Collisional strong-field QED kinetic equations from quantum field theory

Gregor Fauth

Fauth, Berges, Di Piazza (2021)

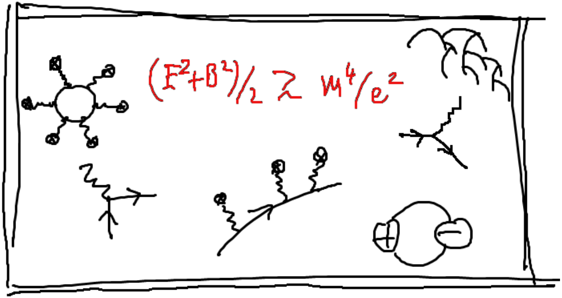
arXiv:2103.13437

Layout

1. An interesting system
2. What did we do?
3. What is 2PI QED and why start there?
4. Our strong-field QED kinetic equations
5. On-shell vs. off-shell in strong fields

An interesting system

Long-time evolution of a box with supercritical field energy:



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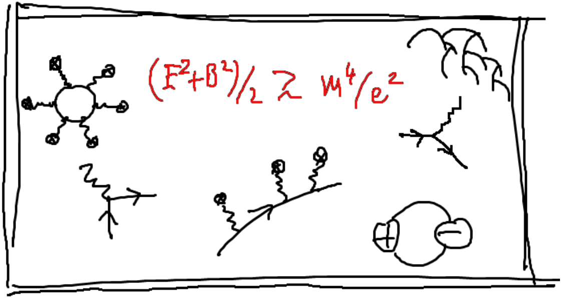

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- Off-central heavy ion collisions, laser systems, highly charged systems, ...

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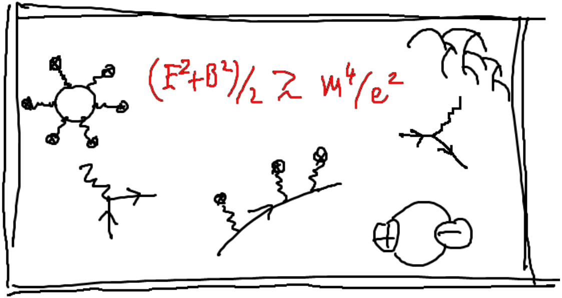

What makes this setting interesting:

- Off-central heavy ion collisions, laser systems, highly charged systems, ...
- Schwinger pair production, Photo pair production, Lorentz force, Plasma oscillations, Particle cascades ...

What are kinetic equations?

What are kinetic equations?

◦ Boltzmann (1872): nonequilibrium equation for many particles

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_X f - \vec{\nabla}_X V(\vec{X}) \cdot \vec{\nabla}_p f = C[f]\,,$$

in terms of the distribution fct. $f(\vec{X}, \vec{p}, t)$

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◦ Dynamical context for particle scattering amplitudes
before formation of asymptotic states, e.g. for $2 \leftrightarrow 2$ collisions:

$$C^{2\leftrightarrow 2}[f](X,\vec{p}) = \int_{\vec{q},\vec{k},\vec{l}} \mathrm{d}\Omega\, |\mathcal{M}|^2 \Big\{[f(X,\vec{p})+1][f(X,\vec{q})+1]f(X,\vec{k})f(X,\vec{l}) - f(X,\vec{p})f(X,\vec{q})[f(X,\vec{k})+1][f(X,\vec{l})+1]\Big\}$$

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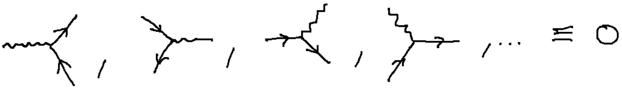
- Derivation provides the correct collision term with all leading order effects.

An interesting system

- Parametrically leading order collisions $\mathcal{O}(e^2)$ are not actually dominant without a macroscopic field because of kinematic suppression:

$$|\mathcal{M}|^2\delta(k-p+q)\delta(p^2-m^2)\delta(q^2-m^2)\delta(k^2) \equiv 0$$



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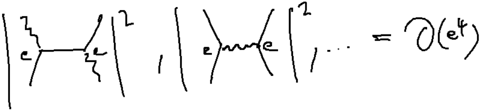
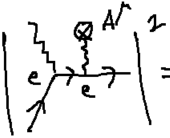
- For vanishing macroscopic field, the leading order is actually $\mathcal{O}(e^4)$, i.e. 2↔2 scatterings between $\gamma\gamma ee$ and $eeee$:



An interesting system

◦ For non-vanishing macroscopic field, the leading order contains $2 \leftrightarrow 1$ scatterings, e.g.



$$= \mathcal{O}\left(e^2 (eA)^2\right)$$

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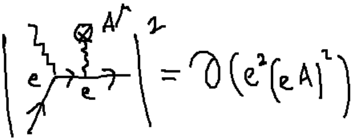
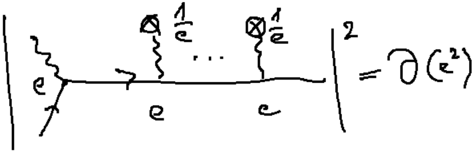
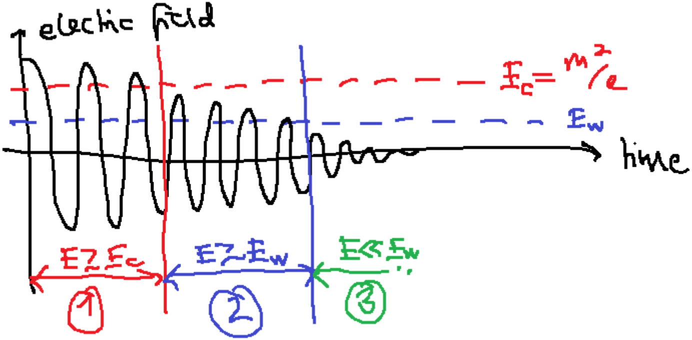
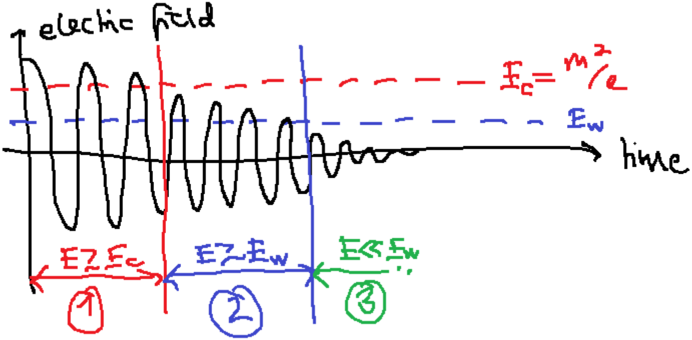


- For strong macroscopic field, $\mathcal{A}^\mu \sim 1/e$, the leading order is $\mathcal{O}(e^2)$ and is even non-perturbative in the coupling to the field:

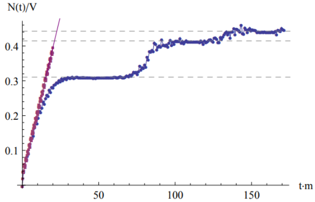



electric field
time
$E_c = m^2/e$
E_w
$E \gtrsim E_c$
$E \gtrsim E_w$
$E \ll E_w$
1
2
3

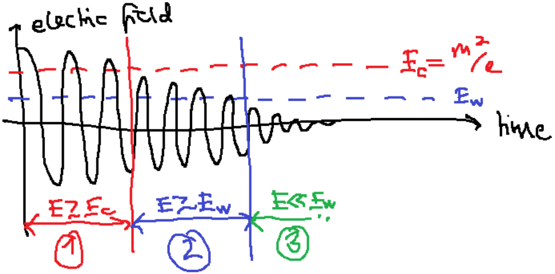


In different regimes, different effects dominate:

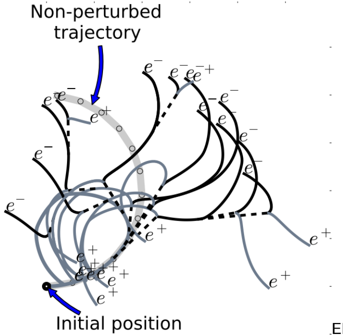
◦ 1: 1-loop (Schwinger-) and $\mathcal{O}(e^2)$-pair production



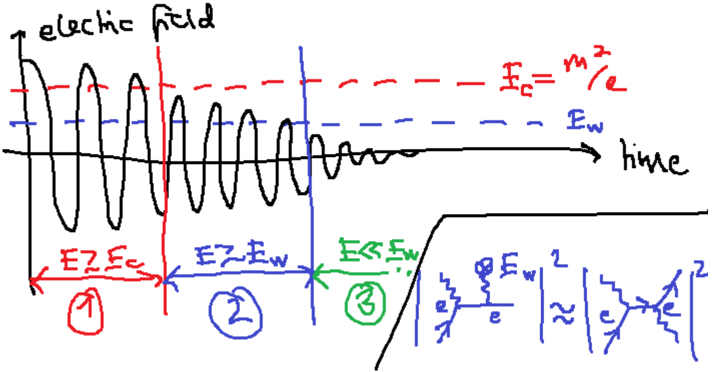
Kasper, Hebenstreit, Berges, arXiv:1403.4849 (2014)



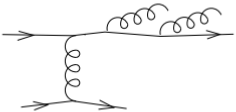
- 2: $1 \leftrightarrow 2$ $\mathcal{O}(e^2)$ collisions (seeded cascades) and Lorentz force



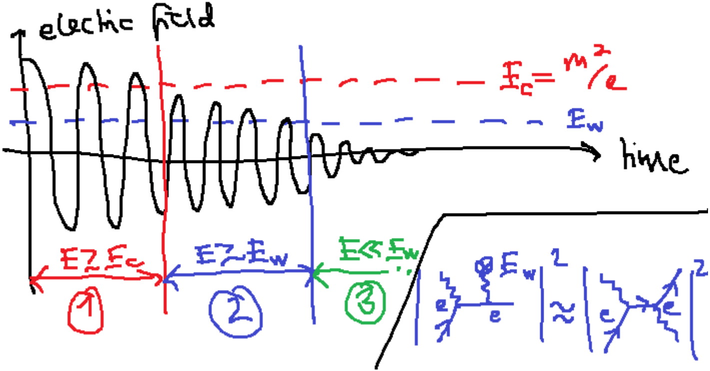
Elkina, Fedotov, Kostyukov, Legkov, Narozhny, Nerush, Ruhl (2011)



◦ 3: $\mathcal{O}(e^4)$ collisions with Landau–Pomeranchuk–Migdal effect



Arnold, Moore, Yaffe, arXiv:hep-ph/0209353 (2003)



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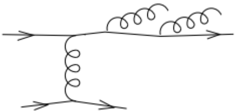

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- Changing between descriptions: problem of matching

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- Our equations do rely on ...
  - ... simple spin tensor structures;
  - ... late time limit, $t_0 \to -\infty$;
  - ... not too large photon occupations $\ll 1/e^2$;
  - ... not too large propagator gradients, $(\partial_p \cdot \partial_X)G \ll 1$.
  - ... no large initial higher correlations;

What is 2PI QED...?

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Self-consistent resummation:

$$
\begin{aligned}
G^{-1} &= G_0^{-1} - \Sigma \\
G &= G_0 + G_0 \Sigma[G_0] G && \text{(Dyson Series)} \\
G_0^{-1} G &= \mathbb{1} + \Sigma[G] G && \text{(self-consistent 2PI)}
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... and why start there?

- Allows for nonequilibrium description that captures long time evolution;
- Proven to facilitate the derivation in other settings;
- Clear separation of spectral and transport phenomena.

What is 2PI QED? ... some notation:

Spectral functions:

$$\rho^{\mu\nu}(x,y) := i\,\langle[A^\mu(x),\, A^\nu(y)]\rangle \ ,$$
$$\rho_\Psi^{AB}(x,y) := i\,\langle\{\Psi^A(x), \bar{\Psi}^B(y)\}\rangle \ .$$

Statistical (transport) functions:

$$F^{\mu\nu}(x,y) := \tfrac{1}{2}\,\langle\{A^\mu(x), A^\nu(y)\}\rangle - \langle A^\mu(x)\rangle\,\langle A^\nu(y)\rangle \ ,$$
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Spectral functions (ρ) encode relevant degrees of freedom,
e.g. in free theory $\rho \sim \delta(p^2 - M^2)$,
statistical functions (F) encode their occupations.

What is 2PI QED?

$$\left[\eta^{\mu}{}_{\sigma}\Box_x - (1-\tfrac{1}{\xi})\partial_x^{\mu}\partial_{\sigma}^{x}\right] F^{\sigma\nu}(x,y) = \int_{t_0}^{x^0} \mathrm{d}z\, \Sigma^{(\rho)}(x,z)^{\mu\gamma} F(z,y)_{\gamma}{}^{\nu} - \int_{t_0}^{y^0} \mathrm{d}z\, \Sigma^{(F)}(x,z)^{\mu\gamma} \rho(z,y)_{\gamma}{}^{\nu} \,,$$

$$\left[\eta^{\mu}{}_{\sigma}\Box_x - (1-\tfrac{1}{\xi})\partial_x^{\mu}\partial_{\sigma}^{x}\right] \rho^{\sigma\nu}(x,y) = -\int_{y^0}^{x^0} \mathrm{d}z\, \Sigma^{(\rho)}(x,z)^{\mu\gamma} \rho(z,y)_{\gamma}{}^{\nu} \,,$$

$$[i\partial\!\!\!/_x - e\mathcal{A}\!\!\!/(x) - m]_{AC} F_{\Psi}(x,y)_{CB} = \int_{t_0}^{x^0} \mathrm{d}z\, \Sigma_{\Psi}^{(\rho)}(x,z)_{AC} F_{\Psi}(z,y)_{CB} - \int_{t_0}^{y^0} \mathrm{d}z \Sigma_{\Psi}^{(F)}(x,z)_{AC} \rho_{\Psi}(z,y)_{CB} \,,$$

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$$\left[\eta^{\mu}{}_{\sigma}\Box_x - (1-\tfrac{1}{\xi})\partial_x^{\mu}\partial_{\sigma}^{x}\right] \mathcal{A}^{\sigma}(x) = -e\,\mathrm{tr}\left\{\gamma^{\mu} F_{\Psi}(x,x)\right\}$$

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In equilibrium: off-shell fluctuation dissipation relation

$$F^{\mu\nu}_{\mathrm{eq}}(p) = -i[\tfrac{1}{2} + f_{\mathrm{B}}(p^0)]\rho^{\mu\nu}_{\mathrm{eq}}(p)$$
$$F^{AB}_{\Psi,\mathrm{eq}}(p) = -i[\tfrac{1}{2} - f_{\mathrm{F}}(p^0)]\rho^{AB}_{\Psi,\mathrm{eq}}(p)$$

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Out of equilibrium: off-shell transport
with Wigner functions $G(X,p) := \int \mathrm{d}^4 s\, e^{ips}\, G(X + \frac{s}{2}, X - \frac{s}{2})$:

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In some circumstances: dynamical reduction to on-shell distributions fcts.

$$f(X,\vec{p}) := \int_0^\infty \mathrm{d}p^0 2p^0 \delta(p^2) f(X,p) = f(X,p) \text{ at } p^0 = |\vec{p}|$$
$$f^{\pm}_{\Psi}(X,\vec{p}) := f^{\pm}_{\Psi}(X,\pm p) \;\text{ at }\; p^0 = \pm\sqrt{|\vec{p}|^2 + m^2}$$

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Strong-field photon kinetic equation

$$\left[\frac{\partial}{\partial X^0} + \frac{\vec{k}}{|\vec{k}|} \cdot \frac{\partial}{\partial \vec{X}}\right] f(X, \vec{k}) = C(X, \vec{k})$$

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... with the photon $\mathcal{O}(e^2)$ collision term

$$\begin{aligned} C(X, \vec{k}) = e^2 \int_p \int_0^\infty \mathrm{d}k^0 \delta(k^2) \Big\{ \mathcal{P}^{\mu}{}_{\mu}(X, p, p-k) - \mathcal{P}_{\xi}(X, p, p-k, k) \Big\} \\ \times \Big\{ \tilde{f}_\Psi(X, p)[1 - \tilde{f}_\Psi(X, p-k)][1 + f(X, k)] \\ - [1 - \tilde{f}_\Psi(X, p)] \tilde{f}_\Psi(X, p-k) f(X, k) \Big\} \end{aligned}$$

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... and the strong-field collision kernel

$$\mathcal{P}^{\mu\nu}(X, p, q) = -\tfrac{1}{4} \mathrm{tr}\{\gamma^\mu i \hat{\rho}_\Psi(X, p) \gamma^\nu i \hat{\rho}_\Psi(X, q)\}\,.$$

... for the gauge-invariant fermion spectral function

$$\hat{\rho}_{\Psi}(x,y) := \mathcal{W}(y,x)\rho_{\Psi}(x,y) \tag{1}$$

with Wilson line $\mathcal{W}(y,x) := e^{ie\int_y^x \mathrm{d}z^{\mu}\mathcal{A}_{\mu}(z)}$ ...

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with the strong-field VGE Annals Phys. 173, 462 (1987) derivatives

$$\nabla_\mu := \frac{\partial}{\partial X^\mu} - e\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\lambda\, \mathcal{F}_{\mu\nu}(X - i\lambda\partial_p)\frac{\partial}{\partial p_\nu}$$

$$\Pi_\mu := p_\mu - ie\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\lambda\,\lambda\,\mathcal{F}_{\mu\nu}(X - i\lambda\partial_p)\frac{\partial}{\partial p_\nu}\,.$$

The scattering kernel is a generalized scattering amplitude

$$\delta(k-p+q)\mathcal{P}(X,p,q,k) \sim |\mathcal{M}|^2$$

$$\mathcal{M}^{\mu}_{\sigma\sigma'}(X,p,q,k) = \int_s e^{iks}\, \bar{u}_{p\sigma}\, \Gamma^{\mu}_{pq}(X+\tfrac{s}{2})\, u_{q\sigma'}$$

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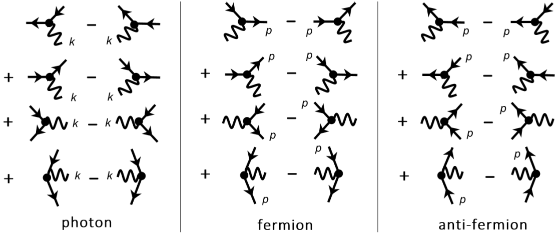
$$\mathcal{M}^{\mu}_{\sigma\sigma'}(X,p,q,k) = \int_s e^{iks}\, \bar{u}_{p\sigma}\, \Gamma^{\mu}_{pq}(X+\tfrac{s}{2})\, u_{q\sigma'}$$

... that contains all possible crossings of $\mathcal{O}(e^2)$ $1 \leftrightarrow 2$ strong-field processes:



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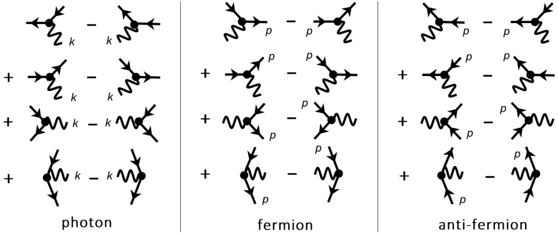

... and establishes contact to strong-field Ward identities in vacuum limit.

Our strong-field QED kinetic equations

Strong-field fermion kinetic equation with force term

$$i\nabla_\mu[\tilde{f}_\Psi(X,p)\hat{\rho}^\mu_\Psi(X,p)] = C_\Psi(X,p)$$

... with a similar collision term ...

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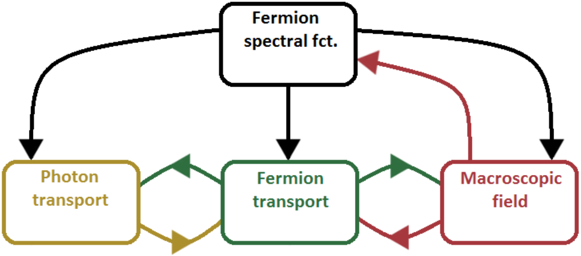
... with a similar collision term ...
... and everything coupled to the Maxwell equation for the macroscopic field

$$\left[\eta^\mu{}_\sigma \Box_X - (1-\tfrac{1}{\xi})\partial^\mu_X \partial^\sigma_X\right] \mathcal{A}_\sigma(X) = j^\mu(X)\,,$$

with the self-consistent U(1) fermion current

$$j^\mu(X) = 2e\int \frac{\mathrm{d}^4 p}{(2\pi)^4}[1-2\tilde{f}_\Psi(X,p)]i\hat{\rho}^\mu_{\Psi,\mathrm{V}}(X,p)\,.$$

Our strong-field QED kinetic equations



On-shell vs. off-shell in strong fields

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$$(i\not{\partial}_x - e\not{A}(x) - m)\rho_{\Psi}(x, y) = \int_{y^0}^{x^0} \mathrm{d}z\, \Sigma_{\Psi}^{(\rho)}(x, z)\rho_{\Psi}(z, y)$$

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- Gives a dynamic spacetime resolved generalized particle picture.

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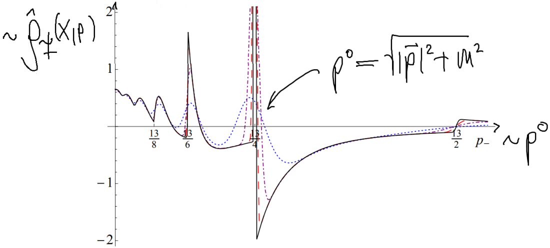
- Lesson from simple solutions: No simple peak structure!



Hebenstreit et al (2017, arXiv:1011.1923)

On-shell vs. off-shell in strong fields

- Explicit solution in external field case, $\mathcal{A}^\mu(x) \simeq \mathcal{A}^\mu_{\mathrm{v}}(n \cdot x)$:

$$\rho_{\Psi,\mathrm{v}}(X,p) = i \int \mathrm{d}l \, \mathcal{K}(X,l;p-ln)\delta(p^2 - m^2 - 2l(n\cdot p))\,\mathrm{sgn}(p^0 - ln^0)$$

$$\mathcal{K}(X,l;q) := \int \mathrm{d}s^- e^{ils^-} e^{i\mathcal{O}(e\mathcal{A})} \ldots (\not{q} + m) \ldots$$

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- Off-shell kinematics instead of $\delta(k - p + q)\delta(p^2 - m^2)\delta(q^2 - m^2) \equiv 0$:

$$k - p' + q' = (l_1 - l_2)n\,,$$
$$p^2 - m^2 = 2l_1(n \cdot p)\,,$$
$$q^2 - m^2 = 2l_2(n \cdot q)\,.$$

On-shell vs. off-shell in strong fields

- Dynamics that remain on-shell need long-lived separation of scales

$$|p_z^*| \gg \hbar |l^*| \Rightarrow \varepsilon_{l^*}(\vec{p}^{\,*}) \simeq \sqrt{|\vec{p}^{\,*}|^2 + m^2}\,.$$

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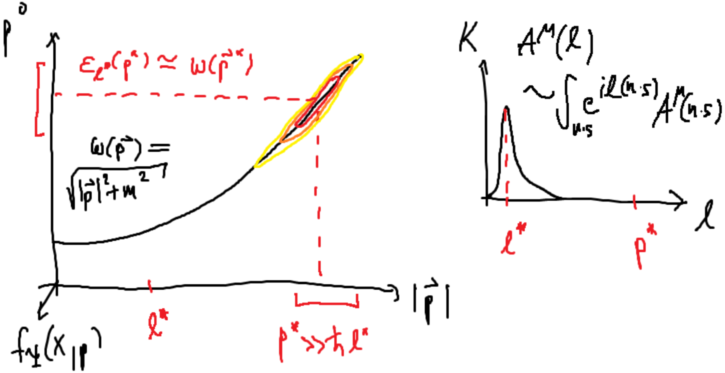

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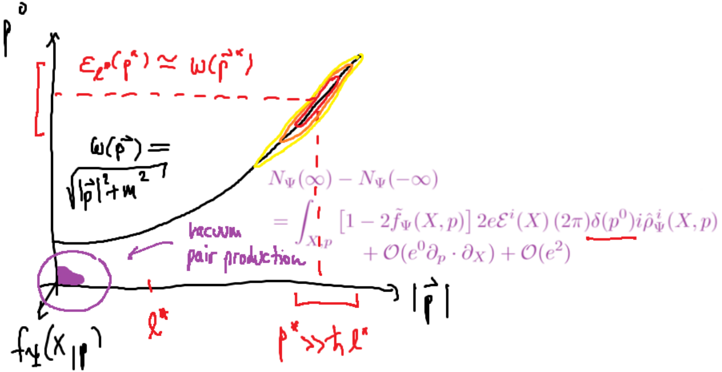

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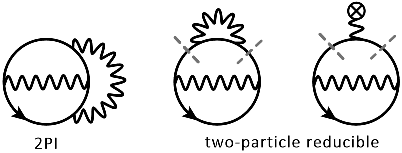
- ... contact between 2PI gauge-fixing and strong-field Ward identities.

Questions?

Appendix

◦ 2PI loop expansion is independent of the macroscopic field:



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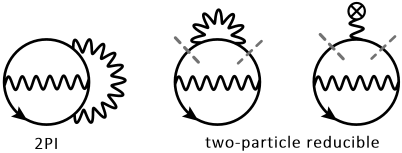
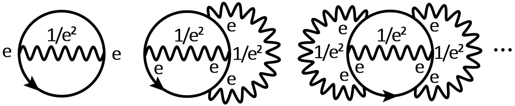


◦ 2PI loop expansion breaks down for too large photon occupations:



What is 2PI QED?

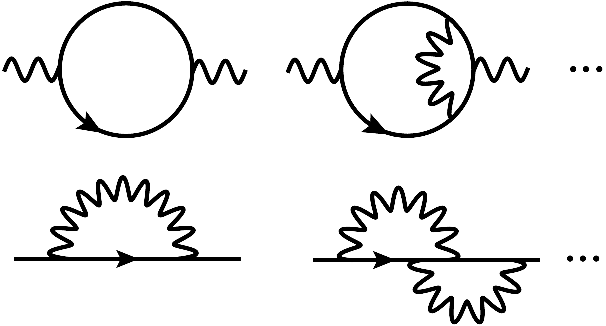
$$\Gamma[\mathcal{A}, D, \Delta] = S[\mathcal{A}] + (\mathrm{Tr}\ln\text{ - terms}) + \Gamma_2[\mathcal{A}, D, \Delta]$$

$$\Gamma_2[\mathcal{A}, D, \Delta] = (\text{sum of all 2PI diagrams with } D, \Delta, \mathcal{A}, V_0)$$

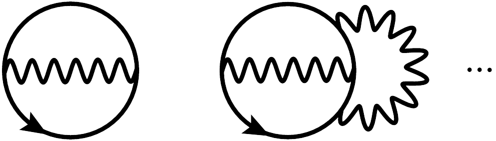
$$\frac{\delta\Gamma}{\delta\mathcal{A}^\mu} = 0 \qquad \frac{\delta\Gamma}{\delta D_{\mu\nu}} = 0 \qquad \frac{\delta\Gamma}{\delta\Delta} = 0$$

$$\Sigma[D, \Delta] := 2i\frac{\delta\Gamma_2}{\delta D} \qquad \Sigma_\Psi[D, \Delta] := -i\frac{\delta\Gamma_2}{\delta\Delta}$$

The self-energies in these equations are loop expanded self-consistently

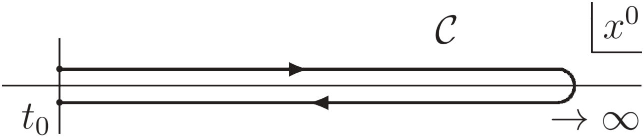


and defined from the 2PI loop expansion of the effective action



$$D^{\mu\nu}(x,y) = F^{\mu\nu}(x,y) - \tfrac{i}{2}\mathrm{sgn}_{\mathcal{C}}(x^0 - y^0)\rho^{\mu\nu}(x,y) \quad (2)$$

$$D^{\mu\nu}(x,y) := \langle \mathcal{T}_{\mathcal{C}} A^{\mu}(x) A^{\nu}(y) \rangle \quad (3)$$



Vacuum theory only needs retarded (or advanced) propagators:

$$D_{\mathrm{R}}^{\mu\nu}(x, y) = \theta(x^0 - y^0)\rho^{\mu\nu}(x, y) \tag{4}$$

$$D_{\mathrm{A}}^{\mu\nu}(x, y) = -\theta(y^0 - x^0)\rho^{\mu\nu}(x, y) \tag{5}$$

What is 2PI nonequilibrium QED?

- Hierarchy of complexity from nonequiblibrium to equilibrium to vacuum

$$\rho(t_0) \xrightarrow{\text{thermaliz.}} \rho_\beta = \frac{e^{-\beta H}}{\mathrm{Tr} e^{-\beta H}} \xrightarrow{\beta \to \infty} |\Omega\rangle \langle\Omega|$$

$$\mathrm{Tr}\left\{\rho(t_0)\mathcal{O}(x)\right\} = g(x) \qquad \mathrm{Tr}\left\{\rho_\beta \mathcal{O}(x)\right\} = const.$$

$$\mathrm{Tr}\left\{\rho(t_0)\mathcal{O}_1(x)\mathcal{O}_2(y)\right\} = g(x,y) \qquad \mathrm{Tr}\left\{\rho_\beta \mathcal{O}_1(x)\mathcal{O}_2(y)\right\} = g(x-y)$$

$$\vdots$$

- Nonequilibrium QED is necessary to describe basic electromagnetic settings with dynamic macroscopic field $\langle A \rangle = \mathcal{A}(x)$ in QFT language.

Structure of our strong-field QED kinetic equations

Kinetic equations are Wigner transformed equations of statistical propagators with spectral functions on their equations of motion

$$
\begin{aligned}
(F\mathrm{LHS})[F] &= (F\mathrm{RHS})[F, \rho_0, F_\Psi, \rho_\Psi[\mathcal{A}]] \\
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(\rho_\Psi\mathrm{LHS})[\rho_\Psi, \mathcal{A}] &= (\rho_\Psi\mathrm{RHS})[F, \rho, F_\Psi, \rho_\Psi] \\
(\mathcal{A}\mathrm{LHS})[\mathcal{A}] &= (\mathcal{A}\mathrm{RHS})[F_\Psi]
\end{aligned}
$$

$$
\int_0^\infty \frac{\mathrm{d}k^0}{(2\pi)} \int \mathrm{d}^4(x-y)\, e^{ik(x-y)} \tfrac{1}{4}\eta_{\mu\nu} \Big[(F\mathrm{EOM})^{\mu\nu}(x,y) - (F\mathrm{EOM})^{\nu\mu}(y,x)\Big]\,,
$$

$$
\int \mathrm{d}^4(x-y)\, e^{ip(x-y)}\, \mathcal{W}(y,x) \tfrac{1}{4}\mathrm{tr}\Big[(F\mathrm{EOM})_\Psi(x,y) - \gamma^0 (F\mathrm{EOM})_\Psi^\dagger(y,x)\gamma^0\Big]
$$

Fermion collision term:

$$C_\Psi(X,p) = e^2 \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \int_{-\infty}^{\infty} \mathrm{d}k^0 \delta(k^2)\mathrm{sgn}(k^0) \times \tag{6}$$
$$\times \Big\{ f_\Psi(X,p-k) f(X,k)[1 - f_\Psi(X,p)]$$
$$- [1 - f_\Psi(X,p-k)][1 + f(X,k)] f_\Psi(X,p) \Big\} \times$$
$$\times \Big\{ \mathcal{P}^\mu{}_\mu(X,p,p-k) - \mathcal{P}_\xi(X,p,p-k,k) \Big\},$$

Gauge-invariant fermion functions

$$\mathcal{W}_\gamma(y,x) := \exp\left(ie \int_\gamma \mathrm{d}z^\mu \mathcal{A}_\mu(z) \right), \tag{7}$$

$$\hat{F}_{\Psi,\gamma}(x,y) := \mathcal{W}_\gamma(y,x) F_\Psi(x,y), \tag{8}$$

$$\hat{\rho}_{\Psi,\gamma}(x,y) := \mathcal{W}_\gamma(y,x) \rho_\Psi(x,y) \tag{9}$$

$$\mathcal{W}(y,x) = \exp\left(ies^\mu \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\lambda\, \mathcal{A}_\mu(X+\lambda s) \right), \tag{10}$$

$$\hat{G}_\Psi(X,p) := \int_s e^{ips}\, \mathcal{W}(y,x) G_\Psi(x,y). \tag{11}$$

$$\hat{G}_\Psi(X,p) = G_\Psi(X, p + e\mathcal{A}(X)) + \mathcal{O}(e^0 \partial_p \cdot \partial_X). \tag{12}$$

Gauge-invariant fermion functions

$$\hat{F}_\Psi(X,p) = -i[\tfrac{1}{2} - \tilde{f}_\Psi(X,p)]\hat{\rho}_\Psi(X,p)\,. \tag{13}$$

With $\mathcal{W}(x,y)\mathcal{W}(y,x) = 1$ and $\mathcal{W}(x,x) = 1$

$$\begin{aligned}\Sigma^{\mu\nu}(x,y) &= e^2 \mathrm{tr}\{\gamma^\mu \Delta(x,y)\gamma^\nu \Delta(y,x)\} \\ &= e^2 \mathrm{tr}\{\gamma^\mu \hat{\Delta}(x,y)\gamma^\nu \hat{\Delta}(y,x)\}\,.\end{aligned} \tag{14}$$

$$\begin{aligned}j^\mu(X) &= -e \lim_{s\to 0} \mathrm{tr}\{\gamma^\mu F_\Psi(X+\tfrac{s}{2}, X-\tfrac{s}{2})\} \\ &= -e \lim_{s\to 0} \mathrm{tr}\{\gamma^\mu \hat{F}_\Psi(X+\tfrac{s}{2}, X-\tfrac{s}{2})\}\,,\end{aligned} \tag{15}$$

$$\mathcal{P}_{\mathrm{v}}^{\mu\nu}(X,p,q,k) = \delta(k^2)\mathrm{sgn}(k^0)\int\frac{\mathrm{d}l_1}{(2\pi)}\int\frac{\mathrm{d}l_2}{(2\pi)} \qquad (16)$$

$$\times\int_{p',q'}\delta(p'^2-m^2)\mathrm{sgn}(p'^0)\delta(q'^2-m^2)\mathrm{sgn}(q'^0)(2\pi)^8$$

$$\times\,\delta(p-p'-l_1 n)\delta(q-q'-l_2 n)\mathcal{Q}^{\mu\nu}(X,l_1,l_2;p',q')$$

with the remaining kernel

$$\mathcal{Q}^{\mu\nu}(X,l_1,l_2;p',q') := \int\mathrm{d}s_1^-\,e^{il_1 s_1^-}\int\mathrm{d}s_2^-\,e^{il_2 s_2^-}$$

$$\times\,\mathcal{T}_{p'q'}^{\mu\nu}(X,s_1^-,s_2^-)\,e^{-i[\mathcal{N}_{p'}(X,s_1^-)+\mathcal{N}_{q'}(X,s_2^-)]}\,. \qquad (17)$$

$$
\begin{aligned}
&-4\mathcal{T}^{\mu\nu}_{p'q'}(X, s_1^-, s_2^-)\\
&:= \mathrm{tr}\left\{\gamma^\mu\left[1+\frac{e\not{n}\not{\mathcal{A}}_{\mathrm{v}}(X+\frac{s_1}{2})}{2(n\cdot p')}\right](\not{p}'+m)\left[1-\frac{e\not{n}\not{\mathcal{A}}_{\mathrm{v}}(X-\frac{s_1}{2})}{2(n\cdot p')}\right]\right.\\
&\left.\gamma^\nu\left[1+\frac{e\not{n}\not{\mathcal{A}}_{\mathrm{v}}(X+\frac{s_2}{2})}{2(n\cdot q')}\right](\not{q}'+m)\left[1-\frac{e\not{n}\not{\mathcal{A}}_{\mathrm{v}}(X-\frac{s_2}{2})}{2(n\cdot q')}\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{N}_q(n\cdot X, n\cdot s)\\
&:= \int_{-\frac{n\cdot s}{2}}^{\frac{n\cdot s}{2}} \mathrm{d}\lambda \left(\frac{e\mathcal{A}(n\cdot X+\lambda)\cdot q}{n\cdot q} - \frac{e^2\mathcal{A}^2(n\cdot X+\lambda)}{2(n\cdot q)}\right).
\end{aligned}
$$

with the field-dependent Dirac matrix

$$\mathcal{K}(X, l; q) := \int \mathrm{d}s^- e^{ils^-} e^{-i\mathcal{N}_q(X^-, s^-)} \tag{18}$$
$$\times \left[\mathbb{1} + \frac{e}{2} \frac{\not{n}\not{\mathcal{A}}_\mathrm{v}(X + \frac{s}{2})}{(n \cdot q)}\right] (\not{q} + m) \left[\mathbb{1} - \frac{e}{2} \frac{\not{n}\not{\mathcal{A}}_\mathrm{v}(X - \frac{s}{2})}{(n \cdot q)}\right]$$

and the field-dependent phase factor

$$\mathcal{N}_q(n \cdot X, n \cdot s) \tag{19}$$
$$:= \int_{-\frac{n \cdot s}{2}}^{\frac{n \cdot s}{2}} \mathrm{d}\lambda \left(\frac{e\mathcal{A}(n \cdot X + \lambda) \cdot q}{n \cdot q} - \frac{e^2 \mathcal{A}^2(n \cdot X + \lambda)}{2(n \cdot q)} \right).$$

$$\int_{p,q} \delta(k-p+q)\,\mathcal{P}^{\mu\nu}_{\mathrm{v}}(X,p,q,k) \tag{20}$$
$$= \int_{p,q,l} \delta(p^2-m^2)\mathrm{sgn}(p^0)\delta(q^2-m^2)\mathrm{sgn}(q^0)\delta(k^2)\mathrm{sgn}(k^0)$$
$$\times\, \delta(k-p+q-ln)\,\mathcal{Q}^{\mu\nu}_{\mathrm{vac}}(X,l,p,q)\,,$$

$$\mathcal{Q}^{\mu\nu}_{\mathrm{vac}}(X,l,p,q) \tag{21}$$
$$:= \int \mathrm{d}s^-\, e^{ils^-}\, \mathcal{T}^{\mu\nu}_{pq}(X,s,-s)\, e^{-i[\mathcal{N}_p(X,s)+\mathcal{N}_q(X,-s)]}\,.$$

$$p^0 = l \pm \varepsilon_l(\vec{p}\,) \Leftrightarrow p^+ = \frac{|\vec{p}_\perp|^2 + m^2}{2p^-} + l \tag{22}$$

$$\varepsilon_l(\vec{p}\,) := (\varepsilon^2(\vec{p}\,) + l^2 - 2lp_z)^{1/2} \tag{23}$$
$$= (|\vec{p}_\perp|^2 + m^2 + (p_z - l)^2)^{1/2}\,.$$

and plane-wave fermion and antifermion distribution functions

$$f_\Psi^-(X, l, \vec{p}\,) := f_\Psi(X, p) \quad \text{at} \quad p^0 = l + \varepsilon_l(\vec{p}\,)\,, \tag{24}$$

$$f_\Psi^+(X, -l, -\vec{p}\,) := 1 - f_\Psi(X, p) \quad \text{at} \quad p^0 = l - \varepsilon_l(\vec{p}\,)\,. \tag{25}$$

Gauge fixing parameter and Ward identities

$$\mathcal{P}_\xi(X,p,q,k) := (1-\xi)\tfrac{1}{k^2}k_\mu k_\nu \mathcal{P}^{\mu\nu}(X,p,q)\,, \tag{26}$$

$$\mathcal{M}^\mu_{\sigma\sigma'}(p,q,k) = \int_x e^{ikx}\,\bar{u}_{p\sigma}\,\Gamma^\mu_{pq}(x)\,u_{q\sigma'}\,, \tag{27}$$

$$\int \mathrm{d}^4X \int_s e^{iks} = \int \mathrm{d}^4x e^{ikx} \int \mathrm{d}^4y e^{-iky}\,. \tag{28}$$

$$\begin{aligned} k\cdot\mathcal{M}_{\sigma\sigma'} = \bar{u}_{p\sigma}\not{n}u_{q\sigma'} \int_l (2\pi)^4\,\delta(k-p+q-ln)\times \\ \times \int \mathrm{d}(n\cdot x)\frac{\partial\left(e^{i\Phi[\mathcal{A}_\mathrm{v}]}\right)}{\partial(n\cdot x)}\,, \end{aligned} \tag{29}$$

$$\begin{aligned} \Phi(n\cdot x) := l\,n\cdot x + \int_{-\infty}^{n\cdot x}\mathrm{d}\lambda\Big(\frac{e\mathcal{A}(\lambda)\cdot q}{n\cdot q} - \frac{e^2\mathcal{A}^2(\lambda)}{2(n\cdot q)}\Big) \\ - \int_{-\infty}^{n\cdot x}\mathrm{d}\lambda\Big(\frac{e\mathcal{A}(\lambda)\cdot p}{n\cdot p} - \frac{e^2\mathcal{A}^2(\lambda)}{2(n\cdot p)}\Big)\,, \end{aligned} \tag{30}$$

$$k - p' + q' = (l_1 - l_2)n\,, \tag{31}$$
$$p - p' = l_1 n\,, \tag{32}$$
$$q - q' = l_2 n\,, \tag{33}$$
$$p^2 - m^2 = 2l_1(n \cdot p)\,, \tag{34}$$
$$q^2 - m^2 = 2l_2(n \cdot q)\,, \tag{35}$$
$$(p' - q')^2 = -2(l_1 - l_2)(n \cdot k)\,. \tag{36}$$