# Confinement in non-Abelian lattice gauge theory via persistent homology

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Cold Quantum Coffee

November 29th, 2022, ITP Heidelberg

IsoQuant SFB1225

STRUCTURES CLUSTER OF EXCELLENCE

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1. Cubical complexes and persistent homology
2. Confinement via different filtrations
3. Conclusions & outlook

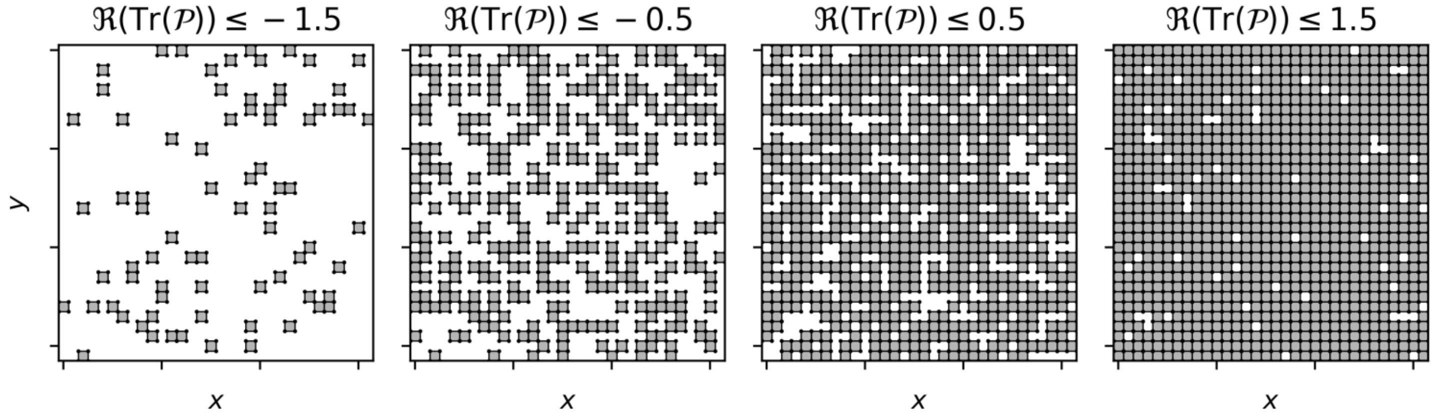
# Cubical complexes

Cubical complex is collection of cubes of different dimensions, **closed under taking boundaries**.

For $f : \Lambda \to \mathbb{R}$ some function on lattice $\Lambda$, sublevel sets $M_f(\nu) := \{x \in \Lambda \,|\, f(x) \leq \nu\}$ form a **filtration**, i.e. a nested sequence of sets "interpolating" between $\emptyset$ and $\Lambda$,

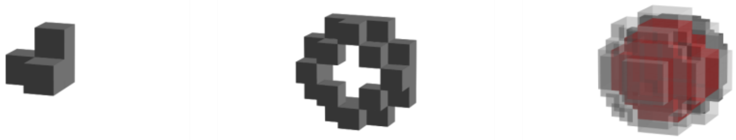
$$M_f(\nu) \subseteq M_f(\mu) \qquad \forall\, \nu \leq \mu$$

"Pixelization" leads to filtration of cubical complexes.



# Homology

Cubical complexes can contain holes of different dimensions (e.g., 0 to 2, from left to right):



Given complex $\mathcal{C}$, homology groups can be computed in different dimensions, $H_\ell(\mathcal{C})$.

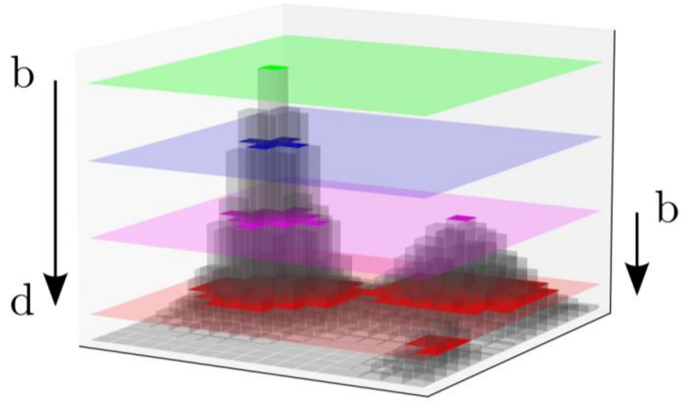
Their Betti numbers count independent $\ell$-dimensional holes:

$$\beta_\ell(\mathcal{C}) := \dim H_\ell(\mathcal{C})$$

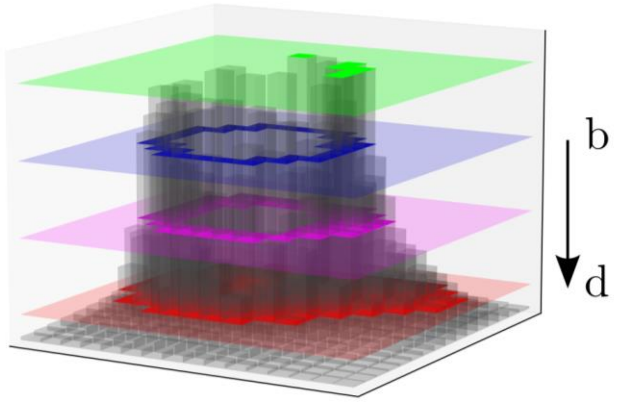
# *Persistent* homology

Homology of the sublevel sets $M_f(\nu)$ generically changes with $\nu$. Holes can be born at **birth** parameter $b$ and die again with **death** $d$, possibly deforming as filtration is swept through. Have with **persistence** $p = d - b$ a measure of dominance of a feature. [Edelsbrunner, Letscher, Zomorodian 2000; Zomorodian & Carlsson 2005]

Example for superlevel sets of a function on a surface:




Dimension 0




Dimension 1

# *Persistent* homology

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**Important properties:**

Persistent homology is stable: Small changes in $f$ result in small changes of persistent homology. [Cohen-Steiner *et al.*, 2007 & 2010; Bauer & Lesnick, 2013]

Well-defined large-volume asymptotics exist for suitable persistent homology descriptors such as (smoothened) Betti numbers, including notions of ergodicity. [Hiraoka, Shirai, Trinh 2018; DS & Wienhard 2020]

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# SU(2) lattice gauge theory simulations

[DS, Urban, Pawlowski, arXiv:2208:03955 (submitted to Scipost Phys.)]

Goal: Can we gauge-invariantly and without a bias towards particular field configurations observe properties of excitations related to confinement via persistent homology?

Carry out **Hybrid Monte Carlo simulations** on 4d Euclidean $32^3 \times 8$ lattice with periodic boundary conditions. No gauge fixing applied. Samples are SU(2)-valued links $U_\mu(x)$, following Wilson action, $\beta = 1/g^2$:

[Duane *et al.*, 1987; Gattringer & Lang 2010]

$$S[U] = \frac{\beta}{2} \sum_{x \in \Lambda} \sum_{\mu < \nu} \mathrm{Tr}[1 - U_{\mu\nu}(x)]$$

Compare multiple times to **cooled configurations** (partially removed UV fluctuations), using standard Wilson flow: [Lüscher, JHEP 2010]

$$\partial_t U_\mu(x,t) = -g^2 (\partial_{x,\mu} S[U(t)]) U_\mu(x,t)$$

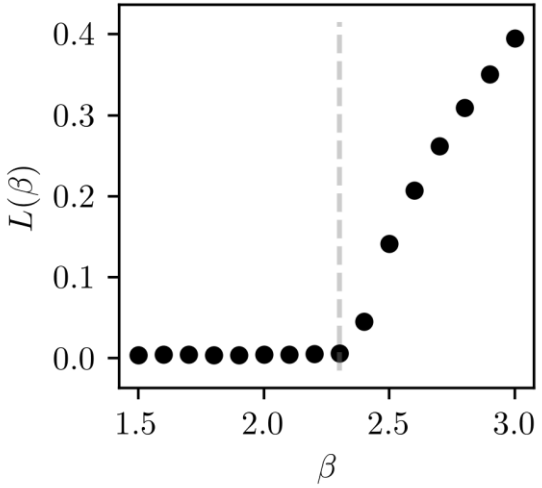
with ini. cond. $U_\mu(x,0)$ given by sampled field configuration without cooling.

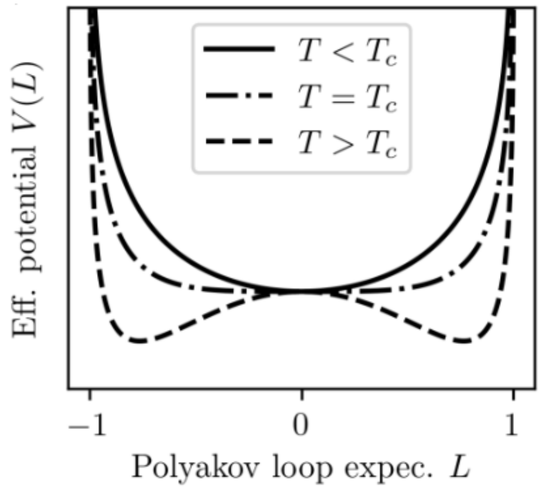
# Common pheno of SU(2) confinement

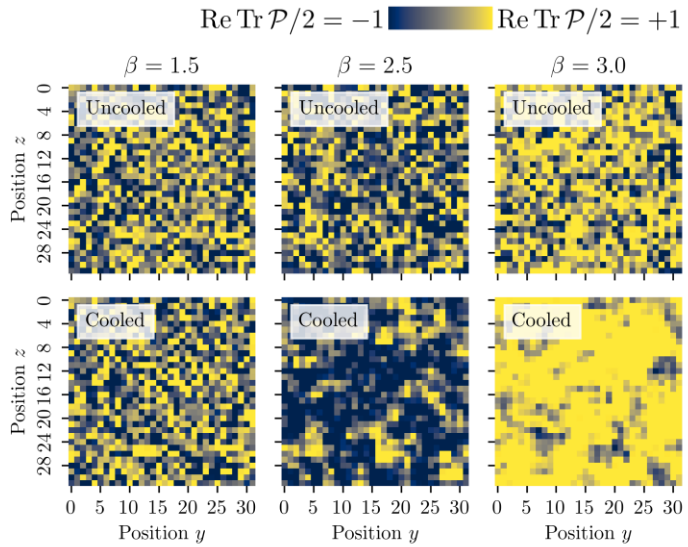
Theory is confining at low $\beta$ as signalled by zero Polyakov loop:

$$P(\mathbf{x}) := \frac{1}{2}\mathrm{Tr}\ \mathrm{P}\prod_{\tau=1}^{N_\tau} U_4(\mathbf{x},\tau), \qquad L := \frac{1}{N_\sigma^3}\langle|\sum_{\mathbf{x}\in\Lambda_s} P(\mathbf{x})|\rangle$$

**Spontaneous center symmetry breaking** in Polyakov loop traces above $\beta_c \simeq 2.3$:




Note that SSB here **above** crit. temperature, different from models such as Ising (there exist heuristic dual descriptions with inverse temperatures for eff. Polyakov models). [e.g., Fukushima & Skokov 2017]

# Instanton-dyons

Evidence for driving via topological excitations, also for chiral symmetry breaking (think of chiral anomaly), require interactions with Polyakov loops (since Polyakov loop can serve as an order parameter).

Top. excitations couple to spatial infinity Polyakov loops (holonomies), which we can diagonalize:

$$\lim_{|\mathbf{x}|\to\infty} \mathcal{P}(\mathbf{x}) = \begin{pmatrix} \exp(2\pi i \mu_1) & 0 \\ 0 & \exp(2\pi i \mu_2) \end{pmatrix}$$

with $\mu_1 \leq \mu_2$ and $\mu_1 + \mu_2 = 0$, related masses of dyons $8\pi^2(\mu_2 - \mu_1)$ and $8\pi^2(1 + \mu_1 - \mu_2)$ .

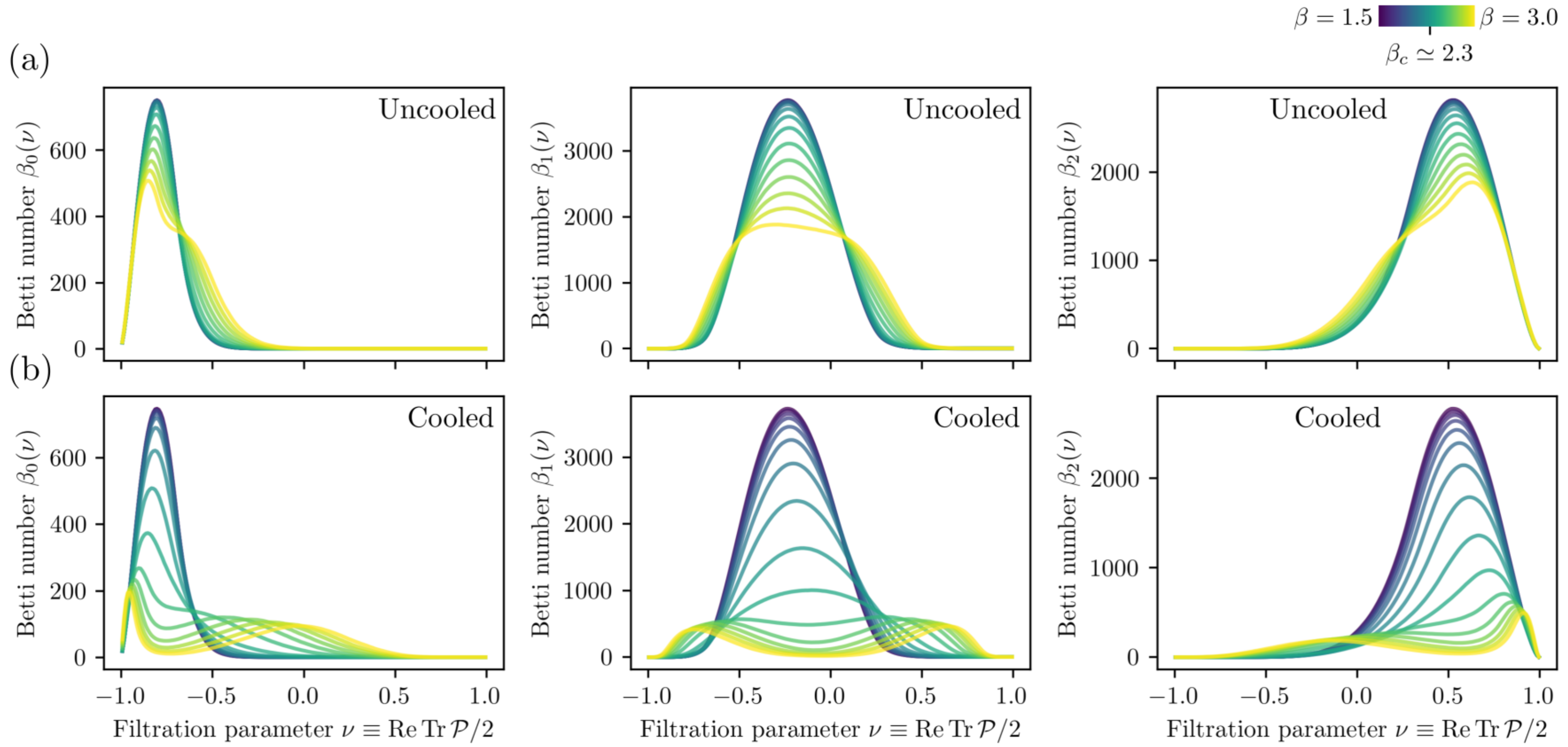
[Kraan & van Baal 1998; Lee & Lu 1998]

Calorons consist of two monopole constituents, so called-dyons (since they are electric and magnetic monopoles). Instanton-dyons are (anti-)self-dual solutions to classical YM eqs.

Ensembles of dyons can account for confinement in theories with trivial gauge group center.

[Diakonov & Petrov 2011]

# Sublevel set filtration of $P(\mathbf{x})$



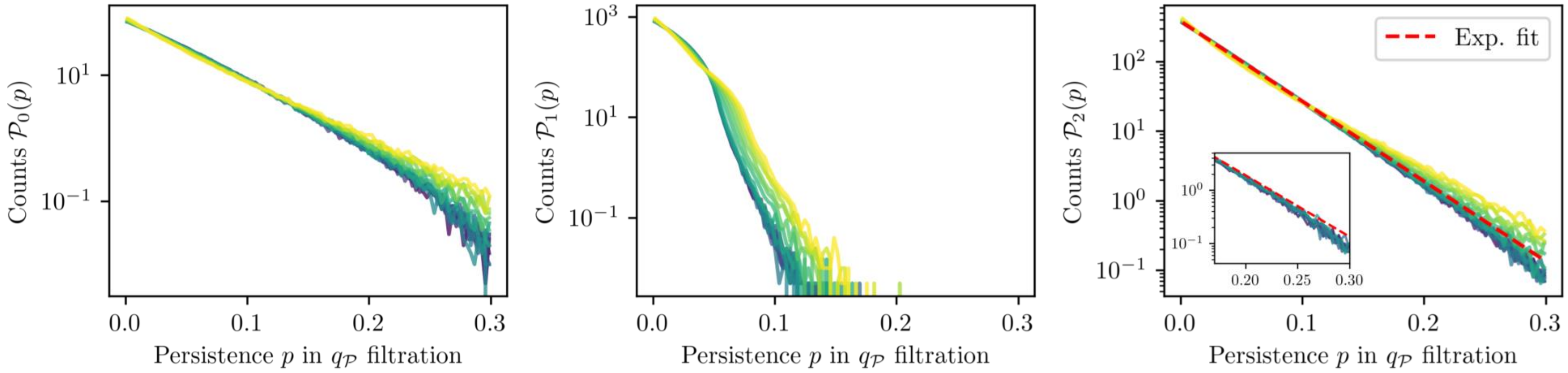

Clear evidence for spontaneously broken center symmetry, effects pronounced by cooling

# Sublevel sets of Polyakov loop topological density

Usual topological density $q \sim \mathrm{Tr}\,\mathbf{E}\cdot\mathbf{B}$ often contains strong UV fluctuation signatures.

Can rewrite topological charge as integral over 3-torus with integrand the Polyakov loop topological density: [Ford *et al.* 1998]

$$q_{\mathcal{P}}(\mathbf{x}) := \frac{1}{24\pi^2}\varepsilon_{ijk}\mathrm{Tr}[(\mathcal{P}^{-1}\partial_i\mathcal{P})(\mathcal{P}^{-1}\partial_j\mathcal{P})(\mathcal{P}^{-1}\partial_k\mathcal{P})]$$




Thus, topological density governed by **local lumps**, reminiscent of monopoles!

Exponential fit yields $\mathcal{P}_2(p) \sim \exp(-26.5p)$

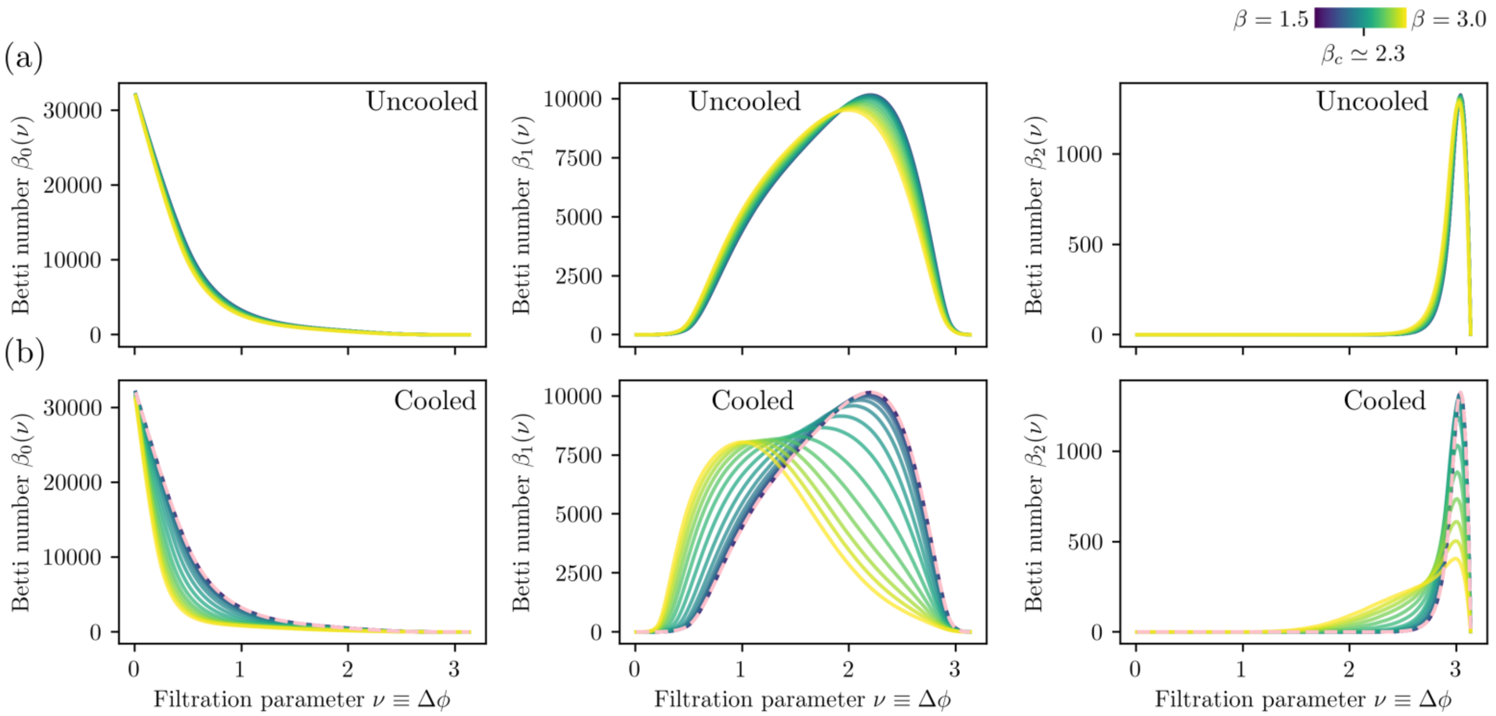
Potential of far-separated instanton dyon-antidyon pair yields 3d action $S_3(r\to\infty) = 8\pi v \simeq 25.1v$ with for both dyons $A_4^a(x\to\infty)\to v\hat{r}_a$ [e.g., Larsen & Shuryak 2016]

**Clear persistence signal of dyons!**

# Angle-difference filtration of holonomy Lie algebra field

Polyakov loop in Lie algebra: $\log \mathcal{P}(\mathbf{x}) = i\phi^a(\mathbf{x})T^a$. Trace $P(\mathbf{x}) = \cos\phi(\mathbf{x})$, $\phi(\mathbf{x}) = \sqrt{\phi^a(\mathbf{x})\phi^a(\mathbf{x})}/2$

Construct angle-difference filtration from differences of $\phi(\mathbf{x})$ between nearest neighbors on lattice, $\pi$-periodic (center-symm.). [Sale, Giansiracusa, Lucini 2022]




Interpretation:

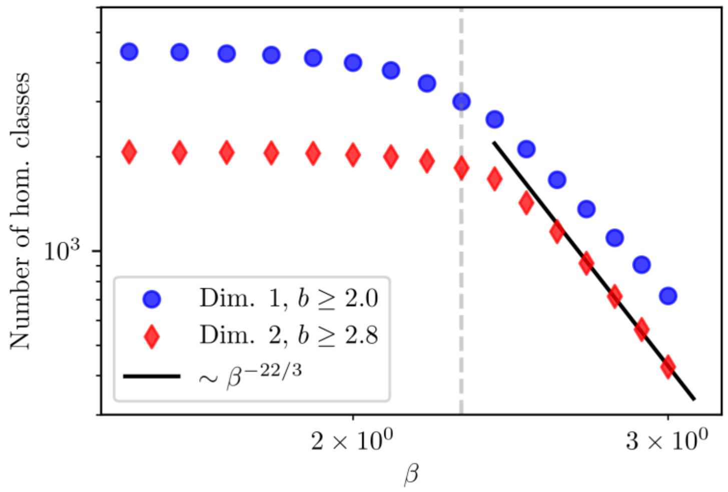
Polyakov loop traces $P(\mathbf{x})$ governed **below $\beta_c$ by small-scale fluctuations** between $\approx \pm 1$.

Thus, many dim-2 features corresponding to these but declining above $\beta_c$, enhanced by cooling.

Dim-1 Betti numbers due to scaffold-like **network of filaments** between these.

# Angle-difference filtration of holonomy Lie algebra field II

Number of dimension-1 and -2 homology classes with large birth:



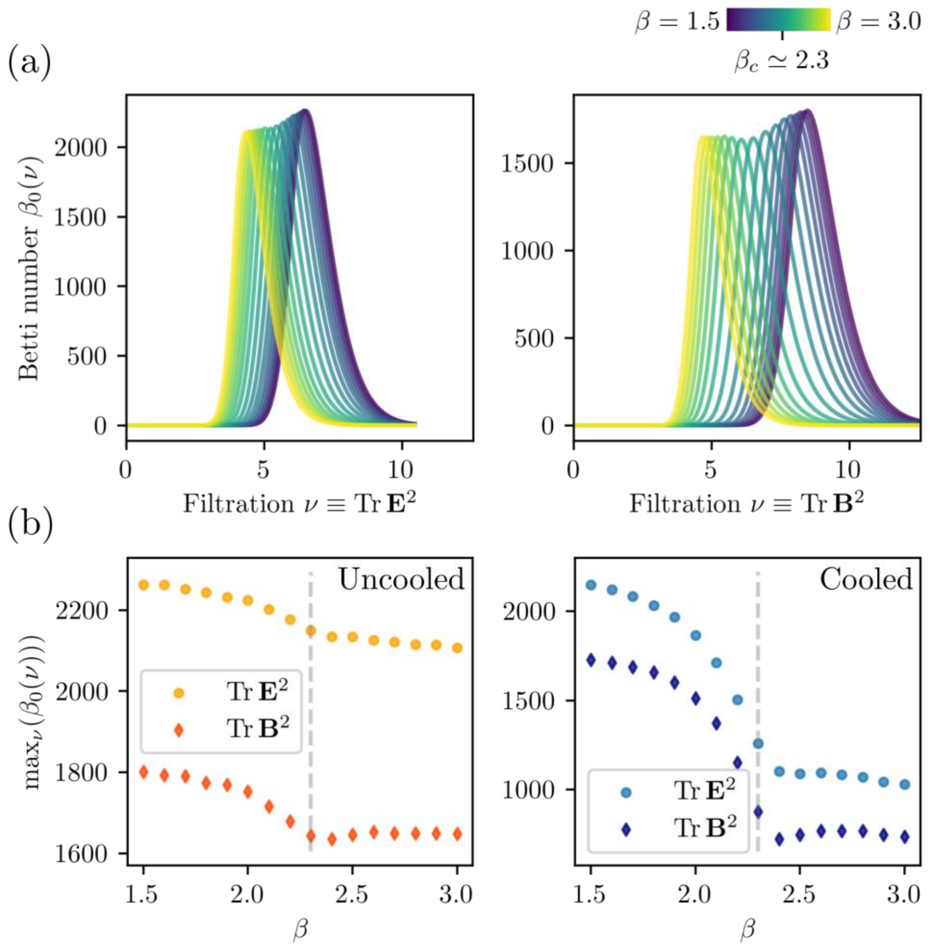

Find in number of lately born homology classes **manifestation of instanton appearance probability**

$$\exp(-S) = \exp(-\frac{8\pi^2}{g^2(T)}) \sim \left(\frac{\Lambda_{\mathrm{UV}}}{T}\right)^b$$

with temperature dependence from one-loop beta function, $b = 11N_c/3$

# Electric and magnetic fields squared

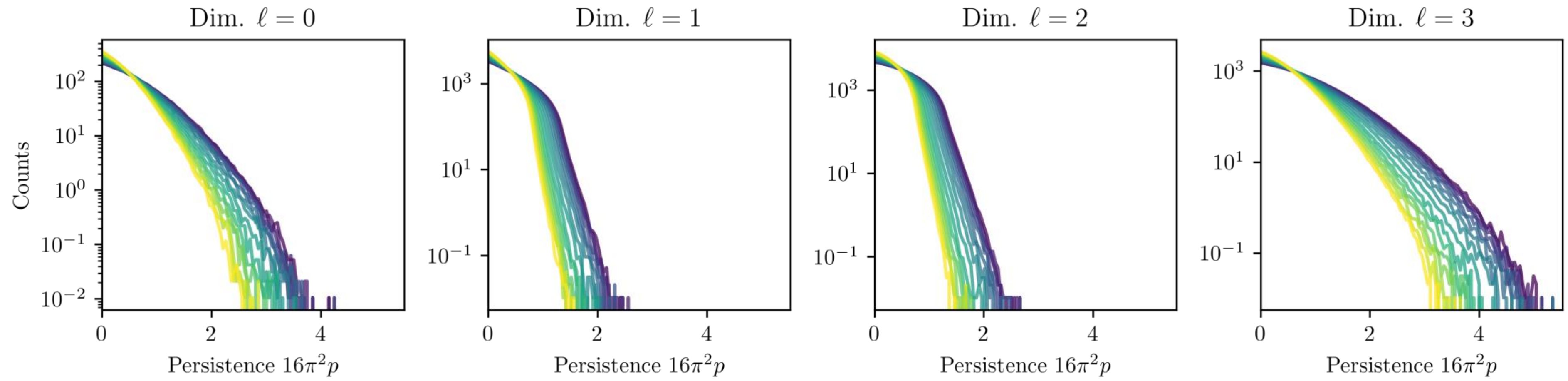
Superlevel set filtration of electric and magnetic fields squared, $\mathrm{Tr}(\mathbf{E}^2(x)), \mathrm{Tr}(\mathbf{B}^2(x))$, shows confinement transition (clover-leaf defs. used):




Differences between $\mathrm{Tr}\,\mathbf{E}^2(x)$ and $\mathrm{Tr}\,\mathbf{B}^2(x)$ due to **electric (Debye) screening outpacing magnetic screening.**

# Superlevel sets filtration of the usual topological density

$$q(x) \sim \mathrm{Tr}\,\mathbf{E}(x) \cdot \mathbf{B}(x)$$




Thus, **local lumps** as in Polyakov loop topological density, but no exponential behavior as for Polyakov loop top. densities

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# Conclusions

- Persistent homology provides sensitive order parameters for critical phenomena in non-Abelian gauge theories.
- Different filtrations allow for versatile investigations of non-perturbative effects.
- Confinement-deconfinement transition can be detected gauge-invariantly via persistent homology observables with interesting characteristics, including links to instantons and dyons.

# Outlook

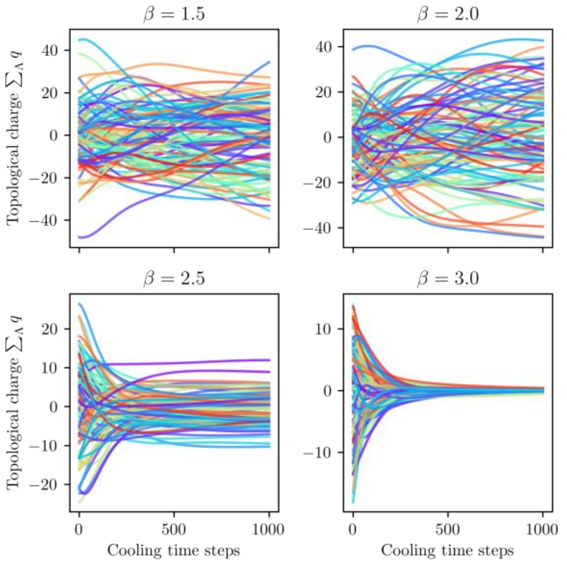
- How about higher-rank gauge groups different from SU(2) and suitable filtrations?
- With regard to neural network architectures designed to gauge equivariantly sample field configurations: Can topological layers make use of the high sensitive of persistent homology to non-local structures?
- How far can a physical interpretation of “homological excitations’’ go?

# Back-up: Hybrid Monte Carlo and Cooling

Hybrid Monte Carlo (HMC): Combine ease of calculation of (Langevin-like) equation of motion method with absence of truncation error in exact Monte Carlo [Duane *et al*. 1987].
Construct Markov process to generate new configuration $\phi'$ from its predecessor $\phi$ with probability $P_M(\phi \mapsto \phi')$; if process is ergodic and satisfies detailed balance $P_S(\phi)P_M(\phi \mapsto \phi') = P_S(\phi')P_M(\phi' \mapsto \phi)$ it converges to a fixed point distribution $P_S(\phi) \sim \exp(-S[\phi])$

$$P_M(\phi \mapsto \phi') = \int [d\pi][d\pi'] P_G(\pi) P_H((\phi,\pi) - (\phi',\pi')) P_A((\phi,\pi) \mapsto (\phi',\pi'))$$

$$\sim \int [d\pi][d\pi'] e^{-\pi^2/2} \delta[(\phi',\pi') - \underbrace{(\phi(\tau_0),\pi(\tau_0))}_{\text{from eom.}}] \min(1, \exp(H(\phi',\pi') - H(\phi,\pi)))$$




Cooling via solving gradient (Wilson) flow eq. $\partial_t U_\mu(x,t) = -g^2(\partial_{x,\mu} S[U(t)])U_\mu(x,t)$ with $\partial_{x,\mu} f(U) = i \sum_a T^a \frac{d}{ds} f(e^{isX^a}U)|_{s=0}$ and

$$X^a(y,\nu) = \begin{cases} T^a & \text{if } (y,\nu) = (x,\mu), \\ 0 & \text{else.} \end{cases}$$

[Lüscher 2010]

# Back-up: 3d action following Larsen & Shuryak 2016

Start with superposed dyon-antidyon pair in particular gauge s.t. resulting configuration fulfils $A_4^a(x \to \infty) \to v\hat{r}_a$ (both dyons need to match v.e.v. of $A_4$ ).

Apply gradient flow to minimize resulting 3d action (3d instead of 4d since $M\overline{M}$ and $L\overline{L}$ config's are time-indep. or time-twisted).

Find quasi-stationary regime with respect to gradient flow, consistent with

$$S_3(r \to \infty) = 8\pi v + (m_1 m_2 - e_1 e_2)\frac{4\pi v}{rv}$$

Raises question: Why do we see only dyon-antidyon pair and not signatures of $4\pi n v$ for larger $n \in \mathbb{N}$? Larger volumes could provide insights.

Actually, the $r \to \infty$ limit contribution $8\pi v$ is there for any pair of (anti-)dyons, but Coulomb-like interaction effects can cancel [Diakonov 2009].

# Back-up: Polyakov loop topological density rewriting

Winding number from field strength tensor: $Q_{\rm top} = \frac{1}{32\pi^2}\int_{T^4} \varepsilon_{\alpha\beta\mu\nu} {\rm Tr}\, F_{\alpha\beta} F_{\mu\nu}$
Fields on 4-torus with extents $N_x, N_y, N_z, N_\tau$ . Start with vector potential $A_\mu$ on 4-torus.
Periodicity of 4-torus manifests in

$$A_\mu(x+N_\nu) = U_\nu^{-1}(x) A_\mu(x) U_\nu(x) + i U_\nu^{-1}(x)\partial_\mu U_\nu(x)$$

with transition functions fulfilling the cocycle condition

$$U_\mu(x) U_\nu(x+N_\mu) = U_\nu(x) U_\mu(x+N_\nu)$$

and transforming under a local gauge transformation $V(x)$ as

$$U_\mu^V(x) = V^{-1}(x) U_\mu(x) V(x+N_\mu)$$

Suppose the transition functions satisfy $U_i(x)=1$ for all $i=1,2,3$ and $U_4=1$. Then, as detailed in [Ford et al. 1998] find

$$Q_{\rm top} = \frac{1}{24\pi^2}\int_{B_4} \varepsilon_{0ijk} {\rm Tr}[(\mathcal{P}^{-1}\partial_i\mathcal{P})(\mathcal{P}^{-1}\partial_j\mathcal{P})(\mathcal{P}^{-1}\partial_k\mathcal{P})].$$