Functional methods for cosmic large-scale structure formation

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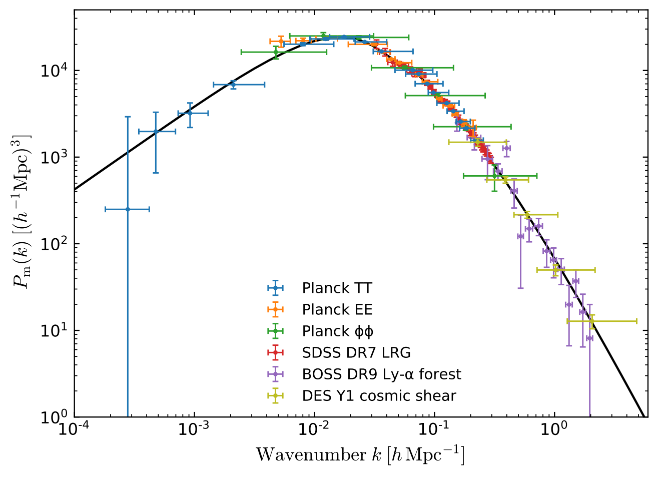
Matter power spectrum



[Planck Collaboration, A&A **641** (2020) A1]

Outline

Kinetic theory approach

Late-time large-scale cosmic structure formation

- Perturbed FLRW space-time

$$\mathrm{d}s^2 = a^2\big[-(1+2\phi)\,\mathrm{d}\tau^2 + (1-2\phi)\,\mathrm{d}\boldsymbol{x}^2\big]$$

- ΛCDM concordance model

$$T^{\mu\nu} = T^{\mu\nu}_{\Lambda} + T^{\mu\nu}_{\mathrm{matter}} + T^{\mu\nu}_{\mathrm{radiation}}$$

- Matter modelled by distribution function $f(\tau, \boldsymbol{x}, \boldsymbol{p})$
  - non-relativistic (cold)
  - collisionless (dark)
- Initial Gaussian random state

Friedmann–Vlasov–Poisson equations

- Friedmann equations

$$\Omega_{\mathrm{m}} + \Omega_{\Lambda} = 1\,, \qquad \frac{\dot{\mathcal{H}}}{\mathcal{H}} = 1 - \frac{3}{2}\,\Omega_{\mathrm{m}}$$

- Vlasov–Poisson equations

$$\partial_\tau f + \frac{p_i}{am}\,\partial_i f - am\,\partial_i\phi\,\frac{\partial f}{\partial p_i} = 0$$

$$\partial_i\partial_i\phi = \tfrac{3}{2}\mathcal{H}^2\Omega_{\mathrm{m}}\left[\int_{\boldsymbol{p}} f - 1\right]$$

Velocity moments & cumulants

- Mass density contrast

$$\rho(\tau, \boldsymbol{x}) = \bar{\rho}(\tau)[1 + \delta(\tau, \boldsymbol{x})]$$

- Normalisation

$$\int_{\boldsymbol{p}} f(\tau, \boldsymbol{x}, \boldsymbol{p}) = 1 + \delta(\tau, \boldsymbol{x})$$

- Momentum flux and stress tensor

$$\int_{\boldsymbol{p}} \frac{p_i}{am} \, f = (1 + \delta) \, u_i$$

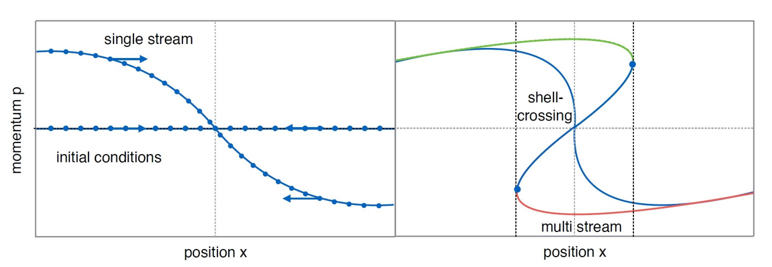
$$\int_{\boldsymbol{p}} \frac{p_i}{am} \frac{p_j}{am} \, f = (1 + \delta)(u_i u_j + \sigma_{ij})$$

Shell-crossing

- Single-stream approximation

$$f(\tau, \boldsymbol{x}, \boldsymbol{p}) \propto [1 + \delta(\tau, \boldsymbol{x})]\, \delta_{\rm D}^{(3)}(\boldsymbol{p} - am\boldsymbol{u}(\tau, \boldsymbol{x}))$$

- Shell-crossing [C. Uhlemann, JCAP **10** (2018) 030]



- Including the velocity dispersion tensor

$$f \propto (1+\delta)\exp\left\{-\frac{1}{2}\left(\frac{p_i}{am}-u_i\right)(\sigma^{-1})_{ij}\left(\frac{p_j}{am}-u_j\right)\right\}$$

- Thermal state

$$f \propto \exp\left\{-\frac{\boldsymbol{p}^2}{2mk_{\mathrm{B}}T}\right\}$$

- Effective theory
  - Cannot describe shell-crossing microscopically
  - Effective description of a multi-stream fluid
  - Captures Maxwell–Boltzmann distribution

Functional methods

Statistical cosmology

- Interested in statistical properties of

$$\psi_a = \Big(\delta, \quad u_i, \quad \sigma_{ij}, \quad \ldots\Big)$$

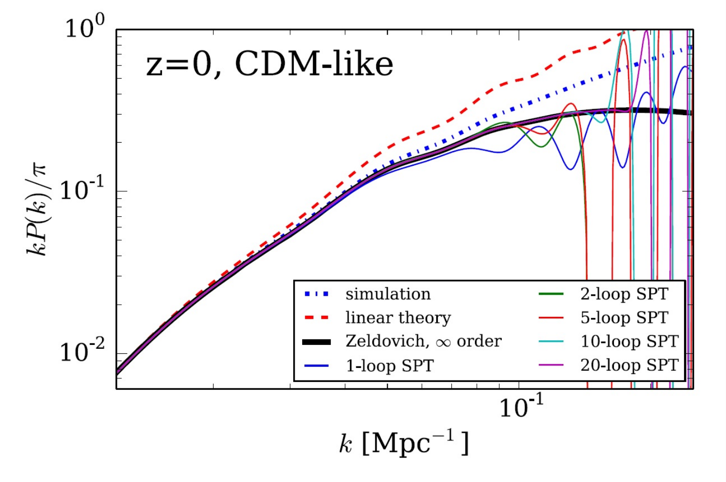
- E.g. covariance function

$$\langle \psi_a(\tau, \boldsymbol{x}) \, \psi_b(\tau', \boldsymbol{x}') \rangle_{\mathrm{c}} = C_{ab}(\tau, \tau', \boldsymbol{x} - \boldsymbol{x}')$$

- Power spectral density

$$C_{ab}(\tau, \tau', \boldsymbol{x} - \boldsymbol{x}') = \int_{\boldsymbol{q}} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \, P_{ab}(\tau, \tau', \boldsymbol{q})$$

Standard perturbation theory



[M. McQuinn & M. White, JCAP **01** (2016) 043]

Martin–Siggia–Rose/Janssen–de Dominicis formalism

- Generating functional

$$\mathrm{e}^{W[J,\hat{J}]} = \int \mathcal{D}\psi \int \mathcal{D}\hat{\psi}\ \mathrm{e}^{-S+J\cdot\psi+\hat{J}\cdot\hat{\psi}}$$

- Bare action

$$S[\psi,\hat{\psi}] = -\,\mathrm{i}\int \hat{\psi}_a\Big[\partial_\tau\psi_a + \Omega_{ab}\psi_b + \gamma_{abc}\psi_b\psi_c\Big] + \int \hat{\psi}_a\Big[\mathrm{i}\,\Psi_a^{\mathrm{in}} + \tfrac{1}{2}P_{ab}^{\mathrm{in}}\hat{\psi}_b\Big]$$

- Connected two-point correlation functions

$$W^{(2)}_{\boldsymbol{AB}} = \delta(\boldsymbol{q}+\boldsymbol{q}')\begin{pmatrix} P_{ab}(\tau,\tau',\boldsymbol{q}) & \mathrm{i}\,G^{\mathrm{R}}_{ab}(\tau,\tau',\boldsymbol{q}) \\ \mathrm{i}\,G^{\mathrm{A}}_{ab}(\tau,\tau',\boldsymbol{q}) & 0 \end{pmatrix}$$

One-particle irreducible effective action

- Effective action

$$\Gamma[\Psi, \hat{\Psi}] = \sup_{J, \hat{J}} \left[J \cdot \Psi + \hat{J} \cdot \hat{\Psi} - W \right]$$

- Effective equations of motion

$$\Gamma^{(1)}_{\boldsymbol{A}} = J_{\boldsymbol{A}}$$

- Two-point functions

$$\Gamma^{(2)}_{\boldsymbol{AB}} W^{(2)}_{\boldsymbol{BC}} = \delta_{\boldsymbol{AC}}$$

- Dyson–Schwinger equation

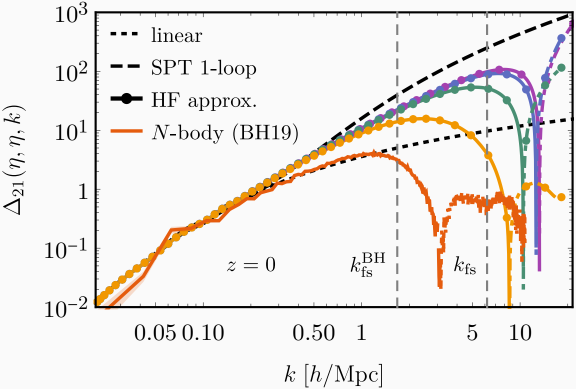
$$\Gamma^{(1)}_{\boldsymbol{A}}[\Psi] = S^{(1)}_{\boldsymbol{A}}\left[\psi = \Psi + \left[\Gamma^{(2)}\right]^{-1} \cdot \frac{\delta}{\delta\Psi}\right]$$

- In the current context

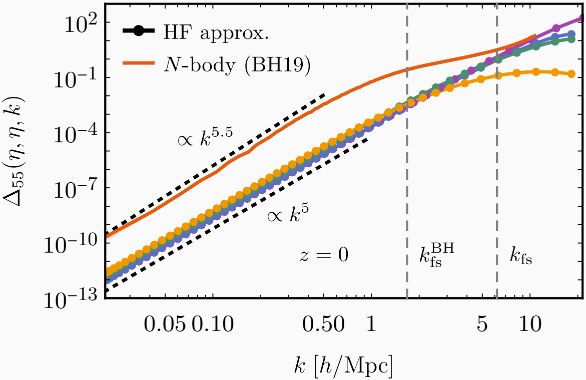
$$\Gamma^{(n)} = \mathrm{Func}\big[\big\{\Gamma^{(m\leq n+1)}\big\}\big]$$

- Closure with e.g.
  - 1PI effective action loop expansion
  - Hartree(–Fock) approximation

Velocity-density cross-spectrum [AE & S. Floerchinger (in prep.)]



Vorticity auto-spectrum [AE & S. Floerchinger (in prep.)]



- Regulate the spectrum of initial fluctuations

$$P^{\mathrm{in}}_{ab}(\boldsymbol{q}) \mapsto P^{\mathrm{in}}_{k,ab}(\boldsymbol{q}) = \theta(k-q)\, P^{\mathrm{in}}_{ab}(\boldsymbol{q})$$

- Scale-dependent effective action

$$\lim_{k\to 0} \Gamma_k = S\,, \qquad \lim_{k\to\infty} \Gamma_k = \Gamma$$

- Renormalisation group flow equation

$$\partial_k \Gamma_k = \tfrac{1}{2}\,\mathrm{Tr}\Big[\Big[\Gamma^{(2)}_k + (P^{\mathrm{in}}_k - P^{\mathrm{in}})\Big]^{-1} \cdot \partial_k P^{\mathrm{in}}_k\Big]$$

Closing the flow equations

- Symmetries and related Ward identities, e.g. extended Galilean invariance [AE & S. Floerchinger, PRD **105** (2022) 023506]

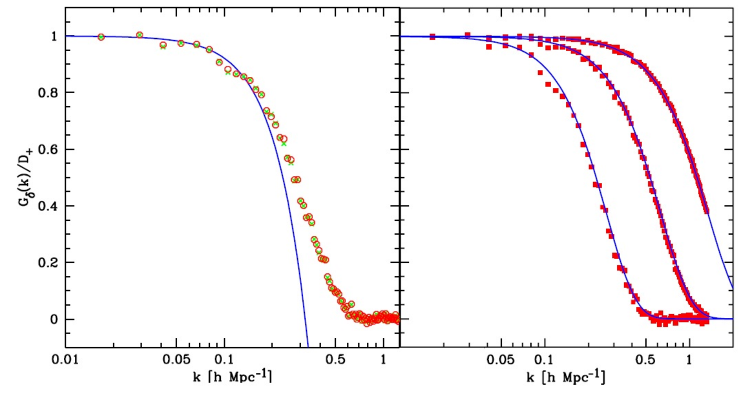
$$\lim_{|\boldsymbol{q}|\to\infty} \partial_k \Gamma_k^{(2)}(\boldsymbol{q}) = \mathrm{Func}\left[\Gamma_k^{(2)}\right]$$

- Effective action ansatz, e.g. time-local dynamics

$$\Gamma_k = -\mathrm{i} \int \hat{\Psi}_a \left[\partial_\tau \Psi_a + \Omega_{k,ab}\Psi_b + \gamma_{k,abc}\Psi_b\Psi_c\right] + \int \hat{\Psi}_a \left[\mathrm{i}\, Q_{k,a} + \tfrac{1}{2} H_{k,ab}\hat{\Psi}_b\right]$$

(Resummed) perturbation theory result

Linear response function [M. Crocce & R. Scoccimarro, PRD **73** (2006) 063520]



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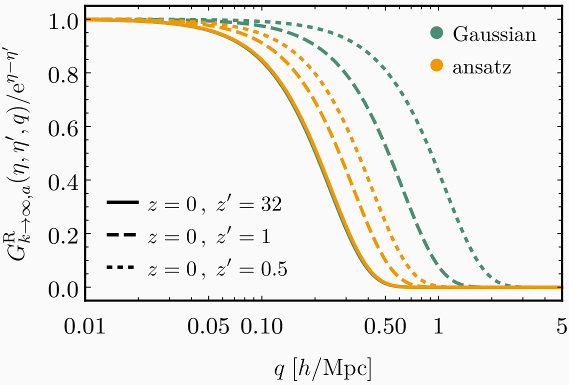

Scale-dependent fields

- Flow of continuity equation

$$\partial_\tau \delta + \underbrace{\partial_i[(1+\delta)u_i] + \partial_i\partial_i F_k(\delta, \boldsymbol{u})}_{\partial_i\left[(1+\delta)\tilde{u}_{k,i}\right]} = 0$$

- Flow equation for $\tilde{\Gamma}_k[\Phi] = \Gamma_k[\Psi_k[\Phi]]$

$$\partial_k \tilde{\Gamma}_k[\Phi] = \partial_k \Gamma_k[\Psi]\Big|_{\Psi=\Psi_k[\Phi]} + \frac{\delta \Gamma_k[\Psi]}{\delta \Psi}\Big|_{\Psi=\Psi_k[\Phi]} \cdot \partial_k \Psi_k[\Phi]$$

- Effective action ansatz

$$\tilde{\Gamma}_k = -\mathrm{i} \int \hat{\Psi}_{k,a} \Big[\partial_\tau \Psi_{k,a} + \Omega_{k,ab}\Psi_{k,b} + \gamma_{k,abc}\Psi_{k,b}\Psi_{k,c}\Big] + \dots$$

Conclusion and outlook

- Dark matter description beyond perfect pressureless fluid approximation to capture shell-crossing
- Generic need for non-perturbative methods to address small-scale physics
- Dyson–Schwinger equation captures shell-crossing and is well-behaved in non-linear regime
- Functional renormalisation group particularly interesting to make use of symmetries and conservation laws