Critical Phenomena at a Many Body Exceptional Point

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Setting the Stage

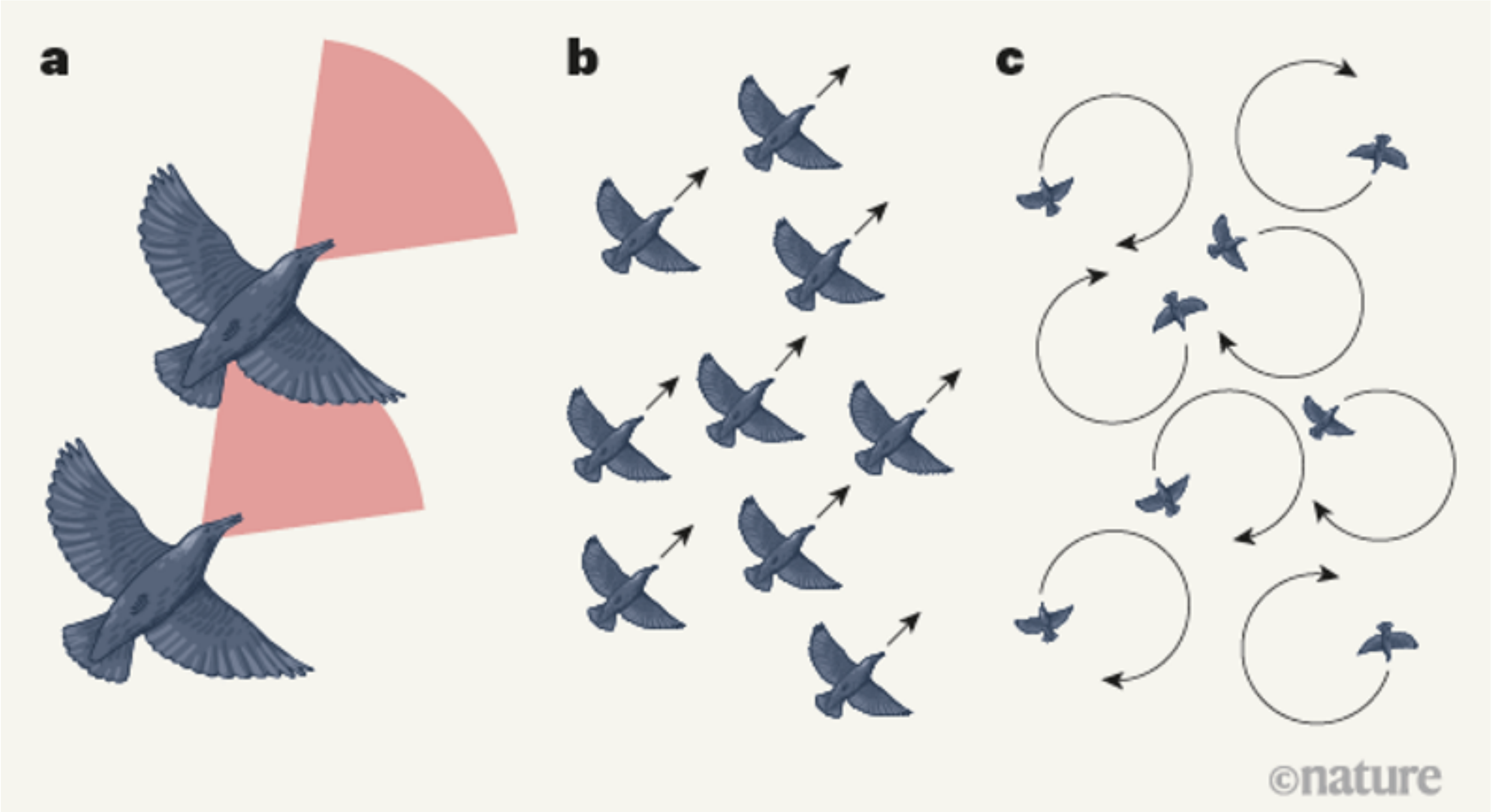
Classical systems out of equilibrium

- Open systems with gain/loss
- Active Matter
- Self Propelled Particles
- …

Nonconservative, nonreciprocal couplings

New Phases, Critical Phenomena,
New Universlity Classes?

Fruchart et al, Nature 2021



Langevin Time Evolution

$$\partial_t \Phi = F[\Phi] + \xi$$

Relaxation to Equilibrium State

$$\partial_t \Phi = -\frac{\delta V[\Phi]}{\delta \Phi} + \xi$$

Exceptional Points

EPs are *spectral singularities*

2 (or more) Eigenvectors coalesce

Rank of (Hamiltonian) Operator decreases

Pole order in Green Function changes

Hamiltonian ⇒ Jordan Block

$$H \sim \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Need non hermitian Hamiltonian ⇒ gain/loss,

nonconservative interactions

Damped Harmonic Oscillator:

$$(\partial_t^2 + 2\gamma\partial_t + \omega_0^2)\phi = 0$$

$$\phi_{1,2} = e^{-i\omega_{1,2}t},\ \omega_{1,2} = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}$$

$\gamma^2 > \omega_0^2$: Overdamped motion

$\gamma^2 < \omega_0^2$: Underdamped motion

$\gamma^2 = \omega_0^2$: Exceptional Point

Solutions: $\phi = e^{-i\gamma t}$, $\phi_{EP} = t\, e^{-\gamma t}$

Exceptional Point Phase Transitions

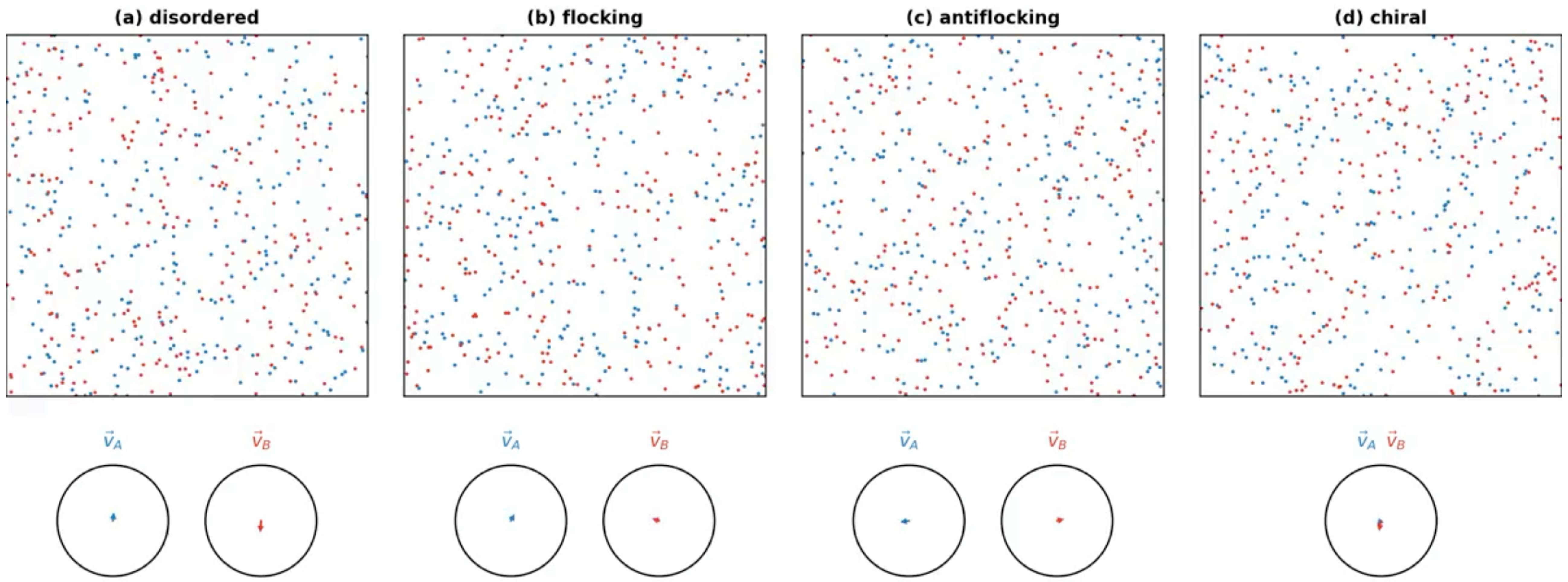
Role of EPs in Many Body Systems?

- Driven Dissipative Systems
- Active Matter
- …

Can EPs become critical

- Exponents?
- Fix Points?

Fruchart et al (Nature, 2021):
Nonreciprocity ⇒Rotating Order
Claim: Phase Transition is exceptional



The Damped U(1) model

Order Parameter Field $\phi \in \mathbb{C}$
Amplitude $\rho = \phi^* \phi$

$U(1) : \phi \to e^{i\alpha}\phi$
$\mathbb{Z}_2 : \phi \to \phi^*$

$$\partial_t^2 \phi + (2\gamma + u\rho)\partial_t \phi + (\omega_0^2 + \lambda\rho)\phi + \xi = 0$$

- Self damping
- Non conservative
- No Potential form

- Ordinary ϕ^4
- Generated by $V = \frac{\lambda}{4}\int_{t,x} \phi^4$

- Gaussian White Noise
- $\langle \xi(t)\xi(t') \rangle = 2T\delta(t - t')$

Steady State Phase Diagram

$$\partial_t^2\phi + (2\gamma + u\rho)\partial_t\phi + (\omega_0^2 + \lambda\rho)\phi = 0$$
$$\phi_{ss}(t) = \sqrt{\rho_{ss}}e^{iEt},\ \rho_{ss},\ E \in \mathbb{R}$$

Disordered Phase, $\omega_0^2 > 0$

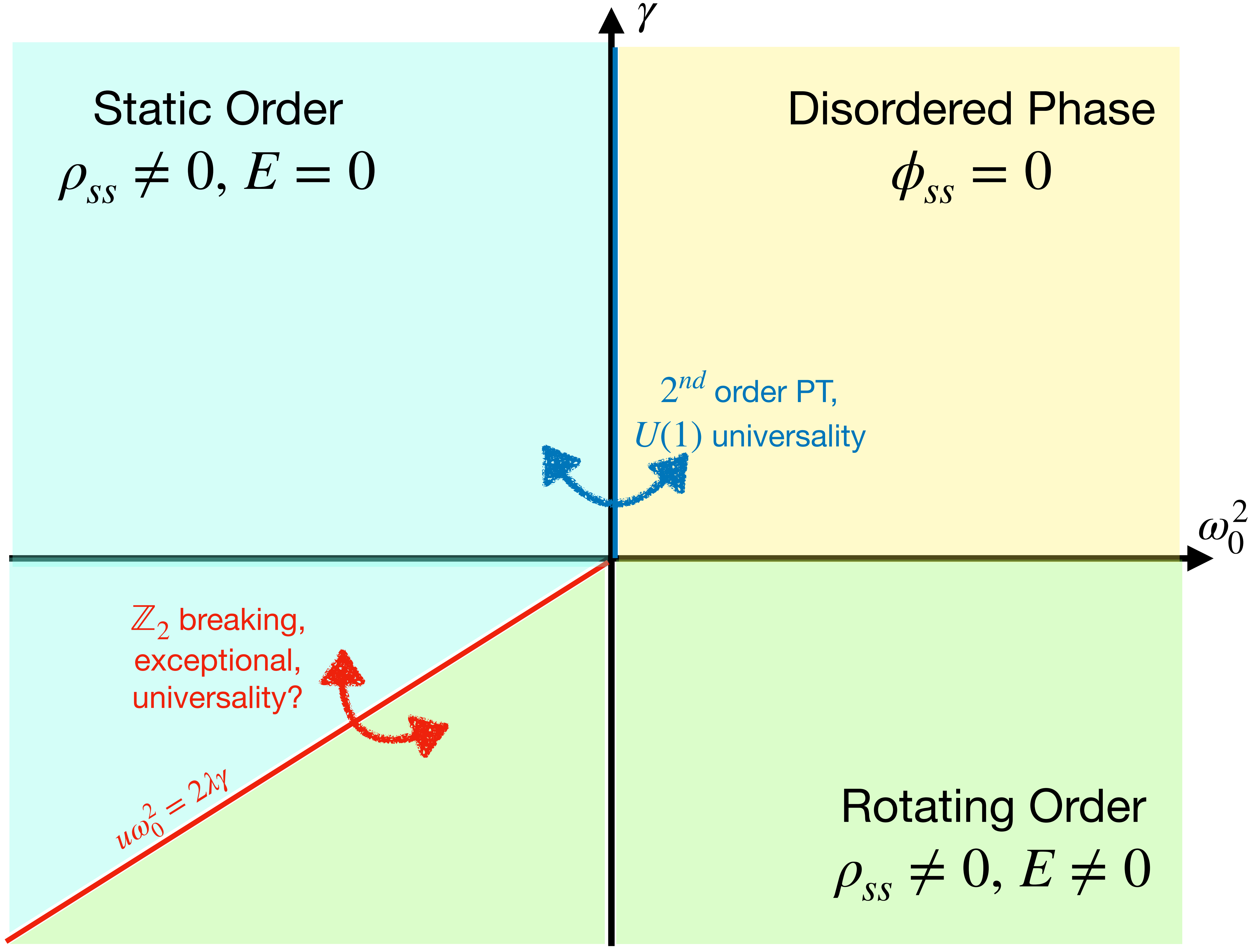
- No symmetries broken, $\phi_{ss} = 0$

Static Order, $\omega_0^2 < 0,\ 2\gamma > u\dfrac{\omega_0^2}{\lambda}$

- $U(1)$ broken, $\mathbb{Z}_2$ in tact,
  $\rho_{ss} = -\dfrac{\omega_0^2}{\lambda}, E = 0$

Rotating Order, $\gamma < 0,\ \omega_0^2 > 2\gamma\dfrac{\lambda}{u}$

- $U(1), \mathbb{Z}_2$ broken,
  $\rho_{ss} = -\dfrac{2\gamma}{u},\ E = \sqrt{\omega_0^2 - 2\gamma\dfrac{\lambda}{u}}$



Steady State Phase Diagram

Exceptional Phase Transition

$$\partial_t^2\phi + (2\gamma + u\rho - Z_1\nabla^2)\partial_t\phi$$
$$+(\omega_0^2 + \lambda\rho - Z_2\nabla^2)\phi + \xi = 0$$

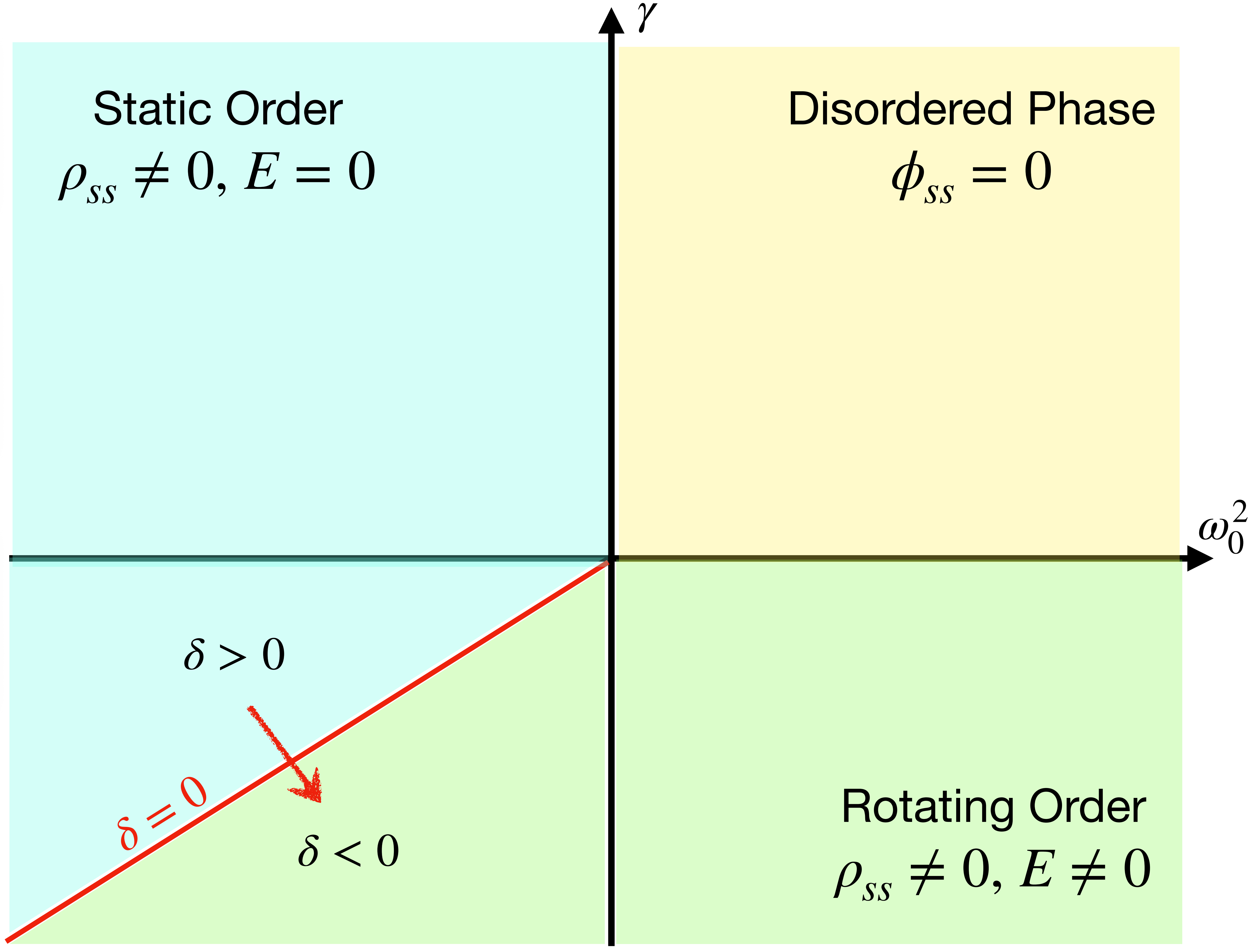
Fluctuations around ordered phase

$$\phi = \sqrt{\rho_{ss} + \rho}\,e^{i\theta} \approx \rho_{ss}(1 + i\theta) + \frac{1}{2}\rho$$

$$\partial_t^2\rho + (\delta - Z_1\nabla^2)\partial_t\rho + (-2\omega_0^2 - Z_2\nabla^2)\rho + \xi_\rho = 0$$
$$\partial_t^2\theta + (\delta - K\nabla^2)\partial_t\theta - v^2\nabla^2\theta + \xi_\theta = 0$$

ρ fluctuations gapped

$\partial_t\theta$ condenses at Critical Exceptional Point $\delta = 0$



Langenvin vs MSRJD Path Integral

Langevin Equation ⟷ **MSRJD Path Integral**

$$\partial_t^2\phi + \gamma\partial_t\phi + M[\phi]\phi + \xi = 0$$

$$\langle \xi(t)\xi(t')\rangle = 2T\delta(t - t')$$

$$\int D\phi D\tilde{\phi} e^{-iS_{MSR}[\phi,\tilde{\phi}]},$$

$$iS_{MSR} = -\int_{t,x} i\tilde{\phi}(\partial_t^2 + \gamma\partial_t + M[\phi])\phi + 2T\tilde{\phi}^2$$

Observables

Generates

Noise Averaged Expectation Values

$$\langle O\rangle_\xi$$

More MSR

In Frequency Space

$$S = \int_{\omega,p} (\tilde{\phi}(-\omega), \phi(-\omega)) \begin{pmatrix} 2T & -\omega^2 - i\gamma\omega + M \\ -\omega^2 + i\gamma\omega + M & 0 \end{pmatrix} \begin{pmatrix} \tilde{\phi}(\omega) \\ \phi(\omega) \end{pmatrix}$$

Green Function

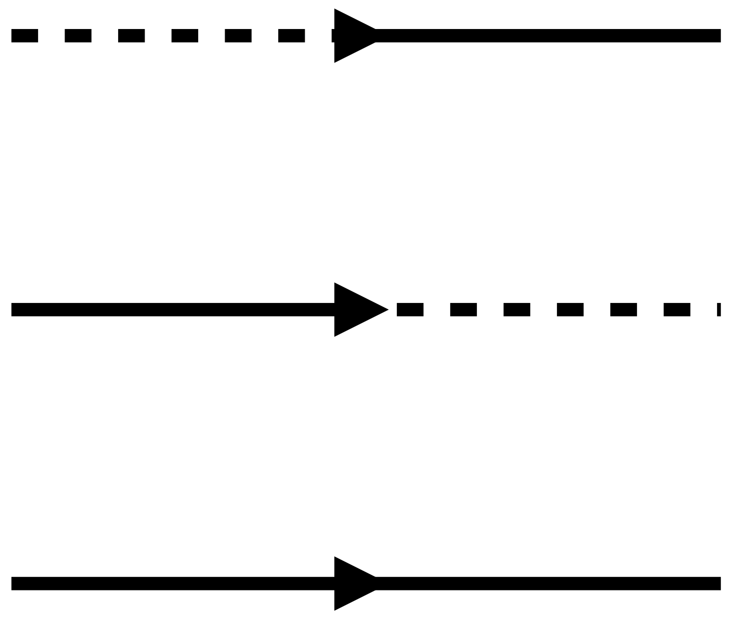
$$G = \begin{pmatrix} 0 & G^A \\ G^R & G^K \end{pmatrix} = \begin{pmatrix} \langle \tilde{\phi}\tilde{\phi} \rangle & \langle \tilde{\phi}\phi \rangle \\ \langle \phi\tilde{\phi} \rangle & \langle \phi\phi \rangle \end{pmatrix},$$

$$G^{R/A} = \chi^{R/A}, \quad G^K = \mathscr{C}$$

$G^A =$

$G^R =$

$G^K =$



MSRJD For Exceptional Phase Transition

Effective Theory for EP transition

$$\partial_t^2\theta + (\delta - K\nabla^2)\partial_t\theta - v^2\nabla^2\theta + \xi_\theta = 0$$

$$S = \int_{\omega,q} \tilde{\theta}\,(-\omega^2 - i\omega(\delta + Kq^2) + v^2q^2)\,\theta + 2T\tilde{\theta}^2$$

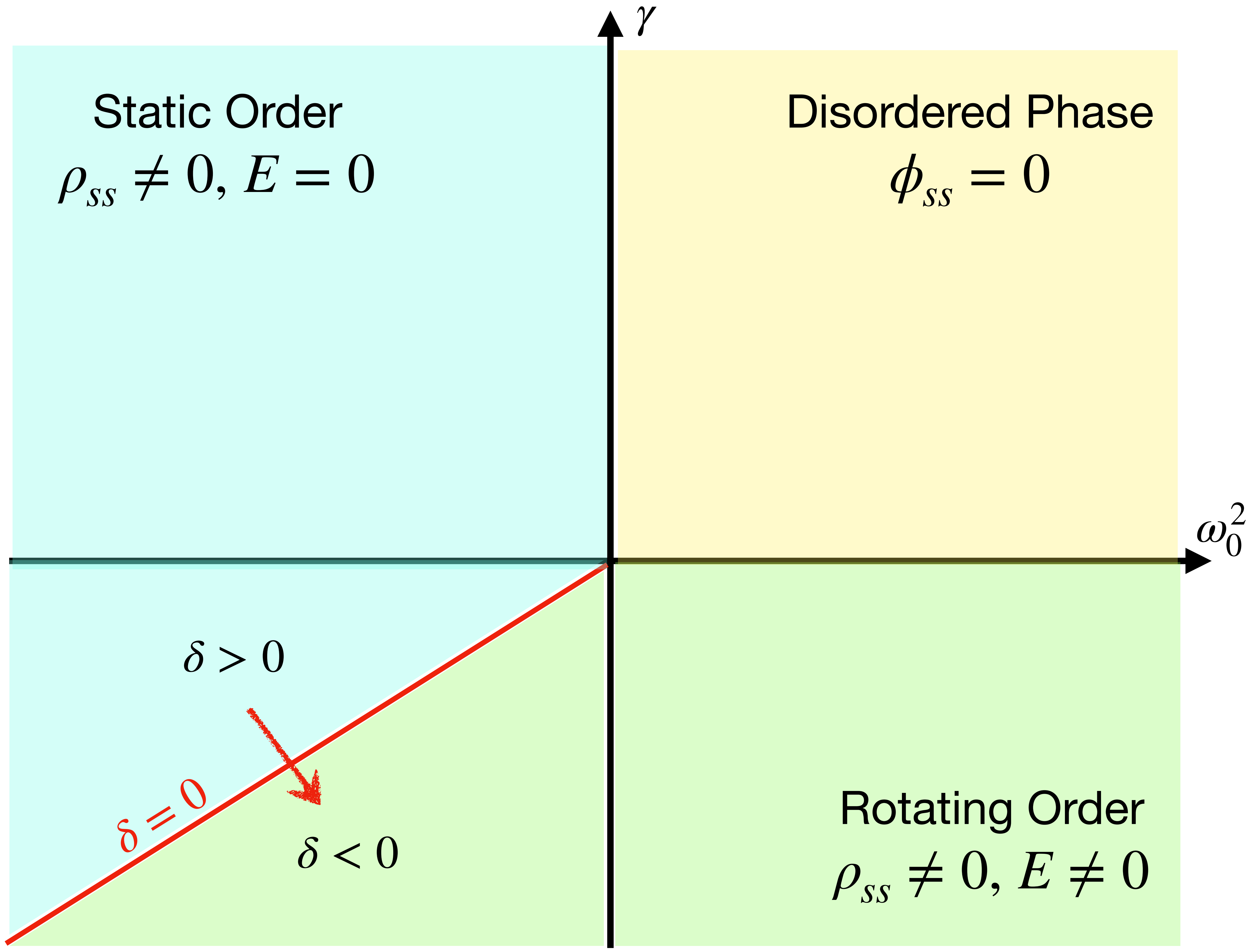
Symmetries

Goldstone

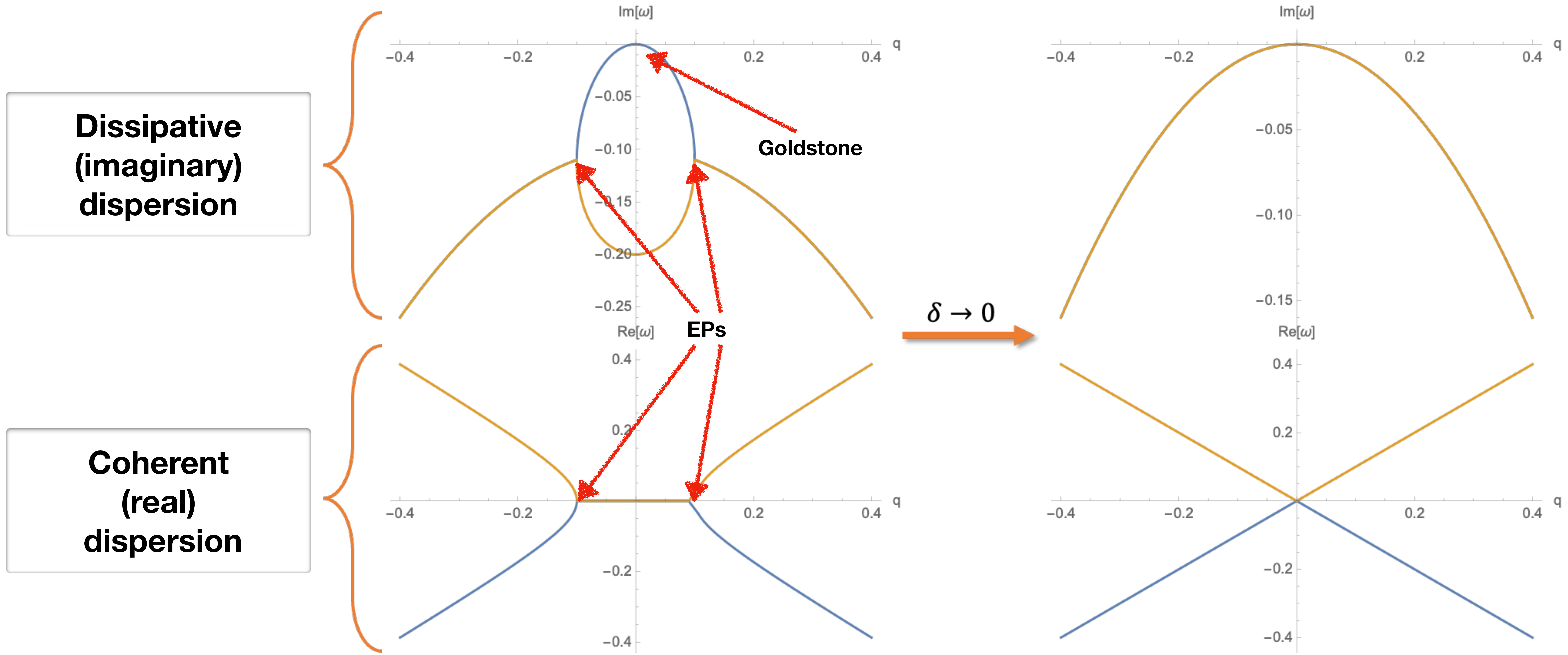
$$\mathbb{Z}_2 : \ (\theta,\tilde{\theta}) \to -(\theta,\tilde{\theta}), \ \ U(1) : \ \theta \to \theta + \alpha$$

Dispersions

$$\omega_{1,2} = -\frac{i}{2}(Kq^2 + \delta) \pm \sqrt{v^2q^2 - \frac{\delta^2}{4}}$$



Dispersions



Correlation Functions

$$G^R(q,t) = \Theta(t) 4\pi e^{-\frac{1}{2}(Kq^2+\delta)t} \frac{\sin(2v|q|t)}{2v|q|}$$

$$G^K(q,t) = \frac{2T\pi}{2(Kq^2+\delta)^2 + v^2q^2} e^{-\frac{1}{2}(Kq^2+\delta)t} \left(\frac{\sin(2v|q|t)}{2v|q|} + \frac{\cos(2v|q|t)}{Kq^2+\delta} \right)$$

Ising Model

$$G^R(q,t) \sim e^{-(q^2+\delta)t}$$

$$G^K \sim \frac{e^{-(q^2+\delta)t}}{q^2+\delta}$$

$\delta = 0,\ q \to 0$

$$G^R \sim t\, e^{-\frac{1}{2}Kq^2 t}$$

$$G^K \sim \frac{e^{-\frac{1}{2}Kq^2 t}}{v^2q^2} \left(t + \frac{1}{Kq^2} \right)$$

Ising Model Exponents

$z = 2,\ \nu = \frac{1}{2},\ G^K \sim q^{-2}$

$t \sim q^{-z} \Rightarrow z_1 = 1,\ z_2 = 2$

$$\xi = \sqrt{\frac{K}{\delta}} \Rightarrow \nu = \frac{1}{2}$$

$$G^K \sim q^{-4}$$

Exceptional Scaling
with time at EP

Couplings and Scaling dimensions

Quadratic Action

$$S^0_{MSR} = \int_{t,x} \tilde{\theta}\,(\partial_t^2 + \partial_t(\delta - K\nabla^2) - v^2\nabla^2)\,\theta + 2T\tilde{\theta}^2$$

Symmetries

$$\mathbb{Z}_2 : \ (\theta, \tilde{\theta}) \to -(\theta, \tilde{\theta}), \ \ U(1) : \ \theta \to \theta + \alpha$$

Engineering dimensions:

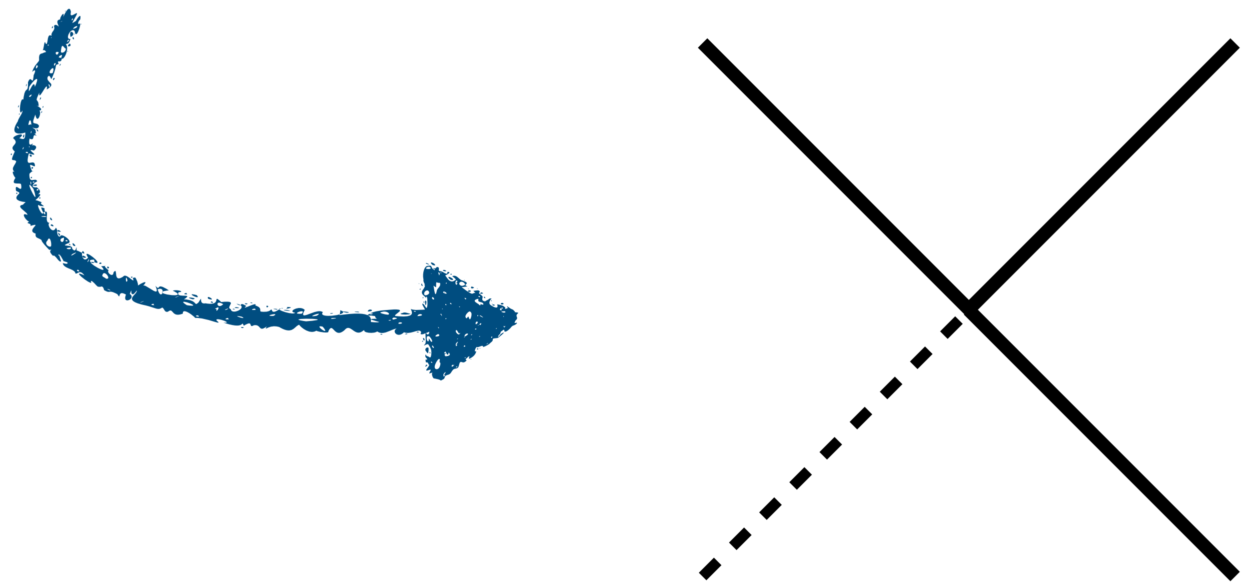
$$[v] = 0,\ [\delta] = 2,\ [K] = 0,\ [T] = 0,$$

$$z_a = 0,\ z_b = 1$$

Allowed Interactions

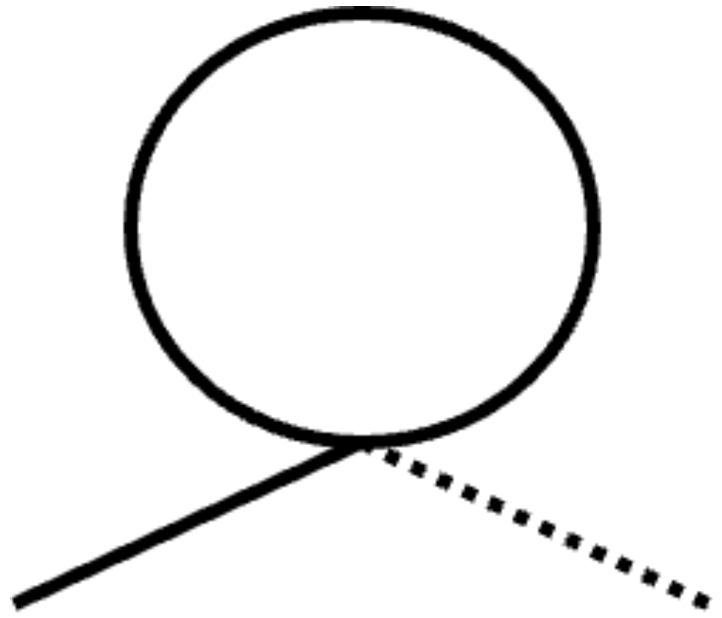
$$S_{int} = g_1 \int_{t,x} \tilde{\phi}(t,x)\Big(\partial_t \phi(t,x)\Big)^3$$

$$+ g_2 \int_{t,x} \tilde{\phi}(t,x)\partial_t\phi(t,x)\Big(\nabla\phi(t,x)\Big)^2$$

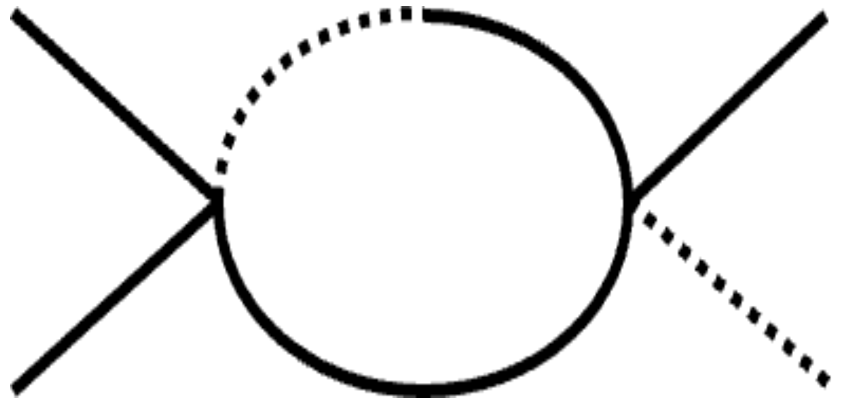


Perturbative β-Functions

Preliminary



$$\beta_\delta = 2\delta + (g_1 + g_2)A_d \frac{1}{2(1+\delta)}$$



$$\beta_{g_1} = (4-d)g_1 + g_1(g_1+g_2)\frac{3A_d}{2(1+\delta)^2}$$

$$\beta_{g_2} = (4-d)g_2 + g_2\left(g_1 + \left(\frac{4}{d}+1\right)g_2\right)\frac{A_d}{2(1+\delta)^2}$$

$$[g_1] = [g_2] = 4-d \Rightarrow d_c = 4$$

Ising Model

$$\beta_\delta = 2\delta + gA_d \frac{1}{2(1+\delta)}$$

$$\beta_g = (4-d)g + g^2 A_d \frac{3}{2(1+\delta)^2}$$

$$[g] = 4-d \Rightarrow d_c = 4$$

g_1: Ising like coupling

g_2: Induces Non Ising Universality

Fix Points and Critical Exponents

Preliminary

$d > d_c$ — Couplings irrelevant — Gaussian FP $g^*, \delta^* = 0$ is stable. Mean Field Exponents hold

$d = d_c - \epsilon$

Ising (Wilson Fisher FP)

$g^*, \delta^* \sim O(\epsilon)$

Linearize β's around FP

$$\Delta_{\delta_{WF}} = 2 - \frac{\epsilon}{3} + O(\epsilon^2),\ \Delta_{g_{WF}} = -\epsilon + O(\epsilon^2)$$

$$\nu = \frac{1}{\Delta_\delta} = \frac{1}{2} + \frac{\epsilon}{12}$$

CEP

Ising FP

$g_2^* = 0,\ g_1^* = g_{WF}^*,\ \delta^* = \delta_{WF}^*$

$$\Delta_\delta = 2 - \frac{\epsilon}{3},\ \Delta_{g_1} = -\epsilon,\ \Delta_{g_2} = \frac{2\epsilon}{3}$$

Exceptional FP

$g_1^* = 0,\ g_2^*, \delta^* \sim O(\epsilon)$

$$\Delta_\delta^{EP} = 2 - \frac{\epsilon}{2},\ \Delta_{g_1}^{EP} = -\epsilon,\ \Delta_{g_2}^{EP} = -\frac{\epsilon}{2}$$

$$\nu^{EP} = \frac{1}{2} + \frac{\epsilon}{8}$$

New Universality Class

Summary

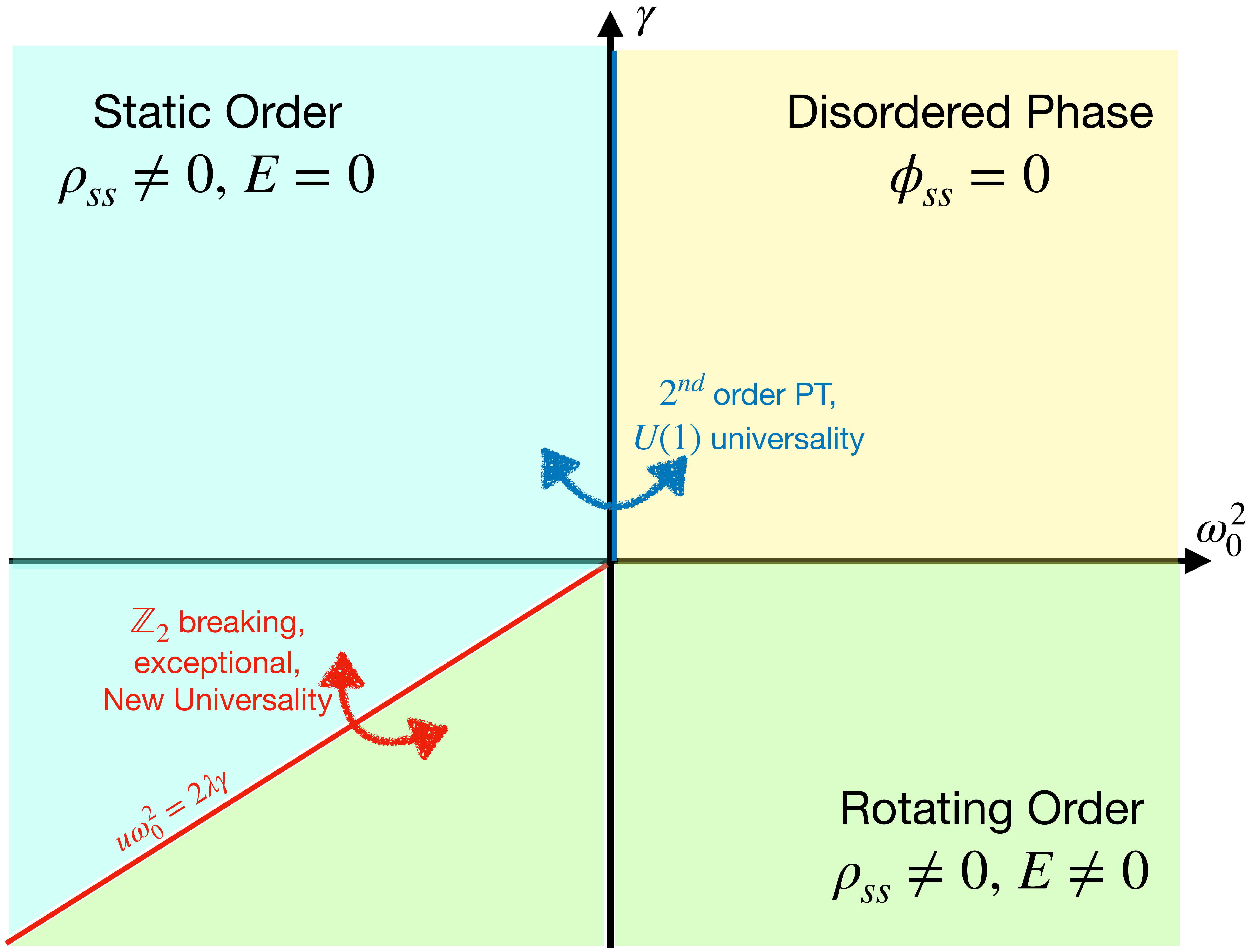
Extended $U(1)$ model with nonconservative nonequilibrium interaction

Found a new genuine nonequilibrium phase with rotating order

Exceptional $\mathbb{Z}_2$ breaking phase transition

Perturbative RG analysis

New NonEq Universality Class



Outlook

- Go beyond 1-Loop perturbation theory
- Generalize to $O(N)$ models
- Analyse Disorder to Rotating Order Phase Transition
- Endpoint of Static and Exceptional PT
- $SU(N) \Rightarrow$ Magnets
- Experimental Realisations