Field Renormalization in QG

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Lattice field theory has provided us with a non-perturbative tool for defining and calculating certain observables. The most impressive results are obtained in QCD.

If we want to extend this success to QG we are facing two major obstacles:

(1) No clear Euclidean-Lorentzian signature dictionary

(2) Not obvious that there exists a QG theory "an sich".

I will ignore (1) and (2) and assume we have a lattice Euclidean field theory of some type.

The exists one example of a theory of fluctuating Euclidean geometries coupled to matter fields: non-critical string theory, which can be solved analytically in the continuum, can be formulated as a lattice theory, solved analytically on the lattice(!) and which can studied by MC simulations. So, seemingly this is the proof of concept that some QG theories satisfy (1) and (2). But even here the devil is in the detail, and this will be the topic of my talk which could also be called “is there a bosonic lattice string theory ?”

The whole enterprise of DT, non-critical strings and matrix models started with the desire to have a non-perturbative formulation of the bosonic string theory.

One has two formulations of the bosonic string theory in $\mathbb{R}^D$, which are believed to be equivalent.

$$Z = \int \mathcal{D}S \, \mathrm{e}^{-kA[S]}, \qquad \text{Nambu-Goto}$$

$$Z = \int \mathcal{D}[g] \int \mathcal{D}_g X \, \mathrm{e}^{-S[X,g]}, \qquad \text{Polyakov}$$

$$S[X,g] = k \int d^2\xi \sqrt{g} \Big(g^{ab} \partial_a X \partial_b X + \Lambda \Big).$$

The Polyakov formulation is just 2d Euclidean QG coupled to D massless Gaussian fields X^μ which live in taget space $\mathbb{R}^D$.

In the case of the Nambu-Goto action the natural lattice theory consists of a hypercubic lattice with link lenth a in $\mathbb{R}^D$ and the string worldsheet is a plaquette surface, the area being the number of plaquettes times a^2. For the Polyakov string a natural lattice theory is one where

$$\int \mathcal{D}[g] \to \sum_T \qquad \text{Discretisation}$$

where the summation is over equilateral triangulations with link length a and where the Gaussian action of X^μ has a natural implementation on the triangulation T as N_T Gaussian fields.

The hypercubic lattice theory is beautiful in its extreme simplicity. When formulated on a lattice in dimensionless units (lattice spacing of length 1) we have

$$Z(\mu) = \sum_S \mathrm{e}^{-\mu A(S)}, \quad , \quad A(S) = \#\ \text{plaquettes in } S$$

S denote connected plaquette surfaces with a fixed topology and fixed boundaries. The partition function has a critical point μ_c. For $\mu < \mu_c$ the partition function is not defined and the continuum limit is obtained for $\mu \to \mu_c$ where large surfaces will dominate.

Here are the results , formulated in dimensionless lattice units. We consider the correlation function $G(n)$ of two plaquette boundaries separated a lattice distance n, and the partition function $G(m,n)$ for a rectangular "Wilson loop" spanned by a minimal surface of $m \times n$ plaquettes.

One can prove that for $\mu > \mu_c$ one has for large n, m an exponential fall off:

$$G_\mu(n) \sim e^{-m(\mu)n}, \qquad G_\mu(m,n) \sim e^{-\sigma(\mu)\, m\, n}.$$

$G(n)$ and $G(m,n)$ only exist in the continuum limit $\mu \to \mu_c$ if

$$m(\mu) \to 0 \quad \text{and} \quad \sigma(\mu) \to 0, \qquad \mu \to \mu_c$$

However, the scaling of $m(\mu)$ and $\sigma(\mu)$ are

$$m(\mu) \sim (\mu - \mu_c)^{1/4} \qquad \sigma(\mu) = \sigma(\mu_c) + \tilde{c}(\mu - \mu_c)^{1/2}, \quad \sigma(\mu_c) > 0.$$

The last result implies that the physical string tension goes to infinity:

$$m(\mu) = m_{ph}\, a(\mu), \qquad \sigma(\mu) = k_{ph}\, a^2(\mu)$$

Thus

$$k_{ph} \sim \frac{\sigma(\mu_c)}{\sqrt{\mu - \mu_c}}\, m_{ph}^2 \quad \to \infty \quad \text{for} \quad \mu \to \mu_c.$$

$$e^{-m(\mu)n} = e^{-m_{\rm ph} L}, \quad L = n\, a,$$

$$e^{-\sigma(\mu)nm} = e^{-k_{\rm ph} A}, \quad A = mn\, a^2$$

This seems in contradiction to the continuum calculation from string theory. Bosonic strings have tachyons, but we can avoid the tachyons if we consider, say, closed strings where one space dimension is compactified to a circle of radius β and the string wraps once around this dimension. We now consider the closed string correlator where two closed strings are separated a distance L. The minimal worldsheet surface has area $L \times \beta$. This situation can clearly be implemented on the hypercubic lattice too and one would have a minimal surface with $n \times m$ links,

$$L = n \cdot a(\mu), \qquad \beta = m \cdot a(\mu)$$

The non-scaling of $\sigma(\mu)$ is still valid, being a bulk property, i.e. K_{ph} is infinite. On the lattice one would have a minimal surface with a few excitations carrying no area (so-called branched polymer excitations).

In contrast continuum string calculations state that the ground state energy for $L \gg \beta$, and where we use notation $d := D - 2$, is given by

$$E_{min} = k_{ph}\sqrt{\beta^2 - \frac{\pi d}{3k_{ph}}}, \qquad F_{min} = LE_{min}(\beta)$$

and the average area of the world sheet can be calculated

$$\langle A \rangle = L\,\frac{\beta^2 - \frac{\pi d}{6k_{ph}}}{\sqrt{\beta^2 - \frac{\pi d}{3k_{ph}}}}$$

In these formulas k_{ph} is just the continuum finite string tension which appears is the Nambu-Goto action. The formula for $\langle A \rangle$ is weird...

The difference between the lattice and continuum calculations is somewhat of a mystery, in the sense that it should not make a difference whether one uses a lattice as a regularization or one uses another regularization with a cut-off which plays the role of the lattice link length a. We decided to address the question, using continuum string calculations, but carefully keeping the dimensionful cut-off at all steps in the calculations. The starting point is the Nambu-Goto action

$$\begin{aligned} S(X) &= k_0 \int d^2\omega \sqrt{\det \partial_a X \cdot \partial_b X} \\ &= k_0 \int d^2\omega \sqrt{\rho} + \frac{k_0}{2} \int d^2\omega \, \lambda^{ab} \left(\partial_a X \cdot \partial_b X - \rho_{ab} \right). \end{aligned}$$

where $\rho_{ab}(\omega)$ and $\lambda^{ab}(\omega)$ are Lagrange multipliers.

Choosing the worldsheet parameter space as $[\omega_L, \omega_\beta]$, the minimal action worldsheet configuration is

$$X^1_{\rm cl} = \frac{L}{\omega_L}\omega_1, \quad X^2_{\rm cl} = \frac{\beta}{\omega_\beta}\omega_2, \quad X^\perp_{\rm cl} = 0,$$

$$[\rho_{ab}]_{\rm cl} = {\rm diag}\left(\frac{L^2}{\omega_L^2}, \frac{\beta^2}{\omega_\beta^2}\right),$$

$$\lambda^{ab}_{\rm cl} = {\rm diag}\left(\frac{\beta\omega_L}{L\omega_\beta}, \frac{L\omega_\beta}{\beta\omega_L}\right) = \rho^{ab}_{\rm cl}\sqrt{\rho_{\rm cl}},$$

$$S[X_{\rm cl}] = k_0 L\,\beta$$

We now write $X = X_{\rm cl} + X_{\rm q}$ and integrate over $X_{\rm q}$ in

$$Z(k_0, L, \beta) = e^{-F(k_0, L, \beta)} = \int \mathcal{D}X^\mu \mathcal{D}\rho_{ab} \mathcal{D}\lambda^{ab}\, e^{-S[X,\rho,\lambda]}$$

and determine the mean field values of the Langrange multipliers ρ and λ from the corresponding effective action. We expect this mean field approximation to be exact in the $d \to \infty$ limit.

This approximation can be compared to the use of Lagrange multipliers in the $O(N)$ non-linear sigma model. Also there one obtains very good results for using mean field for the Lagrange multiplier.

A cut-off Λ enters when we perform the Gaussian integral over $X_{\rm q}$ and we need to calculate the determinant of the operator $\mathcal{O} = -\frac{1}{\sqrt{\rho}}\partial_a \lambda^{ab}\partial_b$. We use proper-time regularization

$$\mathrm{tr}\,\log\mathcal{O} = -\int_{a^2}^{\infty}\frac{d\tau}{\tau}\mathrm{tr}\,e^{-\tau\mathcal{O}}, \quad a^2 \equiv \frac{1}{4\pi\Lambda^2}.$$

a is seemingly a worldsheet cut-off rather than a cut-off in target space, but its eigenvalues only depend on L, β (by diffeomorphism invariance) and

$$a^2 = (\Delta s)^2 = \rho_{ab}\Delta\omega^a\Delta\omega^b = (\Delta X)^2$$

Thus it is also a cut-off in target space, like the hypercubic lattice.

The minimum of the effective action is reached at

$$\bar{\rho}_{11} = \frac{L^2}{\omega_L^2} \frac{\left(\beta^2 - \frac{\beta_0^2}{2c}\right)}{\left(\beta^2 - \frac{\beta_0^2}{c}\right)} \frac{c}{2c-1},$$

$$\bar{\rho}_{22} = \frac{1}{\omega_\beta^2} \left(\beta^2 - \frac{\beta_0^2}{2c}\right) \frac{c}{2c-1},$$

$$\bar{\lambda}^{ab} = c\bar{\rho}^{ab}\sqrt{\bar{\rho}}, \qquad c = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{d\Lambda^2}{2k_0}},$$

where

$$\beta_0^2 = \frac{\pi d}{3k_0}, \qquad c = 1 - \frac{d\Lambda^2}{k_0} + O\left(\frac{\Lambda^4}{k_0^2}\right)$$

For these mean field values of ρ and λ we have (for $L \gg \beta$)

$$F_{\rm mf} = k_0 c\, L\sqrt{\beta^2 - \beta_0^2/c}$$

and the average area of the world sheet will be

$$\langle A \rangle = \int d^2\omega\, \sqrt{\bar{\rho}_{11}\bar{\rho}_{22}} = L \frac{(\beta^2 - \beta_0^2/2c)}{\sqrt{\beta^2 - \beta_0^2/c}}\, \frac{c}{(2c-1)}.$$

For $k_0 \to \infty$ we obtain the minimal surface. For $c = 1$ we obtain the former string result. However, the only physical reasonable result is that when the cut-off $\Lambda \to \infty$ then $c \to 1/2$:

$$c = \frac{1}{2} + \frac{1}{2}\frac{\tilde{k}_{\rm ph}}{d\Lambda^2} \qquad k_0 = 2d\Lambda^2 + \frac{\tilde{k}_{\rm ph}^2}{d\Lambda^2}$$

With these definitions we can define a non-tachyonic two-point function where the two infinitesimal loops are separated a distance L by choosing β as small as possible without $\sqrt{\beta^2 - \beta_0^2/c}$ becoming imaginary for any k_0:

$$\beta = \beta_{min} = \frac{2\pi a}{\sqrt{3}}.$$

Note that β_{min} is of the cut-off scale a and thus the loop around the compactified dimension is of the size of a minimal boundary plaquette loop on the lattice.

$$G(L, k_0) \sim e^{-m_{\rm ph} L}, \qquad m_{\rm ph}^2 = \frac{\pi d}{6} \tilde{k}_{\rm ph}.$$

This defines a tunable lowest mass of our string !

Similarly we can now define the string tension $k_{\rm ph}$ in the limit where $L \gg \beta \gg \beta_{min}$:

$$Z(k_0, L, \beta) = e^{-F(k_0,L,\beta)} = e^{-k_{\rm ph} A_{\rm min} + \mathcal{O}(L,\beta)}, \quad A_{\rm min} = L\,\beta$$

We conclude that the physical string tension $k_{\rm ph}$ is

$$k_{\rm ph} = k_0 c = d\Lambda^2 + \frac{1}{2}\tilde{k}_{\rm ph} + \mathcal{O}(1/\Lambda^2).$$

Thus the physical string tension as defined above diverges as the cutoff Λ is taken to infinity. However, the first correction is finite and behaves as we would have liked $k_{\rm ph}$ to behave, namely as $\tilde{k}_{\rm ph} \propto m_{\rm ph}^2/d$.

We can now formulate the results in dimensionless quantities and we see that we have reproduced the lattice results:

$$\mu = k_0 a^2, \qquad \mu_c = d/2\pi, \qquad n = L/a, \qquad \sigma(\mu) = k_{\rm ph} a^2$$

$$m(\mu)\, n = m_{\rm ph} L, \qquad m(\mu) \sim (\mu - \mu_c)^{1/4},$$

$$\sigma(\mu) = \sigma(\mu_c) + \tilde{c}(\mu - \mu_c)^{1/2}, \qquad \sigma(\mu_c) = \frac{\mu_c}{2} > 0, \quad \tilde{c} = \frac{1}{2\sqrt{\mu_c}}.$$

After 30 years we have finally managed to reproduce the lattice results by a continuum standard string calculation. However, the problem is that these are of course not the standard string results and this is where the link to lattice quantum gravity becomes interesting. How to connect to “ordinary” string theory?

$$F_{mf}(L,\beta,k_0) = k_0 c\, L \sqrt{\beta^2 - \frac{d\pi}{3k_0 c}}, \quad c = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{d\Lambda^2}{2k_0}}$$

Since $k_0 \geq 2d\Lambda^2$ one would be tempted to think that one can ignore the ”tachyonic” term under the square root. It is of the order of the cut off and should not play any role in the limit where $\Lambda \to \infty$. This is precisely true in the limit we have just been discussing.

However, if we just consider X^μ as two-dimensional quantum fields (the 2d QG point of view), the decomposition $X = X_{\rm cl} + X_{\rm q}$ is a decomposition in a background field and quantum fluctuations, and in field theory, integrating out the quantum field will leave us with an effective action depending on the background fields which in general need to be renormalized for the effective action to be finite.

Thus

$$X_{\rm cl} = Z^{1/2} X_R \implies (L,\beta) = Z^{1/2}(L_R,\beta_R)$$

Is there a choice of $Z(k_0)$ and a renormalization of k_0 such that

$$F(L,\beta,k_0) = F_R(L_R,\beta_R,k_R), \qquad L_R,\beta_R,k_R \ \text{ fixed for } \ \Lambda \to \infty.$$

YES! $\qquad Z = \dfrac{2c-1}{c}, \qquad k_R = k_0\, c\, Z = \tilde{k}_{\rm ph}.$

$$Z = 1 - \frac{d\Lambda^2}{2k_0} + \mathcal{O}(k_0^{-2}), \qquad Z(k_0) \to 0 \text{ for } c \to \frac{1}{2}.$$

$$\frac{1}{\tilde{k}_{\rm ph}} = \frac{1}{k_0} + \frac{d\Lambda^2}{k_0^2} + \mathcal{O}(k_0^{-3})$$

With this renormalization we find the continuum string results

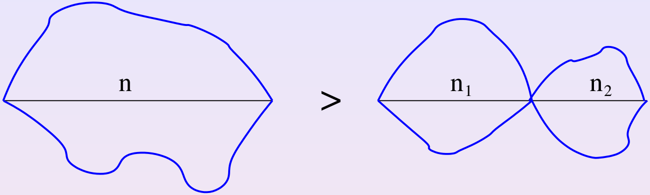
$$F(L_R, \beta_R, k_R) = k_R L_R \sqrt{\beta_R^2 - \frac{\pi d}{3k_R}} \qquad \langle A \rangle = L_R \, \frac{\left(\beta_R^2 - \frac{\pi d}{6k_R}\right)}{\sqrt{\beta_R^2 - \frac{\pi d}{3k_R}}},$$

However, this seems impossible to obtain using the lattice reguarization: on the lattice we have $L = n \cdot a(\mu)$ and the scaling limit is such that L is fixed for $a \to 0$ while $n \to \infty$. Thus $L \gg a$.This is conventional LFT. However the background field renormalization amount to writing

$$L = a \cdot \sqrt{2k_R/\mu_c}\, L_R, \quad \beta = a \cdot \sqrt{2k_R/\mu_c}\, \beta_R.$$

Thus the scaling limit where K_R, L_R and β_R are finite as $a \to 0$ is a limit where L and β are of the order of the cutoff a in contradiction to the LFT limit.

On the lattice it is difficult to understand the existence of a tachyon.



$$G(n) \geq G(n_1)G(n_2) \implies -\lim_{n\to\infty} \frac{\log G(n)}{n} \geq 0$$

Our continuum calculation is in accordance with this before we renormalize the lengths L, β. However when we go from a Gulliver world to a Lilliputian world we introduce a tachyon:

$$k_0 c\, L \sqrt{\beta^2 - \frac{\pi d}{3k_0 c}} = k_R L_R \sqrt{\beta_R^2 - \frac{\pi d}{3k_R}}, \qquad m_{\rm tac}^2 = \frac{3k_R}{\pi d}.$$

$$\langle A_{\text{Gulliver}} \rangle = L \frac{\beta^2 - \beta_0^2/2c}{\sqrt{\beta^2 - \beta_0^2/c}} \frac{c}{(2c-1)} \sim \frac{L\beta}{k_R a^2}$$

$$\langle A_{\text{Lilliput}} \rangle = L_R \frac{\beta_R^2 - \frac{\pi d}{6k_R}}{\sqrt{\beta_R^2 - \frac{\pi d}{3k_R}}} \sim L_R \beta_R$$

In the Gulliver world the wordsheet is unphysical, like the length of a path in the path integral of the free particle is unphysical (and diverge like $L/(m_{ph}a)$). However, in the Lilliputian world the worldsheet has become a physical 2d world which does not fluctuate too much (unless $\beta \to 1/m_{\text{tac}}$).

This situation seems generic in QG if we want to define correlation functions depending on some kind of diffeomorphism invariant distance.

$$G_V(R) \equiv \frac{1}{V} \int \mathcal{D}[g] \int \mathcal{D}_g\phi \; \mathrm{e}^{-S[g,\phi]} \; \delta\Big(\int \sqrt{g} - V\Big)$$
$$\int\int dx\, dy\, \sqrt{g(x)}\, \sqrt{g(y)}\; \phi(x)\phi(y)\; \delta(R - D_g(x,y)).$$

Here the field $[g_{\mu\nu}]$ corresponds to X^μ and like L, β are related to X then R is related to $[g]$ via the geodesic distance D_g, but in a much more complicated way. In fact the geodesic distance operator $\hat{D}_g(x,y)$ is so complicated that one could doubt that it makes any sense.

Again 2d Euclidean quantum gravity comes to our help via DT and via computer simulations and offers some hope that the geodesic distance operator might make sense.

As the very simpest example, let us consider 2d Euclidean quantum gravity and put $\phi = 1$. This can be done in two ways: either there is no field coupled to gravity at all (pure 2d gravity), or we have a field coupled to gravity but we choose to put $\phi = 1$ in the correlator. First we consider pure 2d QG.

$$G_V(R) = \langle S(R) \rangle_V \sim R^{d_h - 1}, \qquad R \ll V^{1/d_h},$$

$$G_V(R) = \frac{1}{V} \left\langle \int dxdy \sqrt{g(x)} \sqrt{g(y)}\, \delta(D_g(x,y) - R) \right\rangle_V$$

One can calculate $G_V(R)$ in the case of 2d Euclidean QG using DT as the lattice theory. Fix Λ rather than V:

$$G_\Lambda(R) = \int_0^\infty dV \, \mathrm{e}^{-\Lambda V} \, G_V(R)$$

The lattice result is (close to the critical point)

$$G_\mu(r) \sim (\mu - \mu_c)^{3/4} \frac{\cosh \left[(\mu - \mu_c)^{1/4} r \right]}{\sinh^3 \left[(\mu - \mu_c)^{1/4} r \right]}$$

while

$$\mu - \mu_c = \Lambda \, a^2(\mu), \quad V = N_T a^2, \quad L = \ell_T a.$$

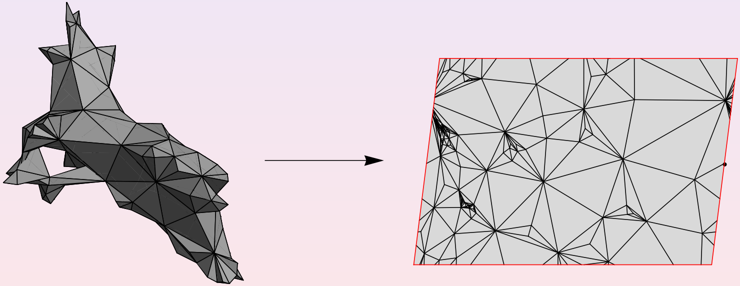
Thus, in order for $G_\mu(r)$ to have a continuum limit for $a(\mu) \to 0$
R has to have a anomalous scaling dimension:

$$G_\Lambda(R) \sim G_\mu(r), \quad R = r\sqrt{a(\mu)} \text{ rather than } R = r\, a(\mu).$$

What happens when a matter field ϕ is coupled to 2d Euclidean quantum gravity? It is not known analytically even in 2d Euclidean quantum gravity, but for for a number of field theories can be addressed by computer simulations.

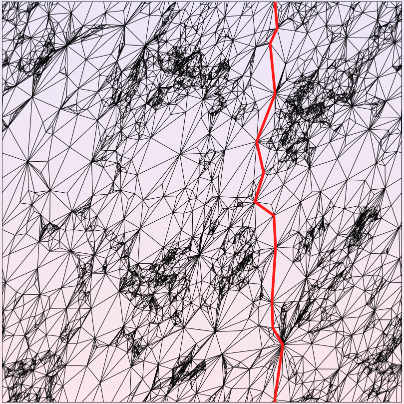
One observes that the anomalous scaling of R is related to the central charge of the conformal field theories couple to 2d quantum gravity and if such a global anomalous scaling exists then it is related to a non-trivial Hausdorff dimension via $V \sim R^{d_h}$

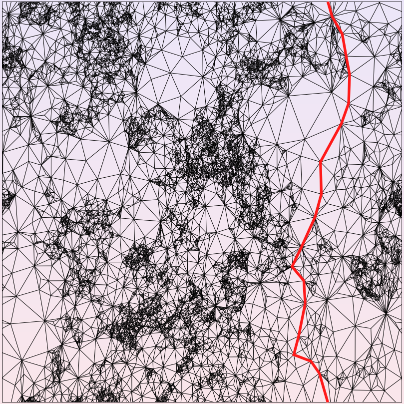
The fractal dimension can be "seen" nicely if the topology is that of a torus, which has the virtue that the shortest non-contractable loop is automatically a geodesic curve. Further, for the torus we have for analytic manifolds harmonic forms which have very nice discretized analogies, and we can use the these to construct a conformal mapping from the abstract triangulation to the complex plane (Timothy Budd and JA)



Since the shortest non-contractable loop is a geodesic:

$$\langle L \rangle_N \sim N^{1/d_h(c)}$$





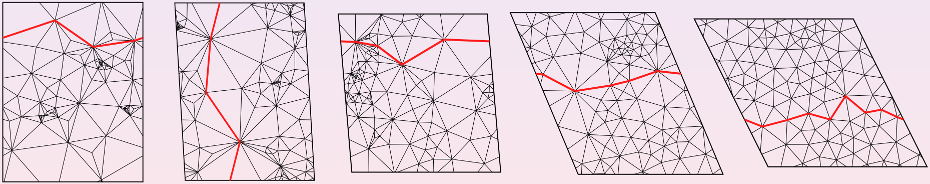
left figure $c = 0$, i.e. $d_h = 4$, right figure $c = -2$, $d_h = 3.56$

$$d_h(c) = 2\frac{\sqrt{49-c}+\sqrt{25-c}}{\sqrt{25-c}+\sqrt{1-c}}, \qquad d_h(0) = 4.$$

(formula of Y. Watabiki)

In the case of a CFT (mass =0) it is indeed a global scale.

c = -5 c = -10 c = -20 c = -40 c = -80



However, how should we really think about a massive correlator of the generic form

$$\langle \Phi\Phi(R) \rangle_V \sim R^{-\alpha} e^{-m_{\rm ph} R} \qquad R \ll \frac{1}{V^{1/d_h}}$$

For large distances one would expect R to be a distance like in gravity without matter (but R in pure gravity already scales anomalous, so $m_{\rm ph}$ has the wrong dimension). For short distances one would expect the propagator to behave like a massless propagator corresponding to a certain central charge, but then the dimension of R is different from the R at long distances. Thus we are seemingly led to a scale dependent Hausdorff dimension. One might feel uncomfortable with this since the Hausdorff dimension was used to define the length scale in the first place.

Maybe the lattice definition of gravity is more subtle than one naively would have expected approaching the problem with an Wilsonian attitude ?