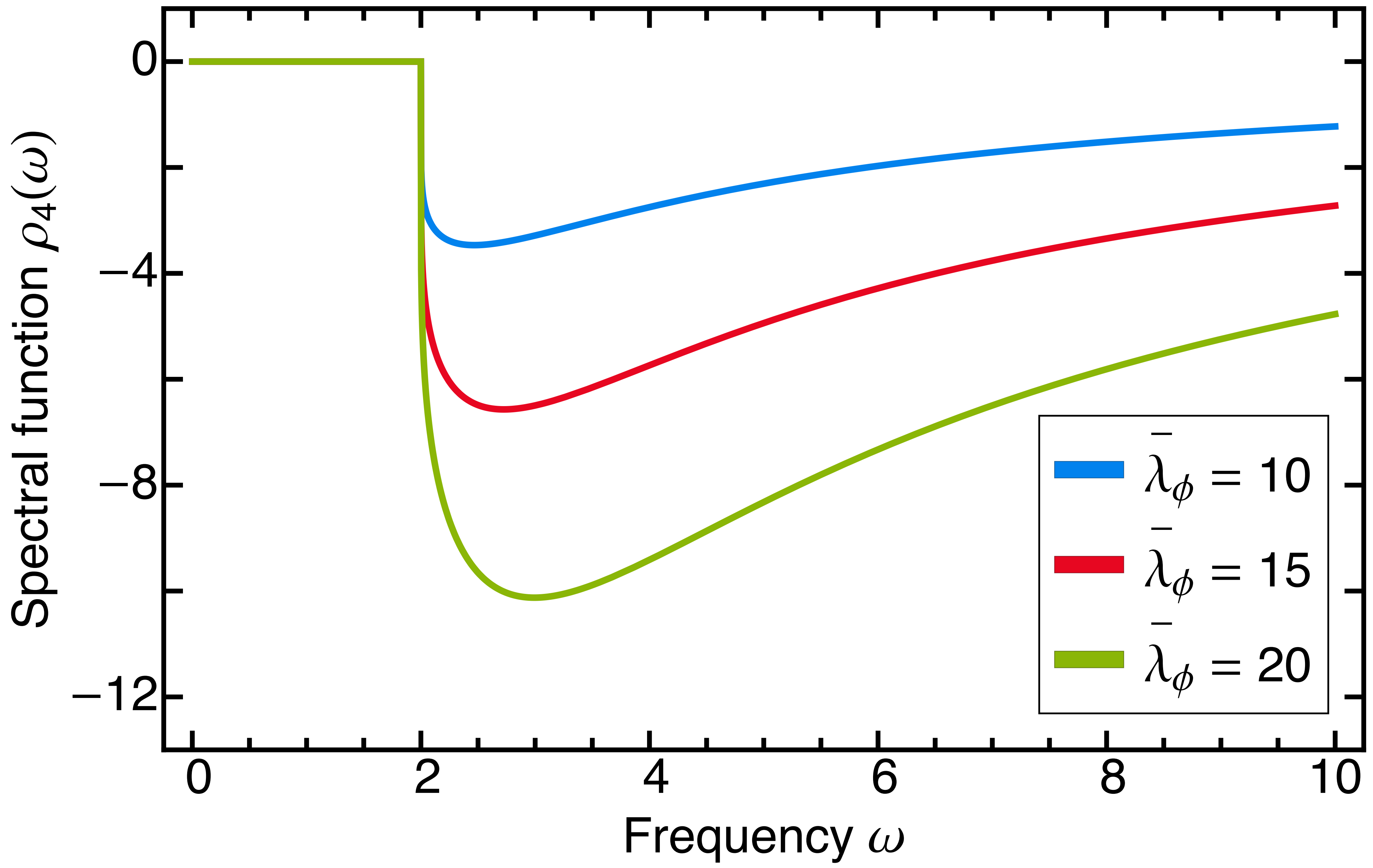
- **Three-point-function** from effective potential on the EoM: $\Gamma^{(3)}(p) = \phi_c\,\Gamma^{(4)}(p)$

ϕ^4-theory in d=2+1 in the skeleton expansion

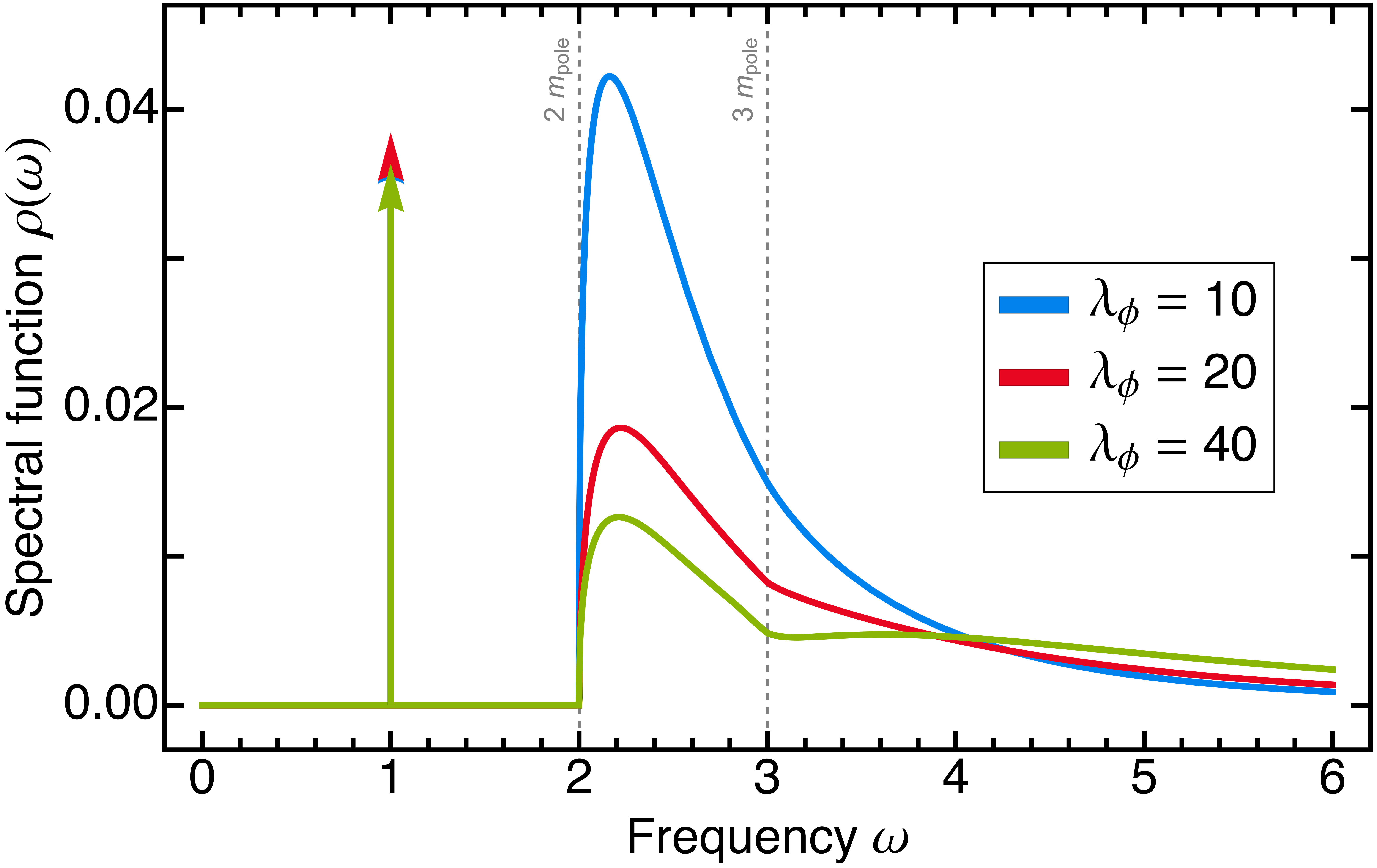
Four-point-function $\Gamma^{(4)}$ has **spectral representation**

$$\Gamma^{(4)}(p) \;=\; \lambda_\phi \;+ \int_\lambda \frac{\lambda\,\rho_4(\lambda)}{p^2+\lambda^2} \quad \text{with} \quad \rho_4(\omega) \;=\; 2\,\mathrm{Im}\,\Gamma^{(4)}(-\mathrm{i}(\omega+\mathrm{i}0^+))$$

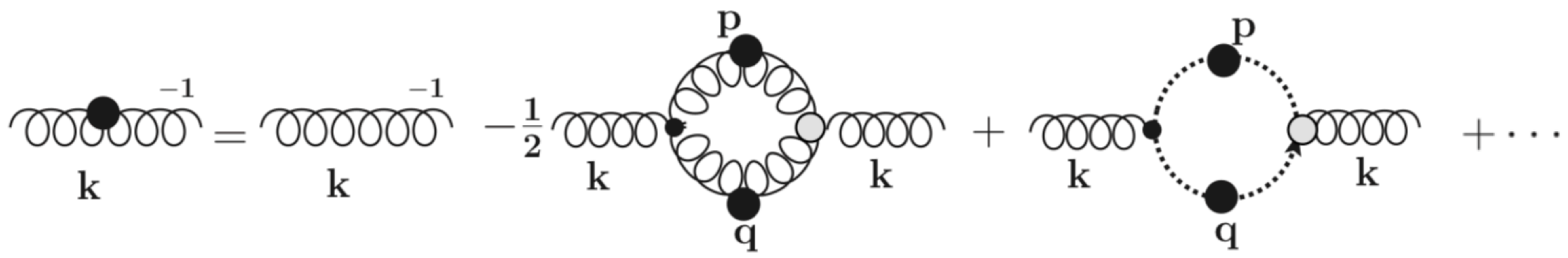




ϕ^4-theory in d=2+1 in the skeleton expansion

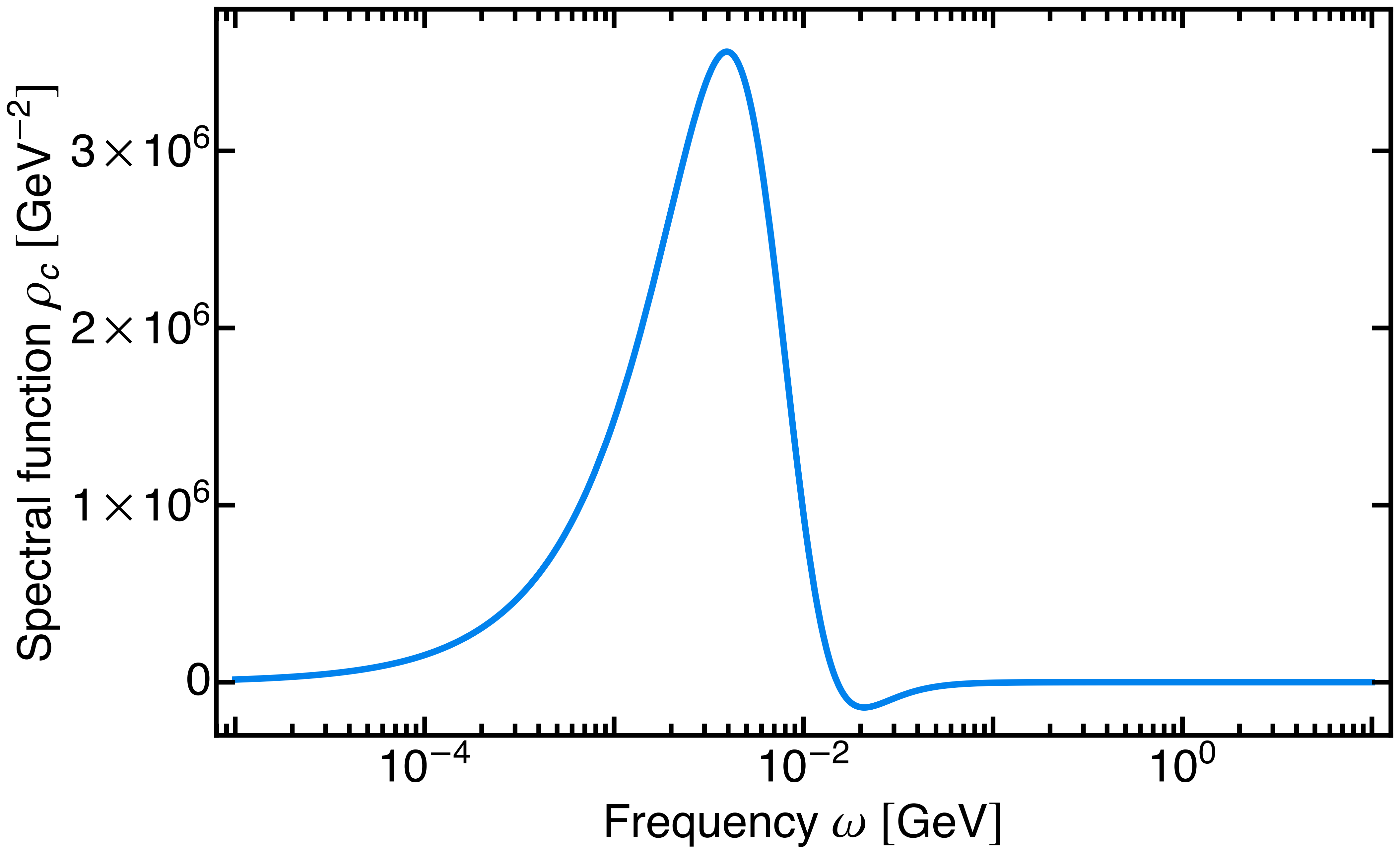


Yang-Mills theory in d=3+1

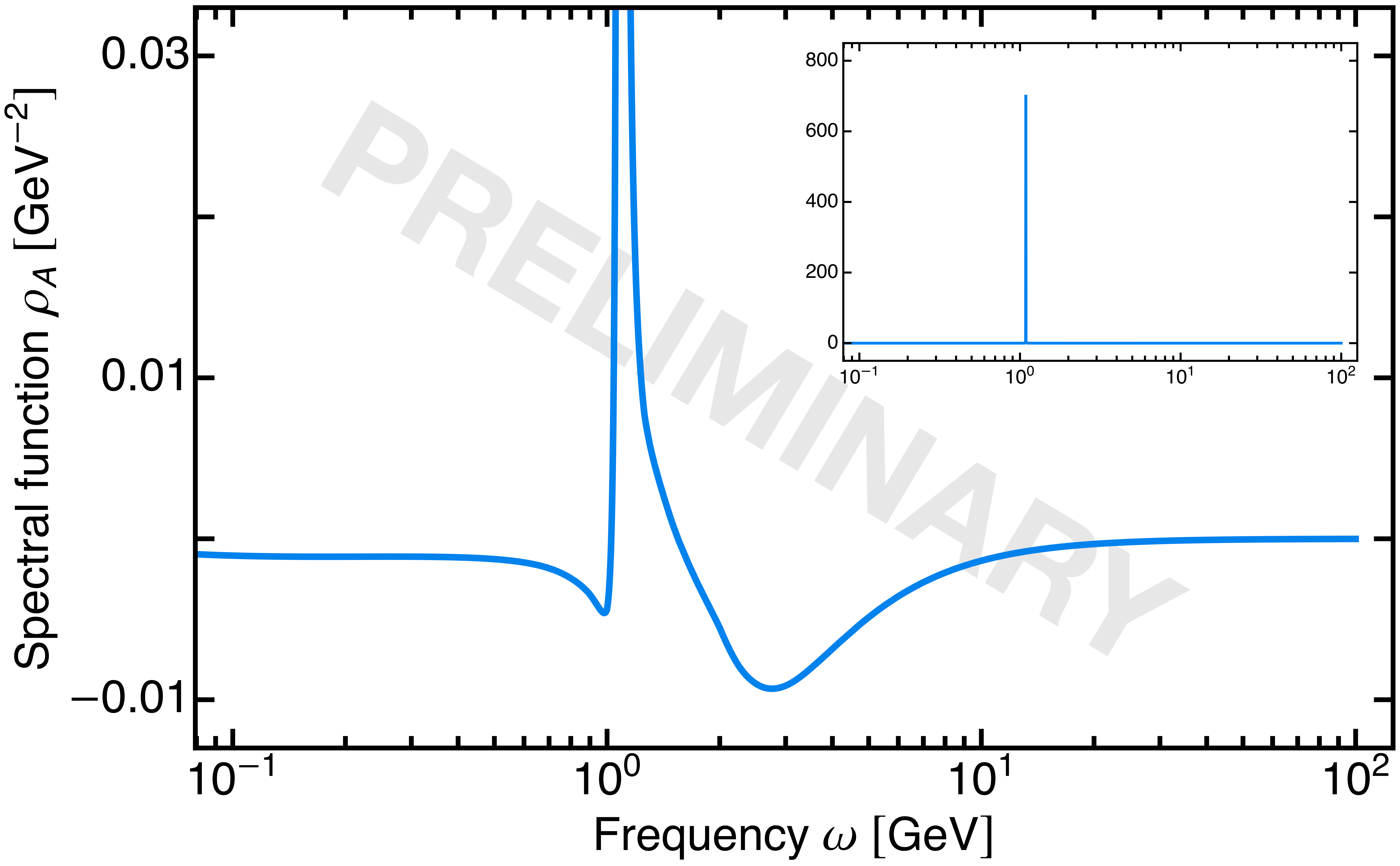


Gluon propagator DSE at one-loop

- **Classical vertices**
- Renormalise via **spectral BPHZ renormalisation**
- Assume **static ghost spectral function** that has **scaling behaviour**

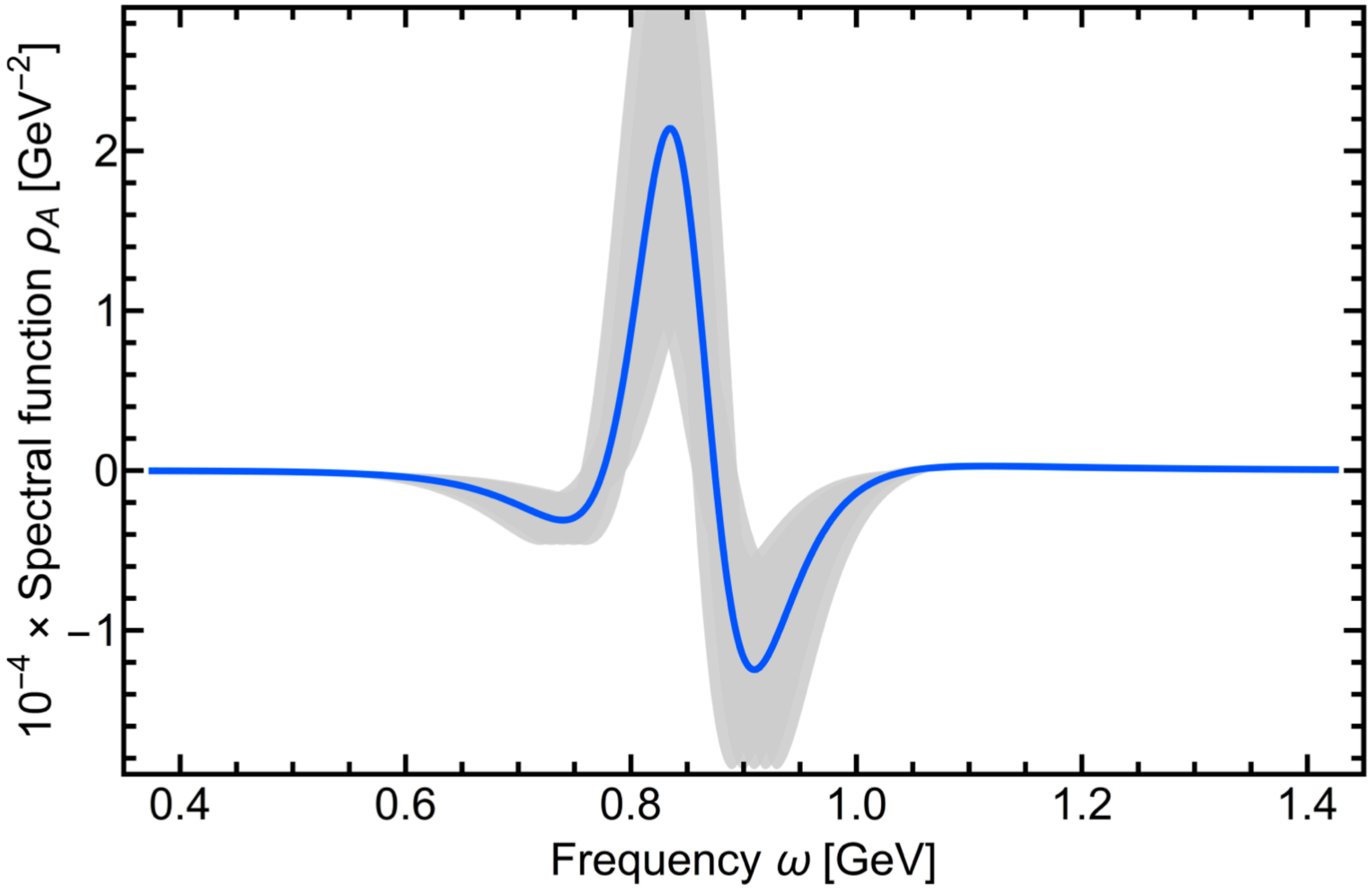


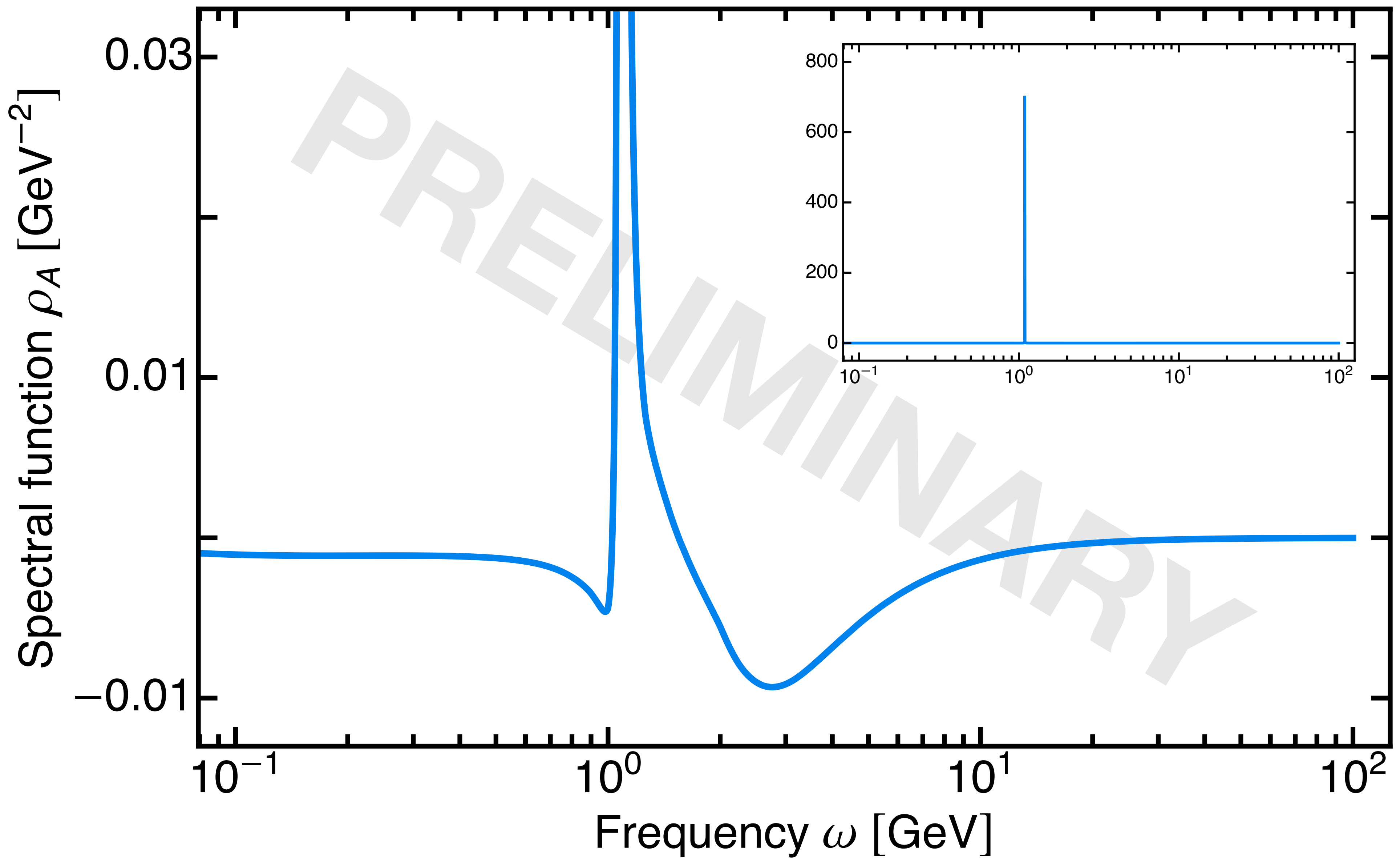
Yang-Mills theory in d=3+1



Yang-Mills theory in d=3+1: Reconstruction

Cyrol, Pawlowski, Rothkopf, Wink, 2018, arXiv:1804.00945





Summary & conclusions

Spectral renormalisation (dimensional & BPHZ)

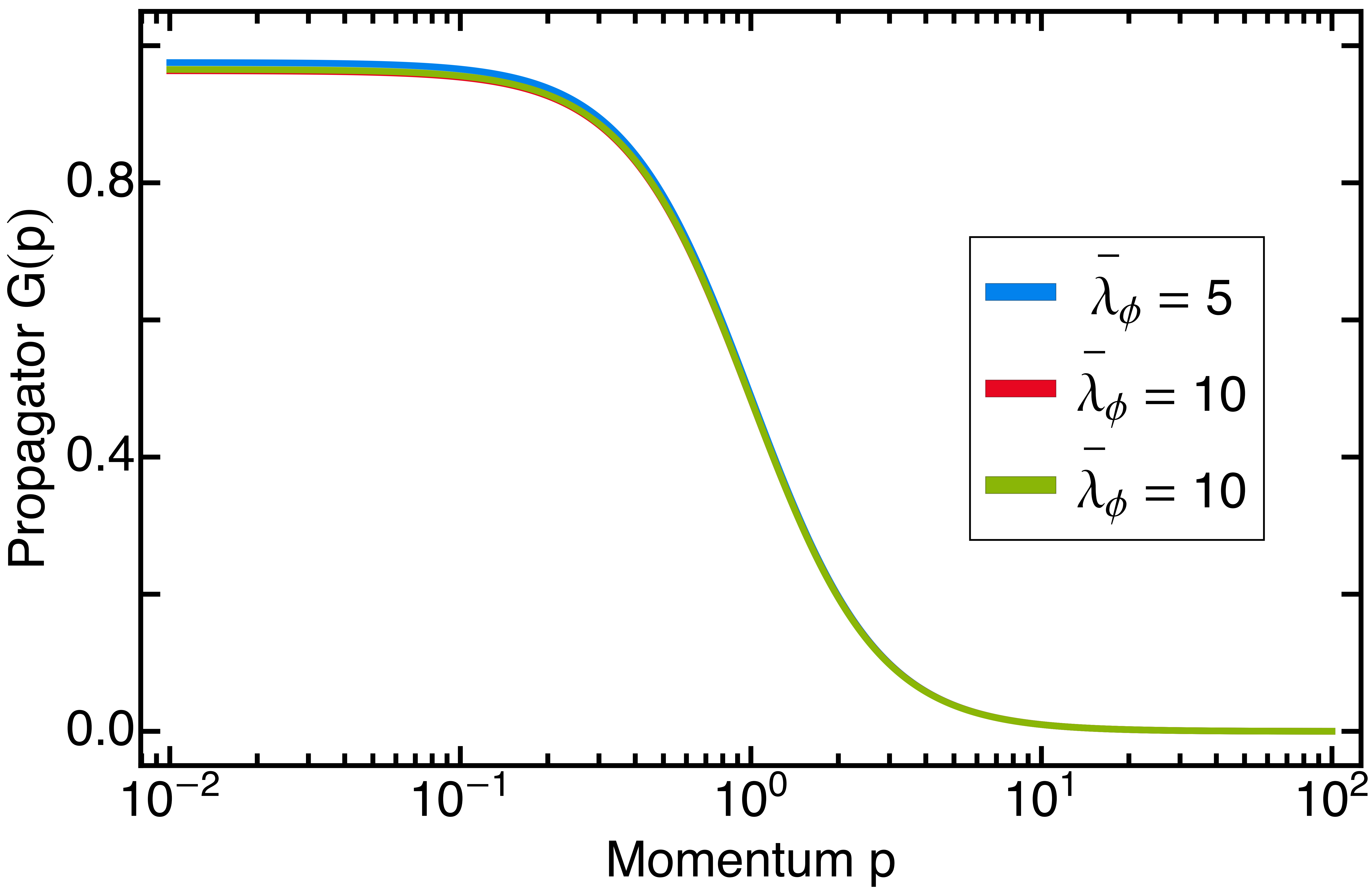
- **novel** renormalisation scheme based on **dimensional** regularisation and **spectral representations**
- Fully **gauge-invariant** and **symmetry-preserving (dimensional),** suitable for fully numerical approach **(BPHZ)**
- allows for **analytic continuation** of renormalised equation by analytic structure due to spectral representations

Scalar ϕ^4-theory in d=2+1: spectral function from the **DSE** with bare vertices and in the skeleton expansion via analytic continuation

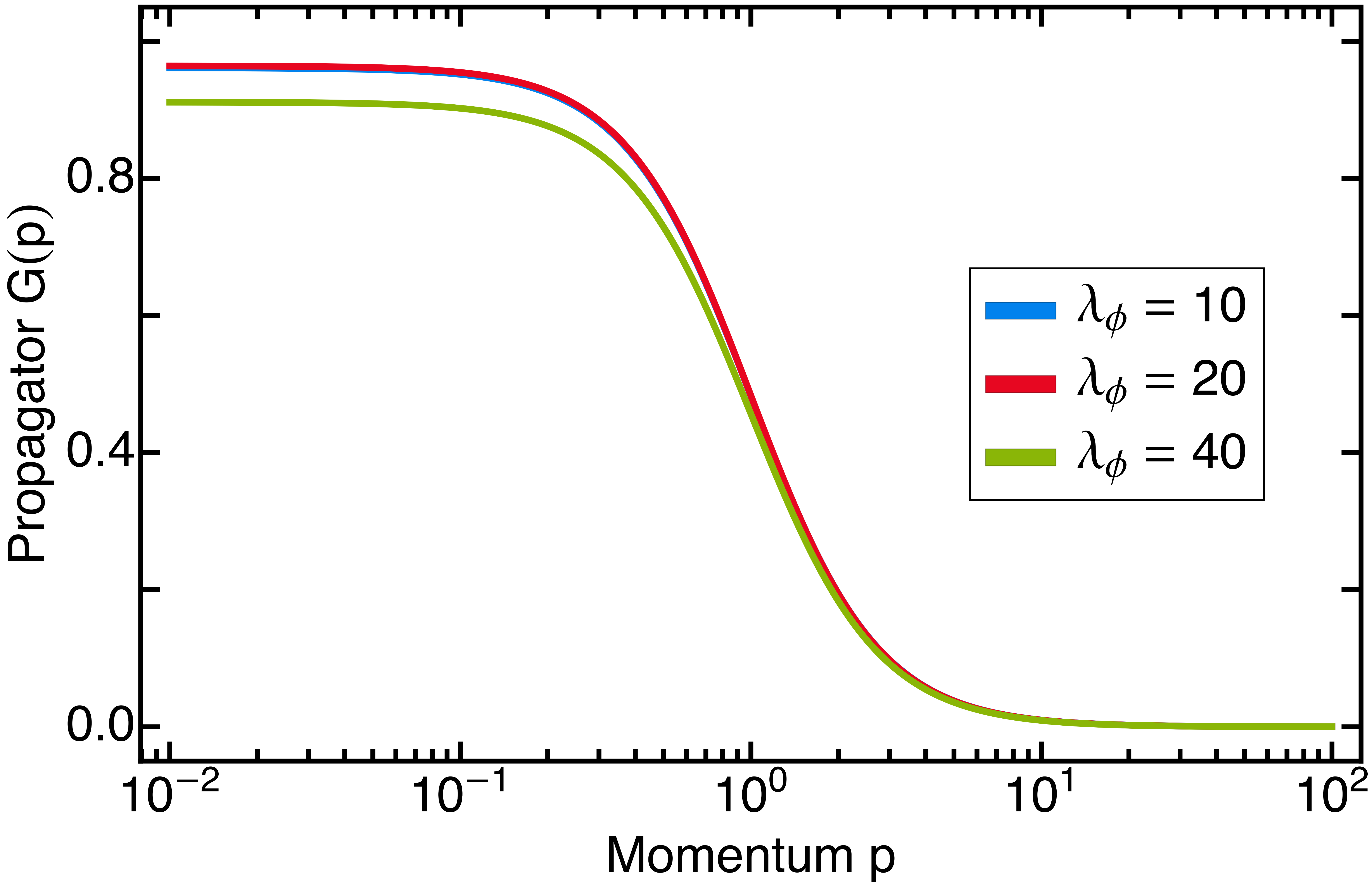
Yang-Mills theory in d=3+1: gluon spectral function from the standalone gluon **DSE** with classical vertices and a scaling-like ghost propagator

Back up

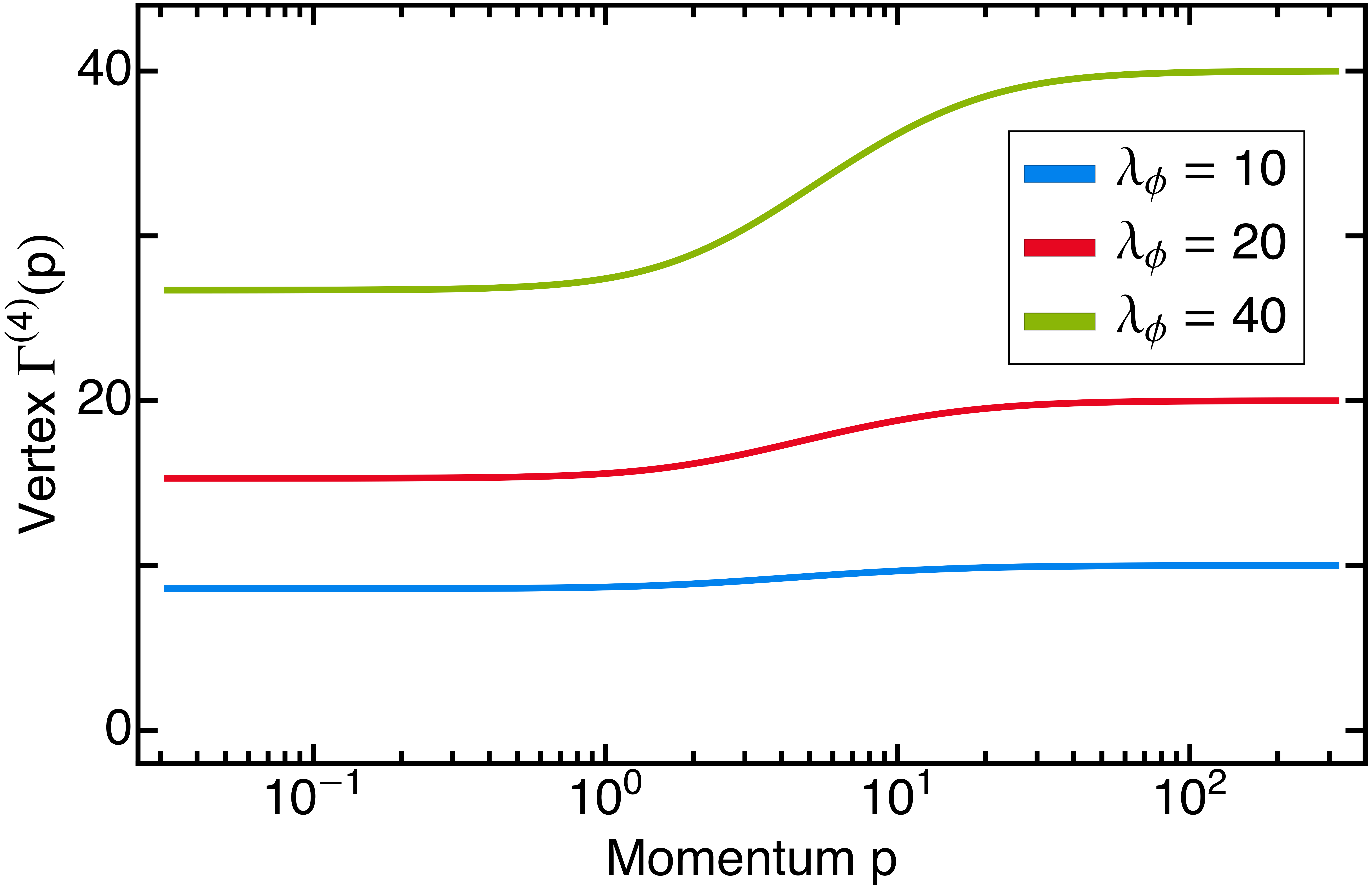
ϕ^4-theory in d=2+1 with classical vertices



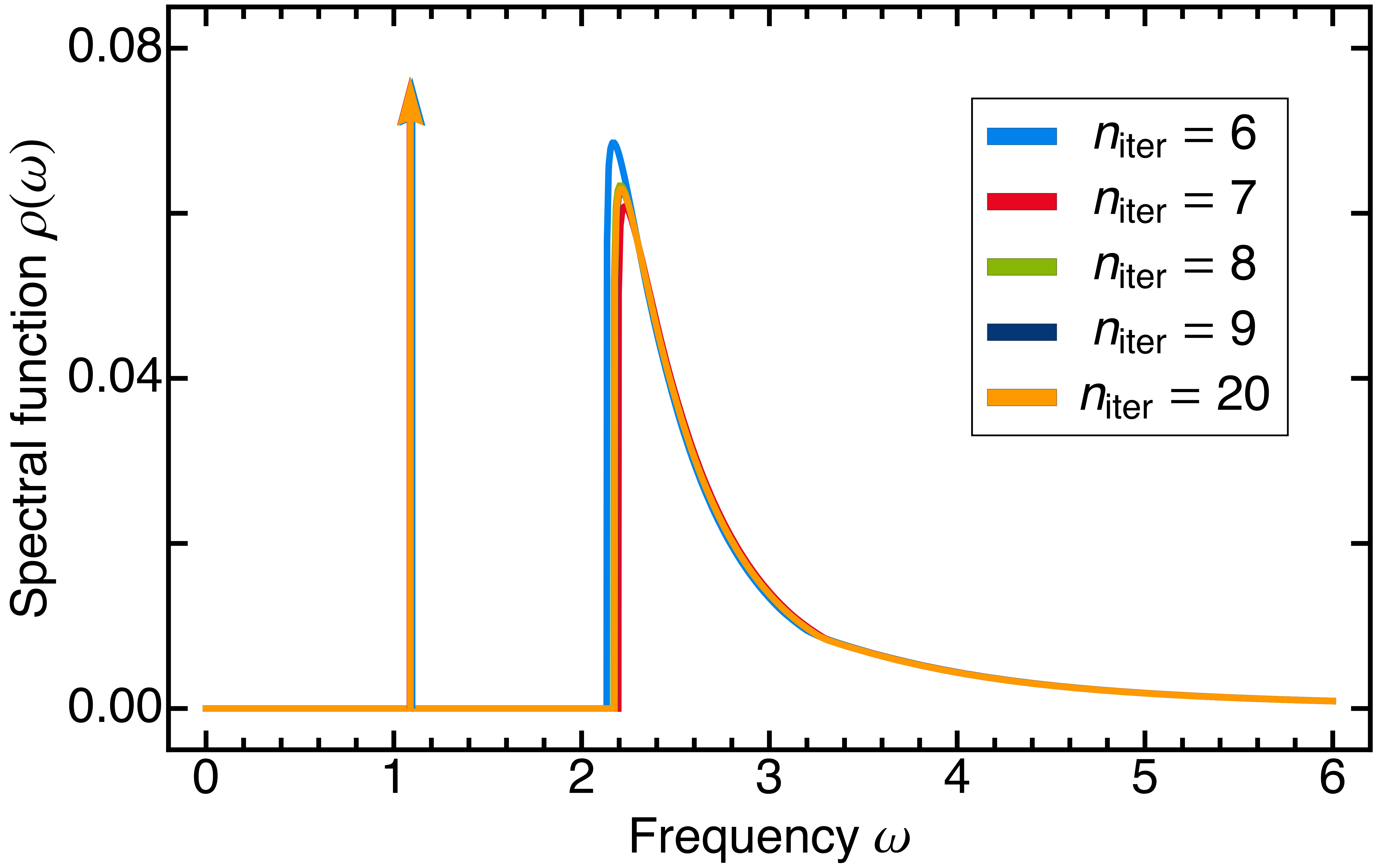
ϕ^4-theory in d=2+1 in the skeleton expansion



ϕ^4-theory in d=2+1 in the skeleton expansion



ϕ^4-theory in d=2+1: Convergence



ϕ^4-theory in d=2+1: Convergence