Modified Anti-de-Sitter Metric, Light-Front Quantized QCD, and Conformal Quantum Mechanics

H.G. Dosch

Institut für Theoretische Physik der Universität Heidelberg

Stan Brodsky*, Guy de Téramond **

*Stanford, USA, **San José, CR

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1) Theory of Strong Interaction. The Problems
2) Modified Anti-de-Sitter metric. A solution?
3) AdS metric, LF Quantized QCD and Conformal Q M.
   Une ménage à trois
Quantum Chromo Dynamics, theory of strongly interacting particles
(hadrons: protons, neutrons, π-mesons . . . .).

Fundamental fields:
**Quarks**: Spin $\frac{1}{2}$ fields with *colour* quantum numbers,
interacting through **Gluons**.
Theory invariant under $SU(3)$ gauge transformations.

Cf. with Quantum Electrodynamics: electrons and photons,
gauge group $U(1)$.

**Confinement**: To the quarks fields correspond no states in
the Fock space of hadrons.

Problem of all realistic Quantum Field Theories:
The only analytically tractable treatment is perturbation
theory around the free fields.

Not adequate for the effects of strong interactions, like
confining the quarks.
But even in QED: Need for a semiclassical treatment of bound state problems (Schrödinger, Dirac equation)

Formidable task: Find semiclassical equations for QCD!

Not completely unrealistic:

- **Qualitative success of non-relativistic quark model** (based on Schrödinger equation)
- **Striking regularities in the hadronic spectra**
  Notably Regge Trajectories

From Donnachie et al. “QCD and Pomeron Physics”
Light Front Quantization

adequate framework in relativistic theories
Dirac, Rev. Mod. Phys. 21, 392 (1949), Brodsky et al. . .

Instant-form: \( x^0 = 0 \) \hspace{1cm} \text{Light Front-form: } x^0 + x^3 = 0

For the Fock state of two massless quarks:

\[
\left(-\partial_\zeta^2 + \frac{4L^2-1}{4\zeta^2} + U(\zeta)\right)\psi(\zeta) = P^2\psi(\zeta)
\]

\( \zeta \): essentially the separation of the quarks in the transverse (1-2) plane, \( L \): longitud. angular momentum

\( U(\zeta) \) comprises all interactions, including those with higher Fock states

Eigenvalues: Squared hadron masses

Gauge-QFT $d = 4 \sim$ Classical gravitational theory $d = 5$

Generating functional of the gauge QFT is given by the minimum of the classical action of the gravitational theory at the 4-dim border of the 5-dim space

**Graviation:** $AdS_5$ metric, $ds^2 = \frac{R^2}{z^2} \left( \sum_{i=0}^{3} dx_i \ dx^i - dz^2 \right)$

Poincaré coordinates. $z = x^5$ holographic coordinate, $z = 0$ border to 4-dim. space $R(1,3)$

QFT: greatly oversymmetrized: conformal supersymmetric gauge theory

**bottom-up approach:** Start from realistic 4-dim Theory.

Here: Light-Front Holography

Consider scalar field in $AdS_5$:

**Action:**

$$S = \int d^4x \, dz \, \sqrt{|g|} \left( g^{MN} \partial_M \Phi(x, z) \partial_N \Phi(x, z) - \mu^2 \Phi(x, z)^2 \right)$$

With $\Phi(x, z) = e^{iPx} \Phi(z)$ e.o.m. can be brought into the form:

$$\left( -\partial_z^2 + \frac{4(\mu R)^2 + 16 - 1}{4z^2} \right) \phi(z) = P^2 \phi(z)$$

Compare with **LF equation**:

$$\left( -\partial_\zeta^2 + \frac{4L^2 - 1}{4\zeta^2} + U(\zeta) \right) \psi(\zeta) = P^2 \psi(\zeta)$$

Same structure: $z \rightarrow \zeta \quad (\mu R)^2 + 4 \rightarrow L^2 \quad$ but $U(\zeta) = 0$

No wonder: $AdS_5$ maximally symmetric, 15 isometries $\rightarrow$ Conf($R^{1,3}$), (10 Poincaré + 4 inversions + dilatation)

Conformal symmetry: No scale $\rightarrow$ no discrete spectrum
Way out: **Distort maximal symmetry!**

\[
S = \int d^4 x \, dz \, \sqrt{|g|} \left( g^{MN} \partial_M \Phi(x, z) \partial_N \Phi(x, z) - \mu^2 \Phi(x, z)^2 \right)
\]

\[
\downarrow
\]

\[
S = \int d^4 x \, dz \, \sqrt{|g|} e^{\varphi(z)} \left( g^{MN} \partial_M \Phi(x, z) \partial_N \Phi(x, z) - \mu^2 \Phi(x, z)^2 \right)
\]

\[
U(z) = \frac{1}{4} (\varphi'(z))^2 - \frac{3}{2} \varphi'(z) + \frac{1}{2} \varphi''(z)
\]

**Soft wall model**


\[
\varphi(z) = \lambda \, z^2, \quad \Rightarrow \quad U(\zeta) = \lambda^2 \zeta^2 - 2\lambda
\]
Choice of \( \varphi(z) = \lambda z^2 \) only motivated by phenomenology. Some deeper reason?
Conformal symmetry.

Classical QCD-Lagrangian with massless quarks conformally invariant.

Need for renormalization $\Rightarrow$ introduces a scale $\Lambda_{QCD}$

Scale dependence of QCD coupling in perturbative regime (small distance = large momentum scales).

$$Q^2 \frac{d\alpha_s(Q^2)}{dQ^2} = - \sum_{i=0} \frac{\beta_i}{4\pi} \alpha_s^{2+i} + \ldots$$

$$\alpha_s(Q^2) = \frac{4\pi}{\beta_0} \log(Q^2/\Lambda_{QCD}) + \ldots$$

from PDG

Indications: At large distances, small $Q$, $\alpha_s$ becomes constant again. Restauration of conformal symmetry?
Fresh look at conformal symmetry

Our goal: Find a semiclassical approximation, i.e. approximate QFT by Quantum Mechanics

Conformal QM = conformal QFT in 1 dimension.

\[ S_{conf} = \frac{1}{2} \int dt \left( \dot{Q}(t)^2 - \frac{g}{Q(2)^2} \right) \]
\[ H = \frac{1}{2} \left( \dot{Q}^2 + \frac{g}{Q^2} \right) \]
\[ P = \frac{\delta S}{\delta Q} = \dot{Q} \rightarrow [Q, \dot{Q}] = i \text{Schrödinger} \]
\[ Q(0) = r, \dot{Q}(0) = -i \partial_r \]
\[ H\psi(r) = \frac{1}{2} \left( -\partial_r^2 + \frac{g}{r^2} \right) \psi(r) \]
again back to free equation.

AdS: \( \left( -\partial_z^2 + \frac{4(\mu R)^2 + 16 - 1}{4z^2} \right) \phi(z) = P^2 \phi(z) \)

LF: \( \text{Big}(-\partial_\zeta^2 + \frac{4L^2 - 1}{4\zeta^2} + U(\zeta)\right)\psi(\zeta) = P^2 \psi(\zeta) \)

But, as stressed by de Alfaro, Fubini and Furlan:
There are besides $H$, the generator of translations $t$ two more constants of motion, namely the two other Noether currents of the conformal action $S_{\text{conf}}$

$D$ for dilatations, $t \rightarrow t(1 + \epsilon)$ and $K$ for special conformal transformation $t \rightarrow \frac{t}{1-\epsilon t}$

Allows to construct a generalized Hamiltonian:

$$G = H + w K + v D$$

translation in $\tau$ with $d\tau = \frac{dt}{1+vt+wt^2}$

Schrödinger

$$G \psi(r) = \frac{1}{2} \left( -\partial_r^2 + \frac{g}{r^2} + w r^2 + \frac{i v}{2} (r \partial_r + \partial_r r) \right) \psi(r)$$

distorted Ads:

$$\left( -\partial_z^2 + \frac{4(\mu R)^2+16-1}{4z^2} + \frac{1}{4}(\varphi'(z))^2 - \frac{3}{2} \varphi'(z) + \frac{1}{2} \varphi''(z) \right) \phi(z) = P^2 \phi(z)$$

$$\text{LF}: \left( -\partial_\zeta^2 + \frac{4l^2-1}{4\zeta^2} + U(\zeta) \right) \psi(\zeta) = P^2 \psi(\zeta)$$

The conformal Lagrangian $S_{\text{conf}}$ implies unambiguously:

$$\varphi(z) = \lambda z^2 \quad U(\zeta) = \lambda \zeta^2$$
Back to geometry;

The conformal group $Conf(R^1)$ is isomorphic to the Lorentz group $S0(2, 1)$ and therefore to the isometries of $AdS_2$.

Best seen by embedding $AdS_2$ into a three-dimensional space:

\[ X_{-1}^2 + X_0^2 - X_1^2 = R^2 \]

Poincaré coordinates:
\[ Z = \frac{R^2}{X_{-1} - X_1}, \quad x^0 = \frac{X_0(X_{-1} - X_1)}{R} \]

Generators of $S0(2, 1)$

Rotation $J^{-10}$, boosts $J^{01}$, $J^{-1,1}$

are the isometries of $AdS_2$
Isomorphism
$SO(2, 1) \sim Conf(R^1)$:

$$aH = J^{-10} - J^{01},$$
$$\frac{1}{a}K = J^{-10} + J^{01},$$
$$D = J^{-1,1}$$

The confining $G$

$$G = H + wK :$$

$$\frac{1 + \theta}{2} a\; G = J^{-10} - \theta J^{01}$$

Schrödinger:

$$G = \frac{1}{2} \left( -\partial_r^2 + \frac{g}{r^2} + \frac{1}{a^2} \frac{1 - \theta}{1 + \theta} r^2 \right)$$