Kähler-Dirac fermions on Euclidean dynamical triangulations

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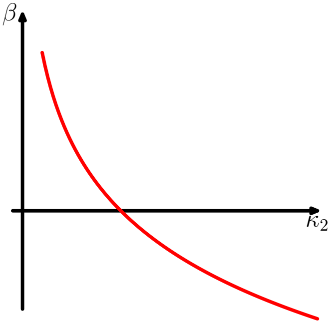
Motivating work

- We work under the asymptotic safety hypothesis
  - $\rightarrow$ gravity is strongly coupled in the UV
- There are many directions.
  - $\rightarrow$ we work with a lattice regularization with EDT.
- Ref. [Laiho et al., 2017] has been looking into Euclidean GR by including a measure term.
- The phase diagram of this model may have a continuous phase transition.

* Take this contingently true, and work out consequences to find problems.



Coupling to matter

- Coupling lattice gravity to scalar matter has been done [de Bakker and Smit, 1994, Hamber and Williams, 1994, de Bakker and Smit, 1997, Ambjørn et al., 2011]
- Coupling lattice gravity to gauge fields has also been explored [Renken et al., 1994, Bilke et al., 1998]
- Lattice fermions on curved lattice spacetimes are less understood [Jurkiewicz et al., 1988, Ambjørn and Varsted, 1991]
- Here we consider the most natural[1] version of fermion fields to couple to lattice gravity.

  $\rightarrow$ **Kähler-Dirac fermions**
- No need to devise a spin-connection
- No need to invent a lattice action

[1] In the sense that construction is easy.

Kähler-Dirac fermions in the continuum

- Simplest view is from the Laplace-de Rham operator:

$$\nabla^2 = (\mathrm{d} - \delta)^2 = -(\mathrm{d}\,\delta + \delta\,\mathrm{d})$$

  with d the exterior derivative, and δ its adjoint.

- Following Dirac's intuition, the Kähler-Dirac operator is

$$D = (\mathrm{d} - \delta)$$

- It's anti-Hermitian
- The spin-connection (and covariant derivative for that matter) are hidden inside d and δ.

Kähler-Dirac fermions in the continuum

Consider the action of $(\gamma^\mu \partial_\mu + m)$ on ψ.

- promote ψ to a 4×4 matrix.
- the γs form a basis for these.
- $\psi = f_0 + f_\mu \gamma^\mu + \frac{1}{2} f_{\mu\nu} \gamma^\mu \gamma^\nu + \dots$
- Any 4×4 matrix would work.
- This gives four identical copies of the Dirac equation.

$$\psi = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix}$$

e.g.

$$\psi_{10} = \psi_2 = f_0 \mathbb{1}_{10} + f_\mu \gamma^\mu_{10} + \frac{1}{2} f_{\mu\nu} \gamma^\mu_{1a} \gamma^\nu_{a0} + \dots$$

Kähler-Dirac fermions in the continuum

- Kähler-Dirac fields are a combination of p-forms

$$\omega = f_0 + f_\mu dx^\mu + \frac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu + \dots$$

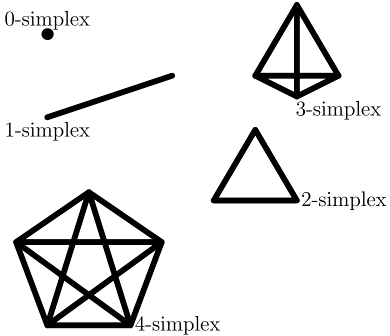
$$\psi = f_0 + f_\mu \gamma^\mu + \frac{1}{2} f_{\mu\nu} \gamma^\mu \gamma^\nu + \dots$$

- $\gamma^\mu \partial_\mu$ acts identically on ψ as $\mathrm{d} - \delta$ on ω.
- However this operator is still valid for any curved space-time
- Unfortunately, in curve spaces:

  Kähler-Dirac operator $\neq$ Dirac operator.
- Nevertheless, when the curvature is negligible:

  Kähler-Dirac operator $=$ Dirac operator[4]

The lattice is built from simplices of various dimensions.



Kähler-Dirac fermions on the lattice

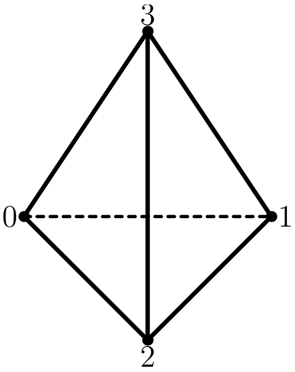
- (co)Homology theory tells us that there is a straightforward transcription to the lattice:

$$\mathrm{d} \mapsto \bar{\mathrm{d}}, \quad \delta \mapsto \bar{\delta}$$

- $\bar{\mathrm{d}}$ is the simplex boundary operator
- Given a set of simplex vertices {0,1,2,3}:

$$\begin{aligned}\bar{\mathrm{d}}\{0,1,2,3\} =& \{1,2,3\} - \{0,2,3\} + \\ & \{0,1,3\} - \{0,1,2\}\end{aligned}$$

  for example.
- $\bar{\delta}$ is its transpose.



$$p\text{-forms} \mapsto p\text{-simplices}$$

Kähler-Dirac fermions on the lattice

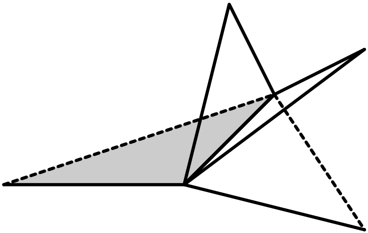
- With this transcription, the continuum results still hold on the lattice:

$$\bar{\nabla}^2 = (\bar{\mathrm{d}} - \bar{\delta})^2 = -(\bar{\mathrm{d}}\bar{\delta} + \bar{\delta}\bar{\mathrm{d}})$$

  is the lattice Laplacian, however in all simplex sectors, not just the 4- and 0-simplex sectors.
- The lattice Kähler-Dirac operator

$$\bar{D} = \bar{\mathrm{d}} - \bar{\delta}$$

  then carries all the same information as the lattice Laplace-de Rham operator.



Kähler-Dirac fermions on the lattice

- We considered the “quenched” approximation
  - $\rightarrow$ A fluctuating background with no back-reaction from the fermions
- We looked at:
  - the eigenvalues
  - correlations
  - condensates

Eigenvalues

We can add a real mass term and consider the spectral decomposition of the Kähler-Dirac operator

$$\bar{D} + m = \sum_{n=1}^{N} (i\lambda_n + m)|n\rangle\langle n|$$

Consider the similarity transform for transposition for $\bar{D}$, Γ,

$$\Gamma \bar{D} \Gamma^{-1} = \bar{D}^T = -\bar{D}$$

Then the Γ anti-commutes with $\bar{D}$ and ensures the eigenvalues come in complex conjugate pairs.

Eigenvalues

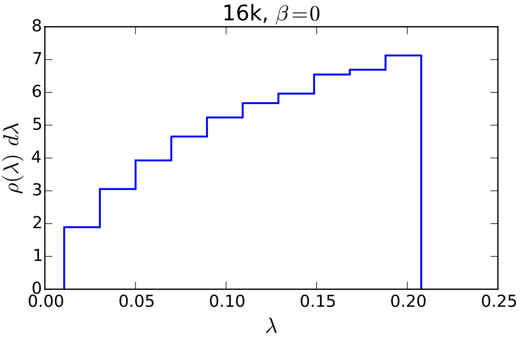
We can look at the eigenvalue density and number

$$d\nu = \rho(\lambda)\, d\lambda$$

whose integral gives the cumulative number of eigenvalues.

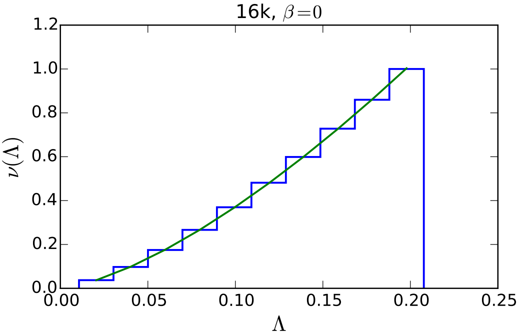
- If $\rho(\lambda) \sim \lambda^{\alpha}$ for small λ,

  $\implies$ Then $\nu(\Lambda) \sim \Lambda^{\alpha+1}$
- We find power-law behavior at small Λ



Eigenvalues

- Fitting
  $\ln \nu = (\alpha + 1) \ln \Lambda + b$.
- These can be combined with the volume scaling (next slide) to possibly extract an effective dimension.



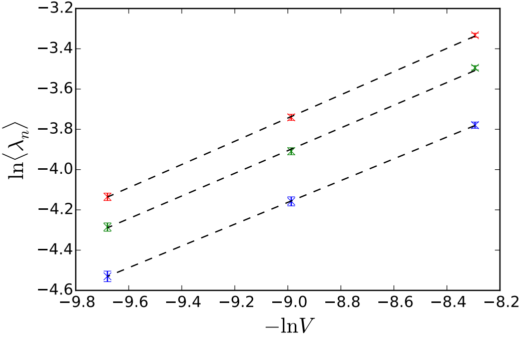
β	4k	8k	16k
0	1.436(2)	1.422(2)	1.457(3)
1.5	1.443(2)	1.384(3)	–

Eigenvalues

In addition we looked at the finite-size scaling of the lowest eigenvalues.

$$\langle \lambda_n \rangle \sim \left(\frac{1}{V}\right)^p$$

- For $\beta = 0$ we find $p \sim 0.55$
- for $\beta = 1.5$ we find $p \sim 0.52$



Eigenvalues

In Ref. [DeGrand, 2009] he looks at how finite-size scaling in the small eigenvalues is related to the power-law behavior of ν.

- If $\langle \lambda_n \rangle \sim (1/R)^{p'}$, then
$$p' = \frac{D_S}{\alpha + 1}$$
- Using the semi-classical radius ($V^{1/4}$) we can extract what we expect to be D_S

β	4k	8k	16k
0	3.16	3.13	3.21
1.5	3.0	2.88	–

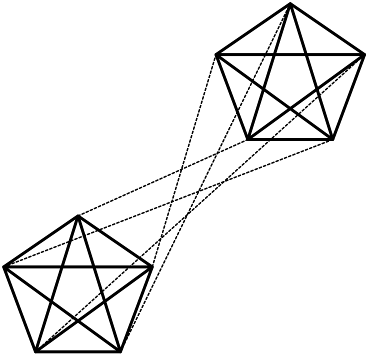
These give the hint that moving down the critical line increases the spectral dimension.

Correlations

- The inverse Kähler-Dirac operator can be written cleanly as

$$K^{-1} = (\bar{D} + m)^{-1} = \frac{K}{K^2}$$

- This matrix records the correlations between simplices
- There are a few possible (legitimate) definitions of distance for the simplices.
- We used smearing over the 4-simplex for its simplicity

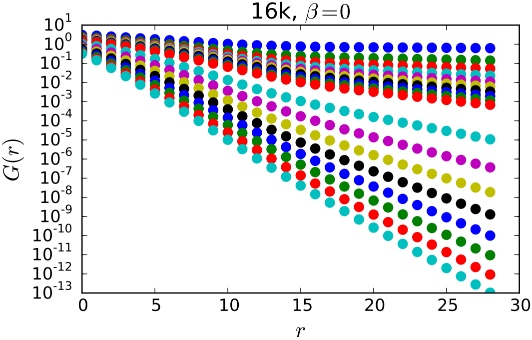


Correlations

- We fit to the ansatz that at long distances:

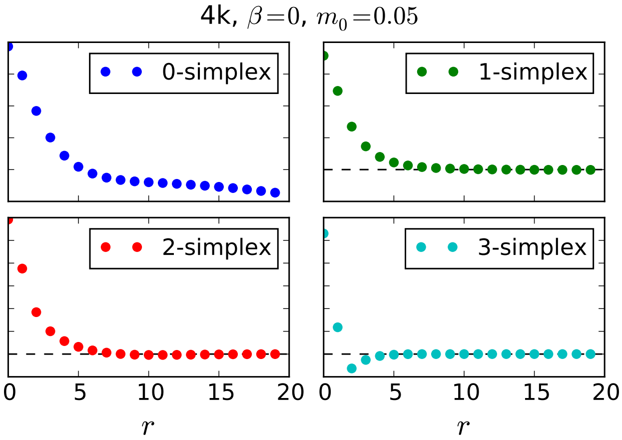
$$G(r) \sim e^{-mr}$$

- we find this empirically for a wide range of probe masses.
- We find the meson propagators also have this form.



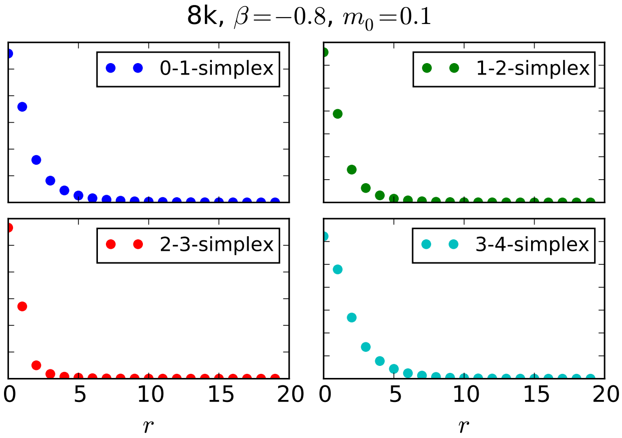
Correlations

- There are nine types of correlations
- Diagonal blocks are given by the inverse Laplacian
- These are p to p-simplex correlators
- off-diagonal blocks correlate p- and $p \pm 1$-simplices



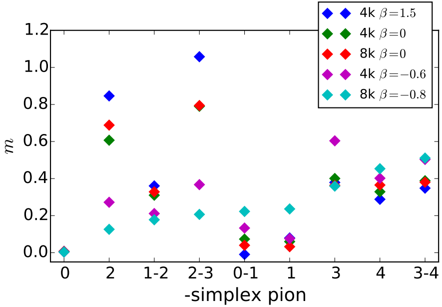
Pion-like correlators

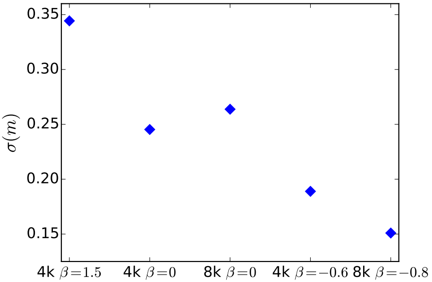
- Off-diagonal correlators of Pion-like mesons.
- $\langle \omega_x \Gamma \bar{\omega}_x \omega_y \Gamma \bar{\omega}_y \rangle \rightarrow \langle (\bar{\omega}_x \omega_y)(\bar{\omega}_x \omega_y)^\dagger \rangle$
- Γ anti-commutes with D ($\bar{D}$)
- All-in-all there are nine pion-like mesons one can make directly from the p-simplex sectors.



Pion-like correlators

The ground state energies and their spreads.





Condensates

We can also consider the diagonal of $(\bar{D}+m)^{-1}$. This is the bilinear condensate.

- We used Z_2 stochastic noise to extract the diagonal [Dong and Liu, 1994]

$$\langle \eta_i \eta_j \rangle = \delta_{ij}, \quad \langle \eta_i \rangle = 0$$

$$(\bar{D}+m)X = \eta \implies \langle \eta X \rangle = \langle \eta (\bar{D}+m)^{-1} \eta \rangle = (\bar{D}+m)^{-1}$$

- We used mini-ensembles of stochastic vectors, and averaged over EDT configs.
- We considered

$$\mathrm{Tr}[(\bar{D}+m)^{-1}] = \mathrm{Tr}[\langle \eta X \rangle] \sim \sum_x \langle \bar{\omega}_x \omega_x \rangle$$

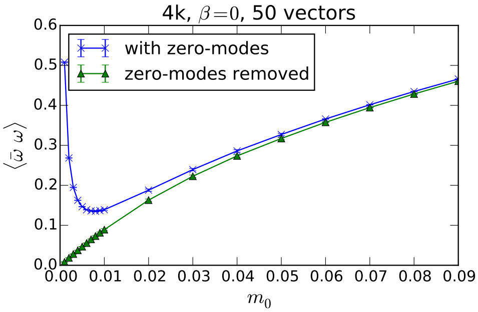
and

$$\mathrm{Tr}[\langle \eta X \rangle^2] \sim \sum_x \langle (\bar{\omega}_x \omega_x)^2 \rangle$$

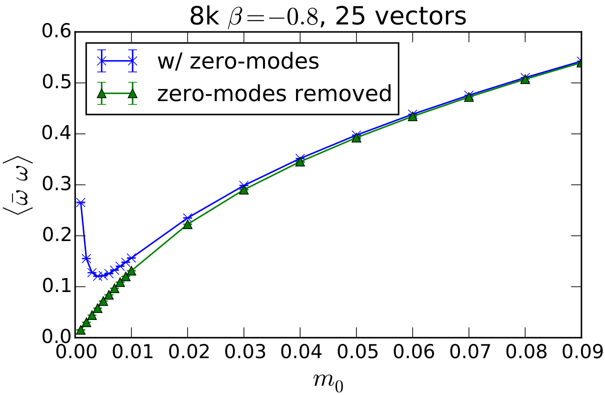
Condensates

- In the un-normalized case we know the small-mass scaling: there are two zero modes which scale $\sim 1/m_0$.
- With the zero-modes removed, the Γ-related symmetry is unbroken spontaneously in the chiral limit [Leutwyler and Smilga, 1992].

$$\langle \bar{\omega}\omega \rangle = \frac{2}{m_0} + 2m_0 \sum_n \frac{1}{\lambda_n^2 + m_0^2}$$



Condensates

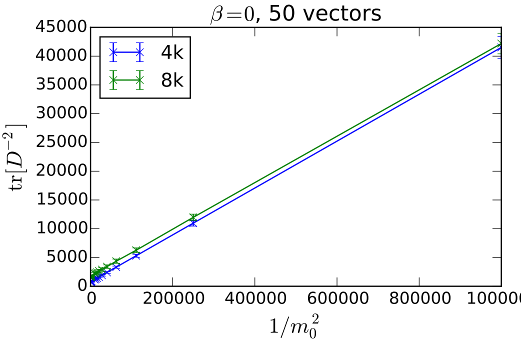


Condensates

- For the four-fermion condensate, it should go $\sim 1/m_0^2$ for small m_0.
- In fact, by naïve power counting, it should contribute to Z.

$$\rightarrow \det[\bar{D} + m_0] \sim m_0^2$$
$$\implies \det[\bar{D} + m_0](\bar{\omega}\omega)^2 \sim 1$$

- This would give a non-zero expectation value for the four-fermi condensate, but leave chiral symmetry unbroken.



Conclusions & future work

- Kähler-Dirac fermions can be put on DTs straightforwardly
- We expect in the large volume, flat spacetime limit similar to Dirac fermions
- As we approach the continuum limit we see degeneracy restored (four copies of Dirac fermions)
- Chiral symmetry is unbroken
- *To do:* Simulations with dynamical fermions

Thank you!

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