The full S-matrix of pure super-Yang-Mills theory at tree-level

Cold Quantum Coffee seminar @ Heidelberg University

Maximilian Rupprecht

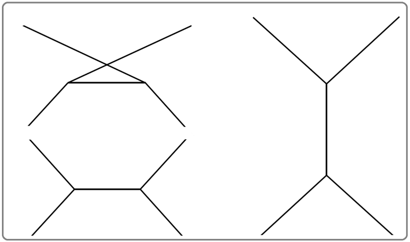
based on my MSc thesis at

The University of Edinburgh

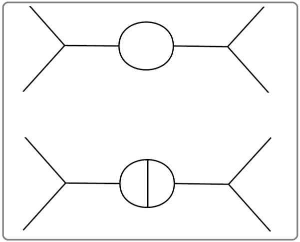
3 November 2020

Basics

- Quantum Field Theory
- Perturbation Theory: Feynman diagrams



tree-level



loops

- n-point amplitude $A = \sum$ diagrams
- S-matrix $S_{\mathrm{fi}} = \langle \phi_{\mathrm{f}} | S | \phi_{\mathrm{i}} \rangle$ via LSZ reduction
- Cross section from $|S_{\mathrm{fi}}|^2$

Overview

Goal: *Express full tree-level S-matrices.*

Here: Pure (super-)Yang-Mills theory.

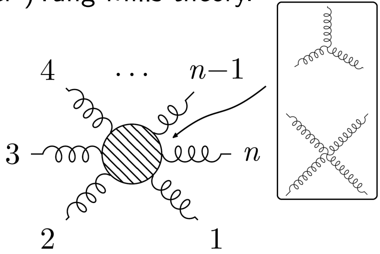


Figure: Gluon amplitude with external momenta $p_1, p_2 \dots, p_n$.

Three main ingredients:

$$\underbrace{\text{MHV amplitudes, BCFW recursions, Twistor theory}}_{\textbf{The RSVW formula}}$$

(Pure) Yang-Mills Theory

Based on some semi-simple Lie algebra

$$[T^a, T^b] = if^{abc}T^c \, .$$

Gluon fields:

$$A_\mu = A_\mu^a T_a \, ,$$

with a sum over the *colour*-index a.
Field-strength:

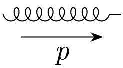
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu]$$

A theory of (generalised) gluons ($\mathrm{SU}(3)$: QCD)

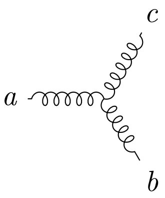
$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$$

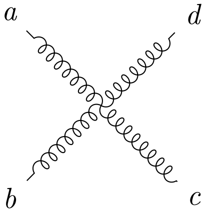
Gluon Self-Interaction

Besides the 'standard' propagator

a, μ  $b, \nu \; = \dfrac{\delta_{ab}\eta^{\mu\nu}}{p^2}\,,$

non-abelian structure allows



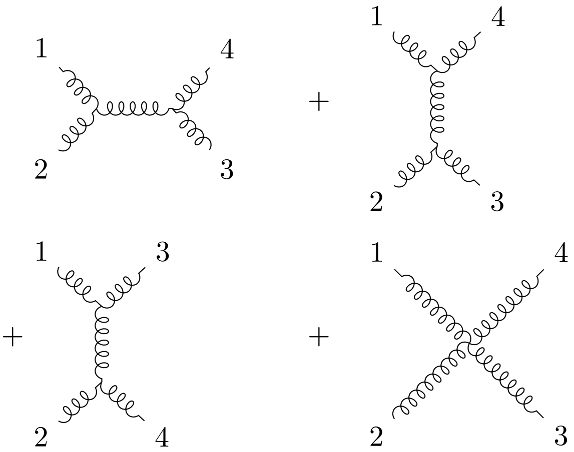


3-pt vertex $\propto gf^{abc}$, 4-pt vertex $\propto g^2 f^{abe} f^{ecd}$.

Traditional approach

Question: What does the n-pt gluon amplitude evaluate to?

eg. 4-pt amplitude:



Gluon Amplitudes (at tree-level)

For large n, too complicated using standard Feynman techniques! Eg. 80's brute force calculation of $gg \to ggg$:

$k_1 \cdot k_4 \, \varepsilon_2 \cdot k_1 \, \varepsilon_1 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5$

from Z. Bern [1]

Colour-ordering

Can replace all structure constants with single traces over generators using

$$f^{abc} \propto \mathrm{Tr}\left(T^a T^b T^c\right) - \mathrm{Tr}\left(T^b T^a T^c\right),$$

and relations like

$$\begin{aligned} f^{abe} f^{cde} \propto \mathrm{Tr}\left(T^a T^b T^c T^d\right) - \mathrm{Tr}\left(T^a T^b T^d T^c\right) \\ - \mathrm{Tr}\left(T^a T^c T^d T^b\right) + \mathrm{Tr}\left(T^a T^d T^c T^b\right). \end{aligned}$$

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We can write eg. for the 4-pt amplitude:

$$\begin{aligned} A_4^{\text{full, tree}} = g^2 (& A_4[1234]\, \mathrm{Tr}\left(T^{a_1} T^{a_2} T^{a_3} T^{a_4}\right) \\ & + \text{permutations of } (234)), \end{aligned}$$

where $A_4[1234]$ is a *colour-ordered partial amplitude*.

Spinor-Helicity Formalism [1]

Based on the fact that $\mathfrak{so}(4) \cong \mathfrak{sl}(2) \times \mathfrak{sl}(2)$:

Can map four-vectors to 2×2 matrices using the Pauli matrices $\sigma^\mu = (1, \sigma^i)$:

$$x_{\alpha\dot{\alpha}} = x_\mu \sigma^\mu_{\alpha\dot{\alpha}} = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}.$$

Note that $\det(x_{\alpha\dot{\alpha}}) = x_\mu x^\mu \implies$ for null vectors write

$$x_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}},$$

with **spinors** λ_α, $\tilde{\lambda}_{\dot{\alpha}}$, where

$\alpha = 0, 1$ lives in $\left(0, \frac{1}{2}\right)$ rep. of $\mathrm{SL}(2)$, 'negative helicity',

$\dot{\alpha} = 0, 1$ lives in $\left(\frac{1}{2}, 0\right)$ rep. of $\mathrm{SL}(2)$, 'positive helicity'.

[1] see e.g. Adamo 2017 [2] or Elvang & Huang 2015 [3].

Parke-Taylor amplitude

Back to gluon amplitudes!

A **pattern** was found for a certain *helicity* configuration:

The 'MHV' n-gluon amplitude (Parke, Taylor 1986 [4]; Proof by Berends, Giele 1988 [5])

$$f(\lambda_i) = \frac{\langle jk \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}, \text{ with } \langle jk \rangle \equiv \lambda_{i\alpha} \lambda_{j\beta} \epsilon^{\beta\alpha},$$

where gluons j and k have negative helicity and the rest have positive helicity.

This motivates the search for new techniques in the study of scattering amplitudes!

MHV amplitudes

Note:

- Can restrict to colour-odered partial amplitudes (stripped of colour-traces)
- Gluons are spin-1 particles!
  $\Rightarrow$ Organise by helicity configurations:

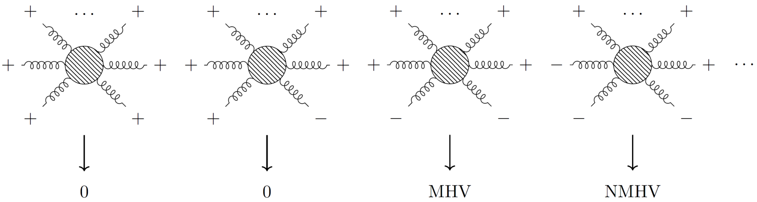


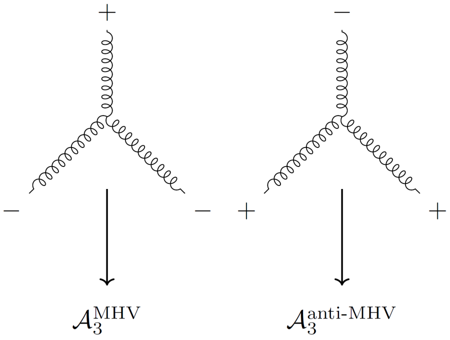
Figure: The MHV (maximal helicity violation) classification of gluon amplitudes.

$$\mathcal{A}_n = \sum_{k=0}^{n-3} \mathcal{A}_n^{\mathsf{N}^k\mathsf{MHV}}$$

Three-point (super-)amplitudes

$$\mathcal{A}_3^{\text{MHV}} = \frac{\delta^{4|8}(P)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \,,$$

$$\mathcal{A}_3^{\text{anti-MHV}} = \frac{\delta^{4}(P)\,\delta^{0|4}\left(\eta_{1a}[23] + \eta_{2a}[31] + \eta_{3a}[12]\right)}{[12][23][31]} \,,$$



using the *spinor-helicity* formalism.

Figure: Three-point amplitudes.

BCFW[2] recursions

Schematically:

$$A_n = \sum_{\{i,j\}} A_\mathrm{L} \frac{1}{P_{ij}^2} A_\mathrm{R}$$

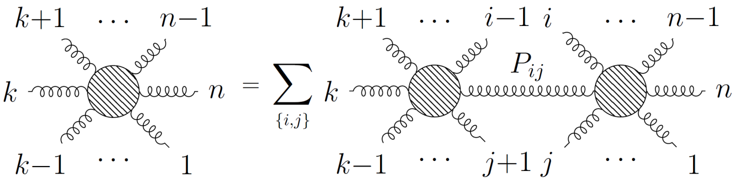


Figure: BCFW recursion. The sub-amplitudes are on-shell!

Allows recursive construction of n-point amplitudes from three-point building blocks!

[2] Britto, Cachazo, Feng, Witten 2005 [6, 7]

Sketch of the Proof 1/2

Consider external momenta $p_1, \ldots, p_n$ with $p_i^2 = 0 \,\forall\, i$.

- Choose two momenta to shift:

$$\hat{p}_k(z) = p_k + zq, \quad \hat{p}_n(z) = p_n - zq, \quad z \in \mathbb{C},$$

with $q^2 = 0$ and $q \cdot p_k = q \cdot p_n = 0$.

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  with $q^2 = 0$ and $q \cdot p_k = q \cdot p_n = 0$.
- Internal momenta are sum of adjacent external momenta: $P_{ij} = p_i + \ldots + p_j$
- Shifted amplitude $\hat{A}_n(z)$ is rational and can only have simple poles from propagators $1/\hat{P}_{ij}(z)^2$ at $z \equiv z_{ij}$
- Use Cauchy's theorem:

$$A_n = \hat{A}_n(z = 0) = \frac{1}{2\pi i} \oint_\gamma \frac{\hat{A}_n(z)}{z} dz$$

Sketch of the Proof 2/2

$$A_n = \hat{A}_n(z=0) = \frac{1}{2\pi i} \oint_\gamma \frac{\hat{A}_n(z)}{z} dz$$

- Can deform contour, enclosing all poles except the one at $z = 0$
- Assuming $\hat{A}_n(z) \to 0$ for $z \to \infty$, residue theorem:

$$A_n = -\sum_{\{i,j\}} \mathrm{Res}_{z=z_{ij}} \frac{\hat{A}_n(z)}{z}$$

- At z_{ij} the propagator $1/\hat{P}_{ij}(z)^2$ goes *on-shell*, factorises amplitude:

$$\mathrm{Res}_{z=z_{ij}} \frac{\hat{A}_n(z)}{z} = -\hat{A}_{\mathrm{L}}\left(z_{ij}\right) \frac{1}{P_{ij}^2} \hat{A}_{\mathrm{R}}\left(z_{ij}\right)$$

Twistor Space[3]

Twistor space $\mathbb{PT}$ is an open subset of $\mathbb{CP}^3$, the three dimensional complex projective space:

$$Z^A = (Z^1, Z^2, Z^3, Z^4) \in \mathbb{C}^4 \setminus \{0\}, \text{ with } rZ^A \sim Z^A,$$

for $r \in \mathbb{C} \setminus \{0\}$.

Can divide Z^A into two Weyl spinors

$$Z^A = (\mu_{\dot{\alpha}}, \lambda^\alpha), \text{ with } \dot{\alpha}, \alpha = 0, 1.$$

To express amplitudes $\tilde{A}(\lambda_i, \mu_i)$ in twistor space, recall spinor-helicity.

[3] see Adamo 2017 [2]

Incidence Relations

Consider the so-called *incidence relations*:

$$\mu_{\dot{\alpha}} + x_{\alpha\dot{\alpha}}\lambda^{\alpha} = 0,$$

for $\dot{\alpha} = 0, 1$.

They define the connection between

$$Z^A = (\mu_{\dot{\alpha}}, \lambda^{\alpha}) \in \mathbb{PT}, \qquad \text{twistor space and}$$

$$x_{\alpha\dot{\alpha}} \in \mathbb{M}, \qquad \text{Minkowski space}$$

coordinates.

The Twistor Correspondence

Visualise the incidence relations $\mu_{\dot\alpha} + x_{\alpha\dot\alpha}\lambda^\alpha = 0$:

1. Fix $x_{\alpha\dot\alpha} \in \mathbb{M}$: IR define complex plane $\mathbb{C}^2 \subset \mathbb{C}^4$. Projective scale $\Rightarrow$ this defines a $X \cong \mathbb{CP}^1 \subset \mathbb{PT}$ (linearly, holomorphically embedded Riemann sphere, a 'line').
2. Fix $X \cap Y = Z \in \mathbb{PT}$. $\Rightarrow (x-y)_{\alpha\dot\alpha}\lambda^\alpha = 0$. $\Rightarrow (x-y)_{\alpha\dot\alpha} = \lambda_\alpha\tilde\lambda_{\dot\alpha}$. So x and y are *null separated*! Z defines a so-called α-plane.

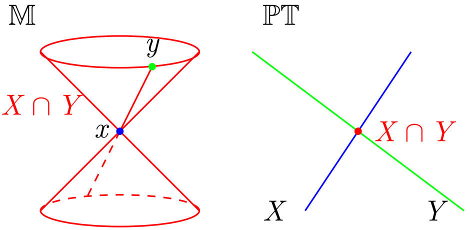
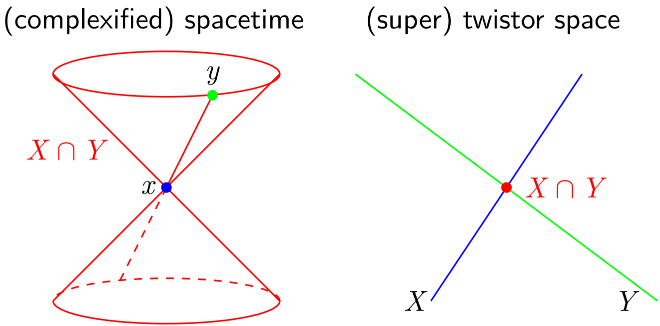


Figure: Schematic illustration of the twistor correspondence. (Original from Adamo 2013 [9])

(Super) twistor space $\mathbb{PT}$



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Super twistor space is described by homogeneous coordinates

$$\mathcal{Z} := (\mu_{\dot\alpha}, \lambda^{\alpha}, \chi^{a}) \in \mathbb{CP}^{3|4},$$

with Weyl spinors $\mu_{\dot\alpha}, \lambda^{\alpha}$ and Grassmann parameters χ^{a}.

The Penrose transform[4]

Use spinor-helicity formalism to decompose field strength

$$F_{\mu\nu} \to F_{\alpha\dot\alpha\beta\dot\beta} =: \epsilon_{\alpha\beta} \underbrace{\tilde F_{\dot\alpha\dot\beta}}_{\text{SD}} + \epsilon_{\dot\alpha\dot\beta} \underbrace{F_{\alpha\beta}}_{\text{ASD}} \,.$$

Obtain independent (zero rest mass) equations for positive and negative helicity sectors:

$$\partial^\mu F_{\mu\nu} = 0 \quad \begin{cases} \partial^{\dot\alpha}_{\beta} \tilde F_{\dot\alpha\dot\beta} = 0\,, & h = +1 \\ \partial^{\alpha}_{\dot\beta} F_{\alpha\beta} = 0\,, & h = -1 \end{cases}$$

> ***The Penrose transform:*** *Solutions to the helicity h z.r.m. equations correspond to representatives of the Dolbeault cohomology group $\mathrm{H}^{0,1}(\mathbb{PT}, \mathcal{O}(2h-2))$ on twistor space.*

[4] Eastwood, Penrose, Wells 1981 [10]

Twistor BCFW

$$\mathcal{A}_n(\mathcal{Z}_1,\ldots,\mathcal{Z}_n) = \sum \int_{\mathbb{C}^*} \frac{\mathrm{d}t}{t} \int_{\mathbb{PT}} \mathrm{D}^{3|4}\mathcal{Z} \times \mathcal{A}_L(\mathcal{Z}_1,\ldots,\mathcal{Z})\,\mathcal{A}_R(\mathcal{Z},\ldots,\mathcal{Z}_n - t\mathcal{Z}_1)\ ,$$

the BCFW recursion in twistor space.

$$\mathrm{Res}_{t=0}\,\mathcal{A}_n(\mathcal{Z}_1,\ldots,\mathcal{Z}_n) = \int \mathrm{D}^{3|4}\mathcal{Z}$$

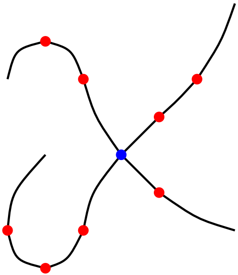

Figure: Essence of the BCFW recursion in twistor space. (Original from Cachazo, Mason, Skinner 2014 [11])

The RSVW[5] formula

$$\mathcal{A}_n = \sum_{d=0}^{\infty} \mathcal{A}_{n,d} = \sum_{d=0}^{\infty} \int \frac{\mathrm{d}^{4(d+1)|4(d+1)}U}{\mathrm{vol}\,\mathrm{GL}(2,\mathbb{C})} \prod_{i=1}^{n} \frac{\mathcal{A}_i(\mathcal{Z}(\sigma_i))\mathrm{D}\sigma_\mathrm{i}}{(i\, i+1)}$$

[5] Roiban, Spradlin, Volovich, Witten 2004 [12, 8]

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Moduli integration over degree d holomorphic maps $\mathcal{Z}(\sigma) = U_{\mathbf{a}_1 \ldots \mathbf{a}_d} \sigma^{\mathbf{a}_1} \cdots \sigma^{\mathbf{a}_d}$ from $\mathbb{CP}^1$ to $\mathbb{PT}$.

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Fixes 4 redundant degrees of freedom in the moduli integration.

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Twistor representatives given by the Penrose transform.

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Twistor representatives given by the Penrose transform.

$\mathrm{SL}(2,\mathbb{C})$-invariant inner product $(ij) := \epsilon_{\mathbf{ab}} \sigma_i^{\mathbf{a}} \sigma_j^{\mathbf{b}}$ and measure $\mathrm{D}\sigma_i := (\sigma_i \mathrm{d}\sigma_i)$ on $\mathbb{CP}^1$.

[5] Roiban, Spradlin, Volovich, Witten 2004 [12, 8]

The RSVW formula

Inductive proof:
RSVW formula contains full tree-level S-matrix of SYM!

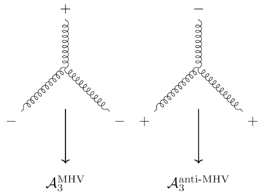
$$\mathcal{A}_{n,d} = \int \frac{\mathrm{d}^{4(d+1)|4(d+1)}\,U}{\mathrm{vol}\,\mathrm{GL}(2,\mathbb{C})} \prod_{i=1}^{n} \frac{\mathcal{A}_i(\mathcal{Z}(\sigma_i))\mathrm{D}\sigma_\mathrm{i}}{(i\,i+1)}$$

- Start with three-point (super-)amplitudes: $\mathcal{A}_{3,1} = \mathcal{A}_3^{\text{MHV}}$ and $\mathcal{A}_{3,0} = \mathcal{A}_3^{\text{anti-MHV}}$
- Show that $\mathcal{A}_{n,d}$ satisfies the BCFW recursion in twistor space (main calculation of the project).
- Assume validity of $\mathcal{A}_n$.
  Induction+BCFW $\Rightarrow$ $\mathcal{A}_{n+1}$ is correct as well!

Summary

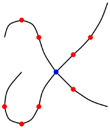
$\mathbb{PT}$

MHV



BCFW

$$\mathrm{Res}_{t^2=0}\,\mathcal{A}_n\left(\mathcal{Z}_1,\ldots,\mathcal{Z}_n\right)=\int \mathrm{D}^{3|4}\mathcal{Z}$$



RSVW

$$\mathcal{A}_n=\sum_{d=0}^{\infty}\mathcal{A}_{n,d}=\sum_{d=0}^{\infty}\int\frac{\mathrm{d}^{4(d+1)|4(d+1)}U}{\mathrm{vol}\,\mathrm{GL}(2,\mathbb{C})}\prod_{i=1}^{n}\frac{\mathcal{A}_i(\mathcal{Z}(\sigma_i))\mathrm{D}\sigma_i}{(i\,i+1)}$$

- n-point gluon amplitudes
- The MHV classification
- The Penrose transform
- RSVW formula
- 3-point amplitudes+BCFW in twistor space

RSVW contains full $\mathcal{N}=4$ $D=4$ SYM S-matrix at tree-level!

Outlook

Similar formulae based on integrals over the moduli space of holomorphic maps from $\mathbb{CP}^1$ to twistor-like spaces:

- $\mathcal{N} = 8$ supergravity (Cachazo, Mason, Skinner 2014 [11])
- Massless scattering in arbitrary dimensions (Cachazo, He, Yuan 2014 [13])
- First results with massive theories (e.g. Cachazo et. al. 2018 [14])
- Maximally supersymmetric $D = 10, 11$ theories (Geyer, Mason 2020 [15])

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Thank you for listening!

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