

Dynamic critical behavior of the Model G chiral transition from the FRG

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Outline

► Introduction

- Chiral phase transition of QCD
- (Dynamic) Critical phenomena and universality classes
- Model G

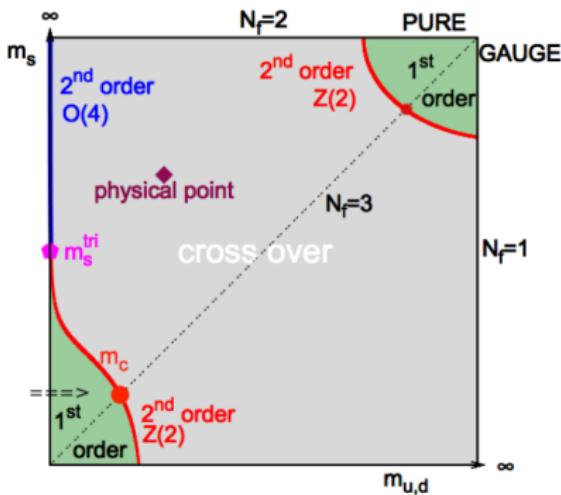
► Our work

- Real-time functional renormalization group formalism
- Model G (dynamic) critical phenomena

First section

Phase transition of QCD

- ▶ Chiral phase transition of two-flavor QCD in the limit of vanishing quark masses
 - Second order
 - O(4) universality class



Columbia plot ↗ de Forcrand, D'Elia,
'17

- ▶ Kinetic part of the action(2-flavors QCD in the chiral limit)

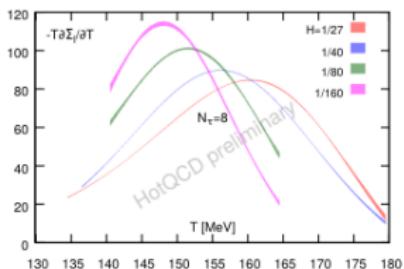
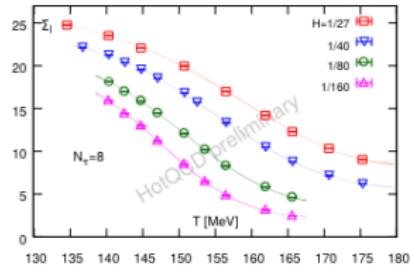
$$\Gamma_{kin} = i \int_x \bar{\psi}_L \gamma^\mu D_\mu \psi_L + \bar{\psi}_R \gamma^\mu D_\mu \psi_R$$

- Symmetry under

$$U(2)_L \times U(2)_R \cong SU(2)_L \times SU(2)_R \times \underbrace{U(1)_A}_{\text{Anomalously broken}} \times \underbrace{U(1)_V}_{\text{Unbroken}}$$

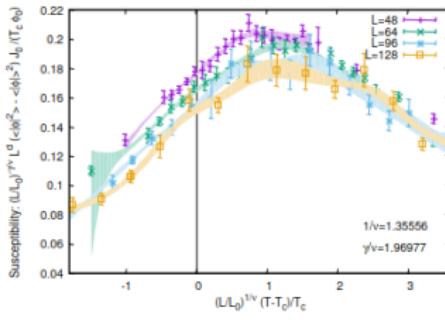
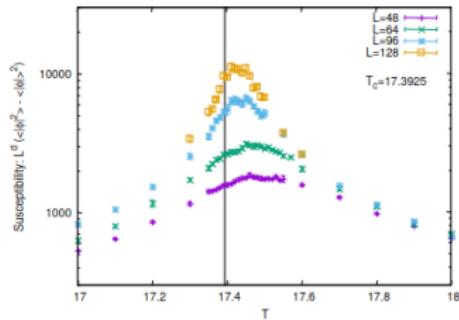
- $\psi_L \rightarrow e^{\frac{i}{2}\vec{\alpha}\vec{\tau}}\psi_L, \psi_R \rightarrow e^{\frac{i}{2}\vec{\beta}\vec{\tau}}\psi_R$
- Order parameter $M^{ij} = <\bar{\psi}_L^i \psi_R^j>$
- $SU(2)_L \times SU(2)_R \cong O(4)$
- $M = \sigma + i\vec{\pi}\vec{\tau}$, with $(\sigma, \vec{\pi})$ O(4) vector

- Below T_c , $\langle \sigma \rangle \neq 0$, $\langle \vec{\pi} \rangle = 0$.
- Symmetry breaking:
 $O(4) \rightarrow O(3) \times Z_2$ or
 $SU(2)_L \times SU(2)_R \rightarrow SU(2)_{L+R}$



Chiral condensate and
temperature derivative from
lattice QCD \nearrow Kaczmarek et
al. '21

- In the vicinity of critical point, the correlation length diverge
 $\xi \sim (\frac{T-T_c}{T_c})^{-\nu}$
- Singular behavior of thermodynamic quantities characterised by critical exponents (universal)

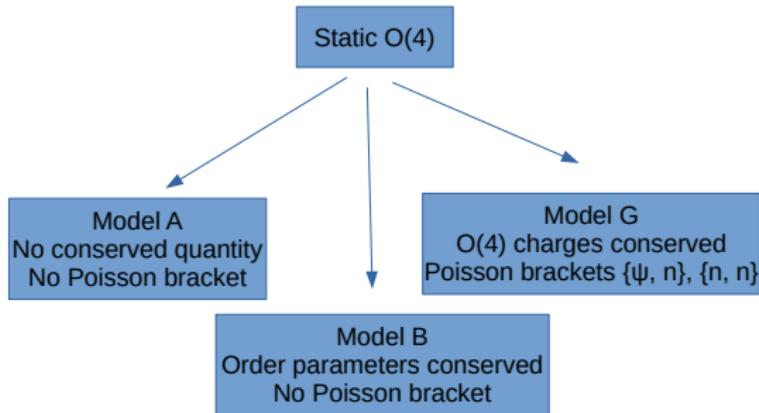


Ferromagnetic susceptibility of O(4) ↗ Schlichting, Smith and von Smekal '19

- Universality classes share the same **spatial dimensions** and **symmetries of the order parameters**

Dynamic critical phenomena

- ▶ Dynamic properties: transport coefficients, relaxation rates, multi-time correlation functions etc. (Requires EoM)
- ▶ Critical slow down: divergence of relaxation rates $\omega(k=0) \sim \xi^{-z}$
- ▶ Classification also depends on: **Conserved quantities, Poisson brackets among order parameters and conserved quantities**
- ▶ Classification of dynamic universality classes ↗ Hohenberg, Halperin '77



Model G

- ▶ Order parameters: $\sigma = \frac{1}{2} \text{Tr} < \bar{q}_L^i q_R^j >$, $\vec{\pi} = -\frac{i}{2} < \bar{q}_L^i q_R^j \vec{\tau}^{ji} >$
- ▶ Conserved charge of O(4): n_{ab} , correspond to isovector charge $n^{ij} = \epsilon^{ijk} \bar{q} \gamma^0 \tau^k q$ and isoaxial charge $n^{0i} = \bar{q} \gamma^0 \gamma^5 \tau^i q$
- ▶ Non-vanishing Poisson brackets:

$$\{\phi_a, n_{bc}\} = \phi_b \delta_{ac} - \phi_c \delta_{ab}$$

$$\{n_{ab}, n_{cd}\} = -\delta_{ad} n_{bc} - \delta_{bc} n_{ad} + \delta_{ac} n_{bd} + \delta_{bd} n_{ac}$$

- ▶ Effective macroscopic EoM:

$$\frac{\partial \phi_a}{\partial t} = \underbrace{-\Gamma_0 \frac{\delta F}{\delta \phi_a}}_{\text{dissipation}} + \underbrace{g\{\phi_a, n_{bc}\} \frac{\delta F}{\delta n_{bc}}}_{\text{mode-coupling}} + \underbrace{\theta_a}_{\text{noise}}$$

$$\frac{\partial n_{ab}}{\partial t} = \underbrace{\gamma \vec{\nabla}^2 \frac{\delta F}{\delta n_{ab}}}_{\text{dissipation}} + \underbrace{g\{n_{ab}, \phi_c\} \frac{\delta F}{\delta \phi_c} + g\{n_{ab}, n_{cd}\} \frac{\delta F}{\delta n_{cd}}}_{\text{mode-coupling}} + \underbrace{\vec{\nabla} \cdot \vec{\zeta}_{ab}}_{\text{noise}}$$

- ▶ The thermal expectation value for observable O

$$\langle O \rangle = \int \mathcal{D}\theta \mathcal{D}\zeta \mathcal{D}\phi \mathcal{D}n P[\theta, \zeta] \delta(\phi - \phi_{\theta, \zeta}) \delta(n - n_{\theta, \zeta}) O[\phi, n]$$

- ▶ Rewrite delta function

$$\delta(\phi - \phi_{\zeta, \theta}) \delta(n - n_{\zeta, \theta}) = \mathcal{J}[\phi, n] \delta(\text{EoM})$$

- ▶ $\mathcal{J}[\phi, n]$ depends on the discretization of the stochastic EoM,
 $\mathcal{J}[\phi, n] = 1$ for Ito discretization.
- ▶ Introduce response fields $\tilde{\phi}, \tilde{n}$

$$\delta(\text{EoM}) = \int D\tilde{\phi} D\tilde{n} \exp(i \int_x \underbrace{-\tilde{\phi}_a \left(\frac{\partial \phi_a}{\partial t} + \dots \right)}_{\text{EoM}} - \underbrace{n_{ab} \left(\frac{\partial n_{ab}}{\partial t} + \dots \right)}_{\text{EoM}})$$

Path integral formalism of model G

► Integrate out ζ, θ , one get the path integral

$$\langle O \rangle = \int \mathcal{D}\phi \mathcal{D}\tilde{\phi} \mathcal{D}n \mathcal{D}\tilde{n} O \exp \{iS\}$$

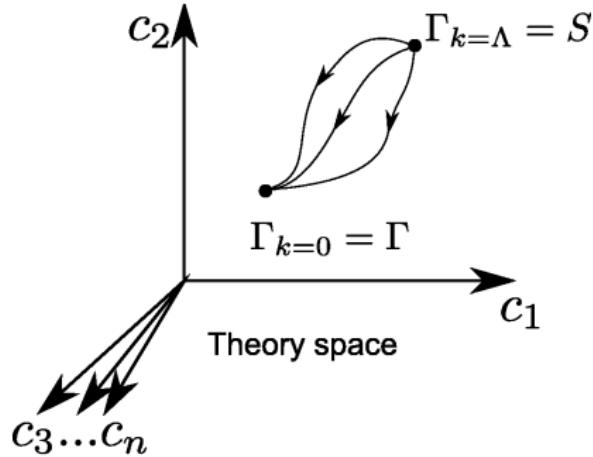
with MSR action given by

$$\begin{aligned} S = & \int_x \left[\underbrace{-\tilde{\phi}_a \left(\frac{\partial \phi_a}{\partial t} + \Gamma_0 \frac{\delta F}{\delta \phi_a} - g\{\phi_a, n_{bc}\} \frac{\delta F}{\delta n_{bc}} \right)}_{\text{EoM}} \right. \\ & - \tilde{n}_{ab} \underbrace{\left(\frac{\partial n_{ab}}{\partial t} - \gamma \vec{\nabla}^2 \frac{\delta F}{\delta n_{ab}} - g\{n_{ab}, \phi_c\} \frac{\delta F}{\delta \phi_c} - g\{n_{ab}, n_{cd}\} \frac{\delta F}{\delta n_{cd}} \right)}_{\text{EoM}} \\ & \left. + iT\tilde{\phi}_a \Gamma_0 \tilde{\phi}_a - iT\tilde{n}_{ab} \gamma \vec{\nabla}^2 \tilde{n}_{ab} \right] \end{aligned}$$

in which free energy given by

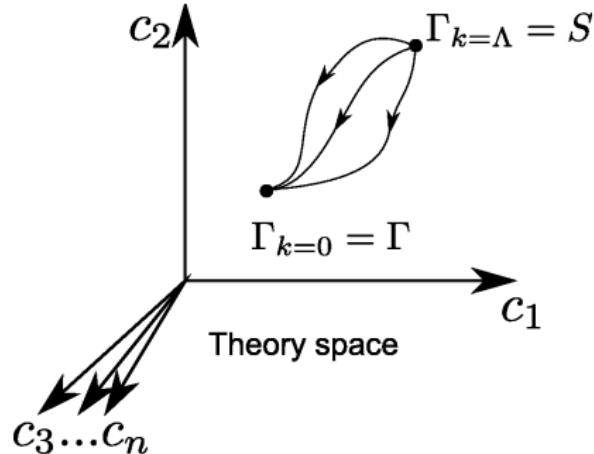
$$F = \int_{\vec{x}} \left\{ \frac{1}{2} (\partial^i \phi_a) (\partial^i \phi_a) + \frac{m^2}{2} \phi_a \phi_a + \frac{\lambda}{4!N} (\phi_a \phi_a)^2 + \frac{1}{2\chi} n_{ab} n_{ab} \right\}$$

- ▶ Scale dependent effective action: Γ_k
- ▶ Regulator R_k : Suppress modes with momentum smaller than FRG scale k
- ▶ $S + \underbrace{\Delta_k S}_{e.g.: \phi_k R_k \phi_k} \rightarrow \tilde{\Gamma}_k$
- ▶ $\Gamma_k = \tilde{\Gamma}_k - \Delta_k S$



Basic idea of functional renormalization group

- ▶ $\Gamma_{k=0}$: full effective action
- ▶ $\Gamma_{k=\Lambda} = S$: bare action
- ▶ The flow of the scale dependent effective action described by Wetterich equation:



$$k\partial_k\Gamma_k = -\frac{i}{2}S\text{Tr}(k\partial_kR_k(\Gamma_k^{(2)}+R_k)^{-1})$$

Basic idea of functional renormalization group

Coupling to external source

- ▶ Traditional way of coupling to external sources:

$$S + \tilde{H}_a \phi_a + \tilde{\mathcal{H}}_{ab} n_{ab} + H_a \tilde{\phi}_a + \mathcal{H}_{ab} \tilde{n}_{ab}$$

- ▶ The EoM becomes

$$\begin{aligned}\frac{\partial \phi_a}{\partial t} &= -\Gamma \frac{\delta}{\delta \phi_a} \left[F - \int_{x'} h'_a(x') \phi_a(x') \right] + g[\phi_a, n_{bc}] \frac{\delta F}{\delta n_{bc}} + \theta_a \\ \frac{\partial n_{ab}}{\partial t} &= \gamma \vec{\nabla}^2 \frac{\delta}{\delta n_{ab}} \left[F - \int_{x'} \mathfrak{h}'_{ab}(x') n_{ab}(x') \right] \\ &\quad + g[n_{ab}, \phi_c] \frac{\delta F}{\delta \phi_c} + g[n_{ab}, n_{cd}] \frac{\delta F}{\delta n_{cd}} + \vec{\nabla} \cdot \vec{\zeta}_{ab}\end{aligned}$$

where we rescale

$$\Gamma h'_a(x) = H_a(x), \quad -\gamma \vec{\nabla}^2 \mathfrak{h}'_{ab}(x) = \mathcal{H}_{ab}(x),$$

- ▶ One cannot recover Boltzmann distribution as a stationary equilibrium distribution. **Traditional way of coupling to external sources doesn't work when mode couplings exist!**

Coupling to external source

- The correct way: $F - H_a \phi_a - \mathcal{H}_{ab} n_{ab}$ which leads to

$$S + \tilde{H}_a \phi_a + \tilde{\mathcal{H}}_{ab} n_{ab} + \tilde{\Phi}_a H_a + \tilde{N}_{ab} \mathcal{H}_{ab}$$

$$\begin{aligned}\tilde{\Phi}_a &\equiv \Gamma_0 \tilde{\phi}_a - g \tilde{n}_{bc} \{n_{bc}, \phi_a\} \\ \tilde{N}_{ab} &\equiv -\gamma \vec{\nabla}^2 \tilde{n}_{ab} - g \tilde{n}_{cd} \{n_{cd}, n_{ab}\} - g \tilde{\phi}_c \{\phi_c, n_{ab}\}\end{aligned}$$

- One recover the Boltzmann distribution as stationary equilibrium distribution.
- Fluctuation-dissipation-relations are recovered.
- 1PI effective action $\Gamma(\phi, n, \tilde{\Phi}, \tilde{N})$

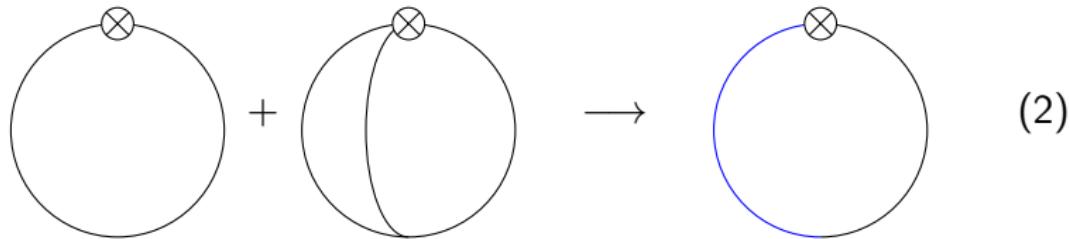
- ▶ Adding the regulator to the free energy

$$F + \frac{1}{2} \int_{\vec{x}\vec{y}} \left(\phi_a(\vec{x}) R_k^\phi(\vec{x}, \vec{y}) \phi_a(\vec{y}) + n_{ab}(\vec{x}) R_k^n(\vec{x}, \vec{y}) n_{ab}(\vec{y}) \right)$$

- ▶ Due to the Poisson brackets structure in the MSR action

$$\Delta_k S = -\tilde{\Phi}_a(\vec{x}) R_k^\phi(\vec{x}, \vec{y}) \phi_a(\vec{y}) - \tilde{N}_{ab}(\vec{x}) R_k^n(\vec{x}, \vec{y}) n_{ab}(\vec{y})$$

- ▶ The diagrammatic flow equation



- ▶ Symmetry of the thermal equilibrium(From which the general FDR for n point function can be derived)



$$\mathcal{T}_\beta \begin{pmatrix} \psi_a(\omega, \vec{p}) \\ \tilde{\Psi}_a(\omega, \vec{p}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\beta\omega & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_a \psi_a(-\omega, \vec{p}) \\ \varepsilon_a \tilde{\Psi}_a(-\omega, \vec{p}) \end{pmatrix}$$

- ε : parity of the field

- ▶ Charge conservation
- ▶ BRST symmetry
 - Ensure that $Z[H, \mathcal{H}] = 1$

► Displacement symmetry

- The MSR action (with physical source terms) can be written as

$$S = \int_x \tilde{\phi}_a i T \Gamma_0 \tilde{\phi}_a - \tilde{n}_{ab} i T \gamma \vec{\nabla}^2 \tilde{n}_{ab} - \tilde{\Phi} \frac{\delta F}{\delta \phi_a} - \tilde{N}_{ab} \frac{\delta F}{\delta n_{ab}} - \tilde{\phi}_a D_t \phi_a - \tilde{n}_{ab} D_t n_{ab}$$

with covariant derivative:

$$D_t \phi_a \equiv \partial_t \phi_a - g(T_{cd})_{ab} \mathcal{H}_{cd} \phi_b$$

$$D_t n_{ab} \equiv \partial_t n_{ab} - g(T_{cd})_{ab,ef} \mathcal{H}_{cd} n_{ef}$$

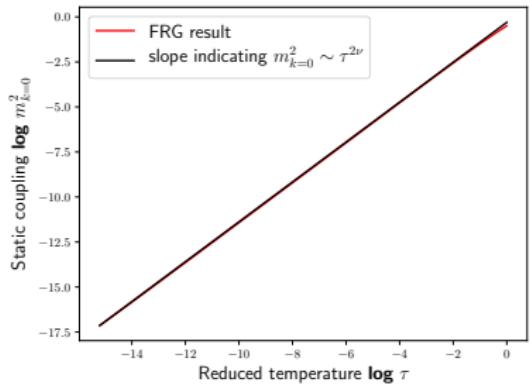
- The MSR action is invariant under uniformly time dependent $O(N)$ rotation.
- From the ward identity of displacement symmetry, one can show g is scale independent: $g_k = g_0$.

► Truncation:

■ Ansatz

$$\begin{aligned} & \Gamma_k[\phi, n, \tilde{\Phi}(\phi, n, \tilde{\phi}_k, \tilde{n}_k), \tilde{N}(\phi, n, \tilde{\phi}_k, \tilde{n}_k)] \\ &= \int_x \left[-\tilde{\phi}_{ka} \left(\frac{\partial \phi_a}{\partial t} + \Gamma_k \frac{\delta F_k}{\delta \phi_a} - g_k^a \{ \phi_a, n_{bc} \} \frac{\delta F_k}{\delta n_{bc}} \right) \right. \\ &\quad - \tilde{n}_{kab} \left(\frac{\partial n_{ab}}{\partial t} - \gamma_k (\vec{\nabla}^2) \frac{\delta F_k}{\delta n_{ab}} - g_k^b \{ n_{ab}, \phi_c \} \frac{\delta F_k}{\delta \phi_c} - g_k^c \{ n_{ab}, n_{cd} \} \frac{\delta F_k}{\delta n_{cd}} \right) \\ &\quad \left. + iT \tilde{\phi}_{ka} \Gamma_k \tilde{\phi}_{ka} - iT \tilde{n}_{kab} \gamma_k (\vec{\nabla}^2) \tilde{n}_{kab} \right] \end{aligned}$$

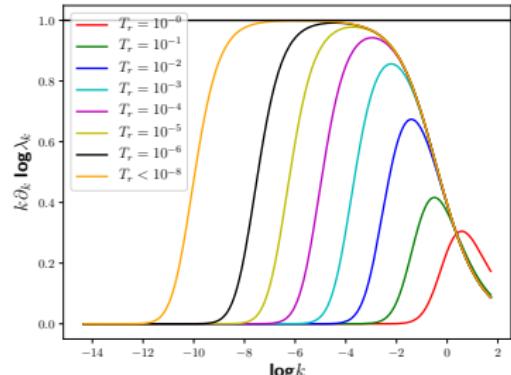
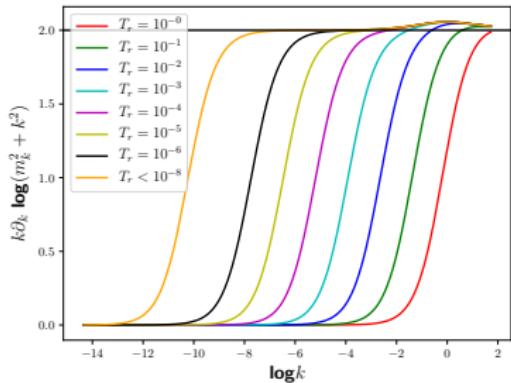
► Same as Euclidean FRG!



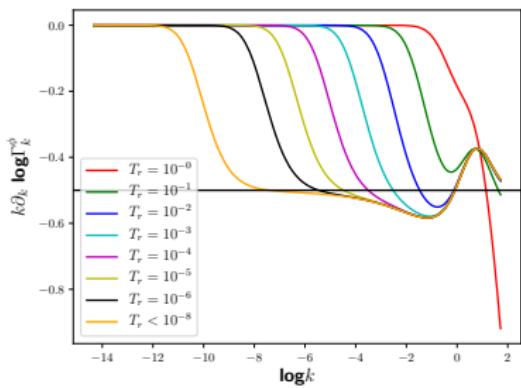
$\nu = 0.553945$ exact value
from our truncation.

$\nu = 0.780(20)$ from LPA.

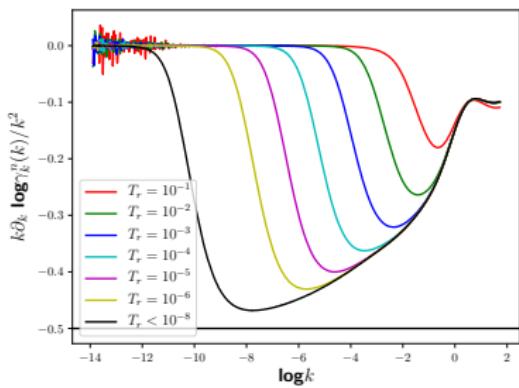
∅ Murgana, Koenigstein and Rischke
'23



Dynamic critical behavior



$\Gamma_k^\phi \sim k^{z\phi-2}$, scaling of the kinetic coefficient of the order parameter ϕ

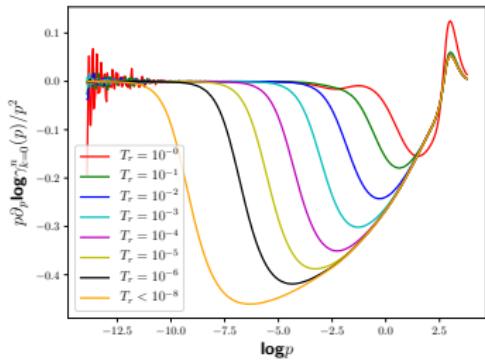


$\gamma_k^n(k)/k^2 \sim k^{z^n-2}$, scaling of the diffusion coefficient of the conserved charge n

- ▶ Scaling hypothesis implies that:

$$D_n(p, \tau) = D_{n,0} \bar{\xi}^{2-z_n} \mathcal{L}(p\bar{\xi}).$$

- ▶ Dimensionless universal scaling function $\mathcal{L}(p\bar{\xi})$ can be extracted

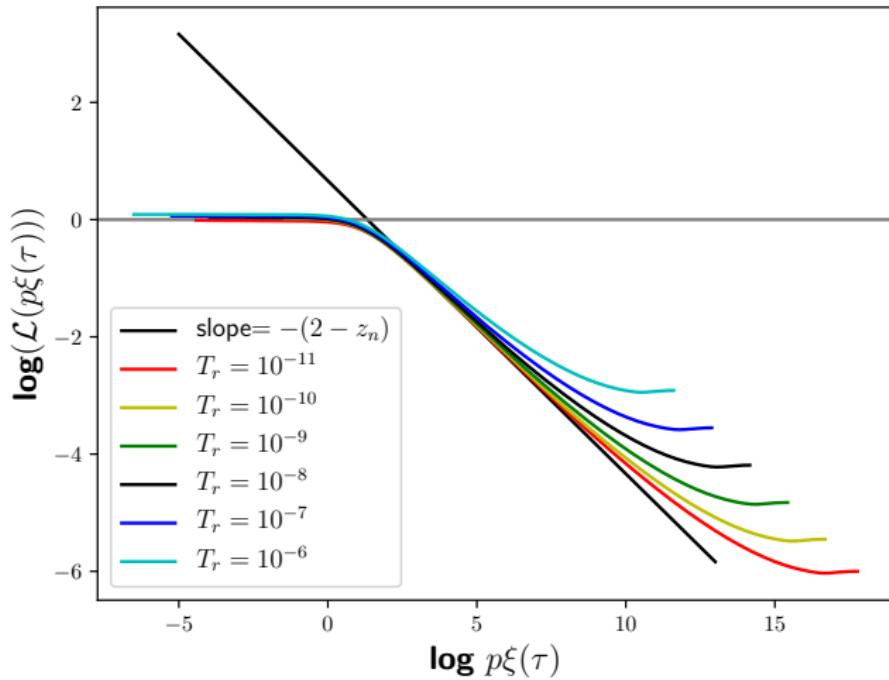


The scaling of diffusion coefficient of the conserved charges

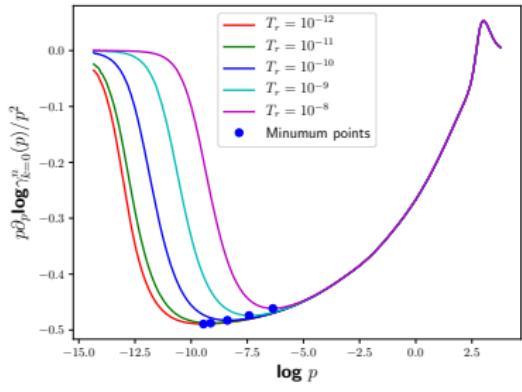
$$D_n(\vec{p}, \tau) = \gamma_{n,k=0}(\vec{p}, \tau) / \vec{p}^2$$

Dynamic critical behavior

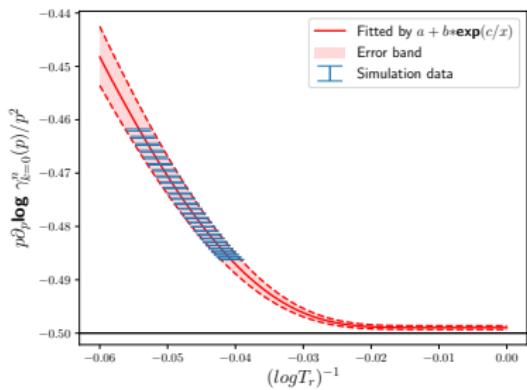
- Dimensionless universal scaling function



Dynamic critical behavior



Scaling of the diffusion coefficient
of the conserved charge at three
spatial dimensions and different
reduced temperatures



Extrapolation of the minimum
points and the extraction of
dynamic critical exponent

Dynamic critical exponent

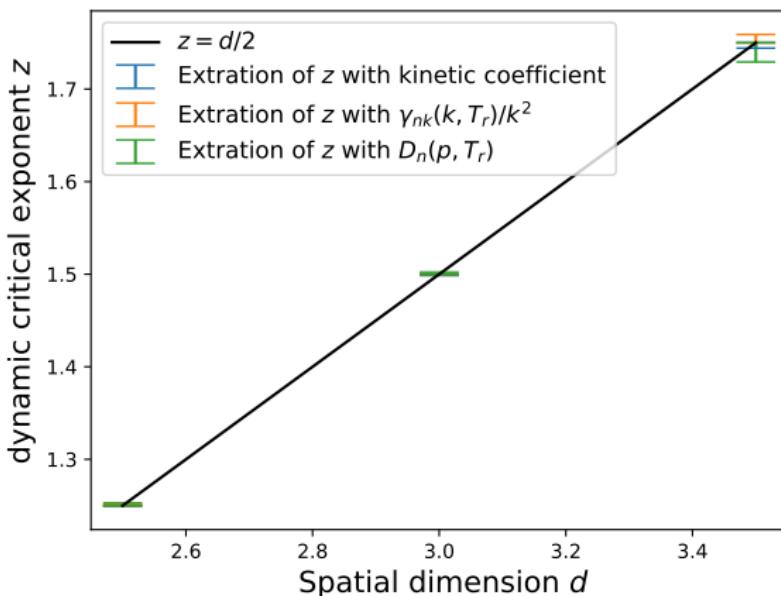


Figure: Extraction of the dynamic critical exponent z for various spatial dimensions d .

Conclusion/Outlook

Conclusion/Outlook

- ▶ What we showed:
 - $O(4)$ static critical behavior recovered
 - Dynamic critical exponent $z^n = z^\phi = d/2$
 - Universal scaling function of the diffusion coefficient of the conserved charges
- ▶ Outlook:
 - Analytic calculation for large N limit
 - Improved truncation, explicit symmetry breaking