Amplitudes in Yang-Mills theory and gravity

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I. Scattering amplitudes
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Part I

Scattering amplitudes
In this talk we are interested in amplitudes of the following theories:

- The zeroth copy: Bi-adjoint scalar theory
- The single copy: Yang-Mills theory
- The double copy: Gravity

We consider tree amplitudes with an arbitrary number of external particles $n$. 
The single copy: Yang-Mills theory

The Lagrangian of a non-Abelian gauge theory:

\[ \mathcal{L}_{\text{YM}} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}, \quad F_{\mu\nu}^a = \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a + g f^{abc} A_{\mu}^b A_{\nu}^c \]

Decompose the tree amplitudes \( \mathcal{A}_n(p, \varepsilon) \) into group-theoretical factors and cyclic-ordered amplitudes \( A_n(\sigma, p, \varepsilon) \):

\[ \mathcal{A}_n(p, \varepsilon) = g^{n-2} \sum_{\sigma \in S_n/\mathbb{Z}_n} 2 \text{Tr} (T^{a_{\sigma(1)} \ldots a_{\sigma(n)}}) \ A_n(\sigma, p, \varepsilon) \]

with

\[ p = (p_1, \ldots, p_n) \quad \text{momenta} \]
\[ \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \quad \text{polarisations} \]
\[ \sigma = (\sigma_1, \ldots, \sigma_n) \quad \text{cyclic order} \]
Lie algebra:

\[
[T^a, T^b] = i f^{abc} T^c, \quad \text{Tr} (T^a T^b) = \frac{1}{2} \delta^{ab}
\]

Multiply commutator relation by \( T^d \) and take the trace:

\[
i f^{abc} = 2 \text{Tr} (T^a T^b T^c) - 2 \text{Tr} (T^b T^a T^c)
\]

Fierz identities for \( U(N) \):

\[
\text{Tr} (T^a X) \text{Tr} (T^a Y) = \frac{1}{2} \text{Tr} (X Y), \quad \text{Tr} (T^a X T^a Y) = \frac{1}{2} \text{Tr} (X) \text{Tr} (Y)
\]
The primitive amplitudes are **gauge-invariant** and each primitive amplitude has a fixed **cyclic order** of the external legs.

The primitive amplitudes are calculated from **cyclic-ordered Feynman rules**:

\[
\begin{align*}
\text{loop} & = -\frac{ig_{\mu\nu}}{p^2} \\
\text{triangle} & = i [g^{\mu_1\mu_2} (p_1^{\mu_3} - p_2^{\mu_3}) + g^{\mu_2\mu_3} (p_2^{\mu_1} - p_3^{\mu_1}) + g^{\mu_3\mu_1} (p_3^{\mu_2} - p_1^{\mu_2})] \\
\text{cross} & = i [2g^{\mu_1\mu_3} g^{\mu_2\mu_4} - g^{\mu_1\mu_2} g^{\mu_3\mu_4} - g^{\mu_1\mu_4} g^{\mu_2\mu_3}] 
\end{align*}
\]
The zeroth copy: Bi-adjoint scalar theory

A scalar field in the adjoint representation of two gauge-groups $G \times \tilde{G}$ with Lagrange density

$$\mathcal{L} = \frac{1}{2} \left( \partial_{\mu} \phi^{ab} \right) \left( \partial^{\mu} \phi^{ab} \right) - \frac{\lambda}{3!} f^{a_1a_2a_3} \tilde{f}^{b_1b_2b_3} \phi^{a_1b_1} \phi^{a_2b_2} \phi^{a_3b_3}$$

Decompose the tree amplitudes $m_n(p)$ into group-theoretical factors and double-ordered amplitudes $m_n(\sigma, \tilde{\sigma}, p)$:

$$m_n(p) = \lambda^{n-2} \sum_{\sigma \in S_n/\mathbb{Z}_n} \sum_{\tilde{\sigma} \in S_n/\mathbb{Z}_n} 2 \text{Tr} \left( T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}} \right) 2 \text{Tr} \left( \tilde{T}^{b_{\tilde{\sigma}(1)}} \ldots \tilde{T}^{b_{\tilde{\sigma}(n)}} \right) m_n(\sigma, \tilde{\sigma}, p)$$

The permutations $\sigma$ and $\tilde{\sigma}$ denote two cyclic orders.
Double-ordered amplitudes

Flip: exchange two branches at a vertex.

Two diagrams with different external orders are equivalent, if we can transform one diagram into the other by a sequence of flips.

The double-ordered amplitude $m_n(\sigma, \tilde{\sigma}, p)$ is computed from the Feynman diagrams compatible with the cyclic orders $\sigma$ and $\tilde{\sigma}$.

Feynman rules:

\[
\begin{align*}
\text{Feynman rule 1} & \quad = \frac{i}{p^2} \\
\text{Feynman rule 2} & \quad = i
\end{align*}
\]
Let us consider (small) fluctuations around the flat Minkowski metric

\[ g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \]

with \( \kappa = \sqrt{\frac{32\pi}{G}} \) and consider an effective theory defined by the Einstein-Hilbert Lagrangian

\[ \mathcal{L}_{\text{EH}} = -\frac{2}{\kappa^2} \sqrt{-g} R. \]

The field \( h_{\mu\nu} \) describes a graviton.

The inverse metric \( g^{\mu\nu} \) and \( \sqrt{-g} \) are infinite series in \( h_{\mu\nu} \), therefore

\[ \mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{GF}} = \sum_{n=2}^{\infty} \mathcal{L}^{(n)}, \]

where \( \mathcal{L}^{(n)} \) contains exactly \( n \) fields \( h_{\mu\nu} \).

Thus the Feynman rules will give an infinite tower of vertices.
Feynman rules for gravity

External edge:

\[\varepsilon_{\mu_1}(k) \varepsilon_{\mu_2}(k)\]

Internal edge:

\[\frac{1}{2} \left( \eta_{\mu_1 \nu_1} \eta_{\mu_2 \nu_2} + \eta_{\mu_1 \nu_2} \eta_{\mu_2 \nu_1} - \frac{2}{D-2} \eta_{\mu_1 \mu_2} \eta_{\nu_1 \nu_2} \right) \frac{i}{k^2}\]

Vertices:

= long expression

= even longer expression

plus Feynman rules for 5-graviton vertex, 6-graviton vertex, etc.
Graviton amplitudes

The graviton amplitudes are un-ordered, we simply factor out the coupling:

$$\mathcal{M}_n(p, \varepsilon, \tilde{\varepsilon}) = \left( \frac{\kappa}{4} \right)^{n-2} M_n(p, \varepsilon, \tilde{\varepsilon})$$

\[ p = (p_1, \ldots, p_n) \quad \text{momenta} \]
\[ \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \quad \text{first set of spin-1 polarisation vectors} \]
\[ \tilde{\varepsilon} = (\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_n) \quad \text{second set of spin-1 polarisation vectors} \]

\[ \varepsilon_j^+ \tilde{\varepsilon}_j^+ \text{ and } \varepsilon_j^- \tilde{\varepsilon}_j^- \text{ describe the two polarisation states of the spin-2 graviton with index } j. \]
Amplitudes

We consider the double ordered bi-adjoint scalar amplitudes \( m_n(\sigma, \tilde{\sigma}, p) \), the single ordered Yang-Mills amplitudes \( A_n(\sigma, p, \epsilon) \) and the un-ordered graviton amplitudes \( M_n(p, \epsilon, \tilde{\epsilon}) \).

All these amplitudes can be computed from Feynman diagrams.

\[
m_n(\sigma, \tilde{\sigma}, p) = i(-1)^{n-3+n_{\text{flip}}(\sigma, \tilde{\sigma})} \sum_{\text{trivalent graphs } G \text{ compatible with } \sigma \text{ and } \tilde{\sigma}} \frac{1}{D(G)}, \quad D(G) = \prod_{\text{edges } e} s_e,
\]

\[
A_n(\sigma, p, \epsilon) = \text{long expression},
\]

\[
M_n(p, \epsilon, \tilde{\epsilon}) = \text{even longer expression}.
\]
Part II

Review of recent developments

1. Jacobi-like relations (BCJ numerators)

2. The scattering equations (CHY representation)

3. KLT relations

4. Positive geometries and canonical forms

5. Intersection theory
Part II.1

Jacobi-like relations
Jacobi relation:

$$[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0,$$

In terms of structure constants:

$$(i f_{abe}) (i f_{ecd}) + (i f_{bce}) (i f_{ead}) + (i f_{cae}) (i f_{ebd}) = 0.$$ 

Graphically:

$$
\begin{align*}
\text{1} & \quad \text{2} & \quad \text{3} \\
& \quad \downarrow \\
\text{4} \\
\end{align*}
\quad +
\begin{align*}
\text{2} & \quad \text{3} & \quad \text{1} \\
& \quad \downarrow \\
\text{4} \\
\end{align*}
\quad +
\begin{align*}
\text{3} & \quad \text{1} & \quad \text{2} \\
& \quad \downarrow \\
\text{4} \\
\end{align*}
= 0.$$
In Yang-Mills theory we have a *three-valent* and a *four-valent* vertex.

We may always re-write a four-valent vertex in terms of two three-valent vertices:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
4
\end{array}
\end{array}
\end{array}
\end{array}
=\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
4
\end{array}
\end{array}
\end{array}
\end{array}
+\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
4
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

This is not unique!
We may write the Yang-Mills amplitude in a form

\[ A_n(\sigma, p, \varepsilon) = i(-1)^{n-3} \sum_{\text{trivalent graphs}} \frac{N(G)}{D(G)}, \]

with numerators \( N(G) \) satisfying anti-symmetry relations and Jacobi relations:

\[
\begin{align*}
\begin{array}{ccc}
1 & 2 & 3 \\
 & & \\
4 & & \\
\end{array} & + & \\
\begin{array}{ccc}
2 & 3 & 1 \\
 & & \\
4 & & \\
\end{array} & + & \\
\begin{array}{ccc}
3 & 1 & 2 \\
 & & \\
4 & & \\
\end{array} &= 0
\end{align*}
\]

\[
N(G_1) + N(G_2) + N(G_3) = 0
\]

Bern, Carrasco, Johansson, '10

**BCJ numerators**
Multi-peripheral graphs

Combining the anti-symmetry of the vertices and the Jacobi identity one has

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2 \\
3
\end{array}
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
2 \\
3
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
3 \\
2
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \\
4
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \\
4
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \\
4
\end{array}
\end{array}
\end{array}
\end{array}
\]

We may express all BCJ-numerators in terms of the BCJ-numerators of multi-peripheral graphs (or comb graphs):

\[
\begin{array}{c}
\begin{array}{c}
\alpha_2 \\
\alpha_3 \\
\alpha_{n-1}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \\
\ldots
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
n
\end{array}
\end{array}
\]
Double copy and colour-kinematics duality

If the Yang-Mills amplitude is written in terms of BCJ-numerators $N(G)$ and group-theoretical factors $C(G)$

$$\mathcal{A}_n(p, \varepsilon) = i (-1)^{n-3} \frac{n-2}{n-3} \sum_{\text{trivalent graphs } G} \frac{C(G) N(G)}{D(G)},$$

then

$$\mathcal{M}_n(p, \varepsilon, \tilde{\varepsilon}) = i (-1)^{n-3} \left( \frac{\kappa}{4} \right)^{n-2} \sum_{\text{trivalent graphs } G} \frac{N(G) \tilde{N}(G)}{D(G)},$$

and of course

$$m_n(p) = i (-1)^{n-3} \lambda^{n-2} \sum_{\text{trivalent graphs } G} \frac{C(G) \tilde{C}(G)}{D(G)}.$$

Bern, Carrasco, Johansson, '10
We may construct an effective Lagrangian, which gives directly BCJ-numerators

$$\mathcal{L}_{YM} + \mathcal{L}_{GF} = \sum_{n=2}^{\infty} \mathcal{L}^{(n)},$$

$\mathcal{L}^{(2)}$, $\mathcal{L}^{(3)}$ and $\mathcal{L}^{(4)}$ agree with the standard terms and $\mathcal{L}^{(n \geq 5)}$ are a complicated zero.

The effective Lagrangian is not unique.

Tolotti, S.W, '13
Part II.2

The scattering equations
The Riemann sphere

The Riemann sphere is the complex plane plus the point at infinity:

\[ \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \]

Each \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{C}) \) acts on \( z \in \hat{\mathbb{C}} \) through a Möbius transformation:

\[ g \cdot z = \frac{az + b}{cz + d}. \]

Mark \( n \) distinct points \((z_1, \ldots, z_n)\) on \( \hat{\mathbb{C}} \).

The moduli space of genus 0 curves with \( n \) distinct marked points is denoted by

\[ \mathcal{M}_{0,n} = \left\{ z \in \hat{\mathbb{C}}^n : z_i \neq z_j \right\} / \text{PSL}(2, \mathbb{C}). \]

\( \mathcal{M}_{0,n} \) is an affine algebraic variety of dimension \((n - 3)\).
The scattering equations

Set

\[ f_i(z, p) = \sum_{j=1, j \neq i}^{n} \frac{2p_i \cdot p_j}{z_i - z_j}. \]

The scattering equations:

\[ f_i(z, p) = 0, \quad 1 \leq i \leq n. \]

Only \((n - 3)\) equations of the \(n\) equations are independent.

Two solutions which are related by a Möbius-transformation are called equivalent solutions.

There are \((n - 3)!\) inequivalent solutions not related by a Möbius-transformation.
The CHY representation

There exists two functions $C(\sigma, z)$ and $E(p, \varepsilon, z)$ on $\hat{C}^n$ such that

$$m_n(\sigma, \tilde{\sigma}, p) = i \oint_{\mathcal{C}} d\Omega_{\text{CHY}} C(\sigma, z) C(\tilde{\sigma}, z),$$

$$A_n(\sigma, p, \varepsilon) = i \oint_{\mathcal{C}} d\Omega_{\text{CHY}} C(\sigma, z) E(p, \varepsilon, z),$$

$$M_n(p, \varepsilon, \tilde{\varepsilon}) = i \oint_{\mathcal{C}} d\Omega_{\text{CHY}} E(p, \varepsilon, z) E(p, \tilde{\varepsilon}, z).$$

Details on the definition of the measure $d\Omega_{\text{CHY}}$:

$$d\Omega_{\text{CHY}} = \frac{1}{(2\pi i)^n} \frac{d^n z}{d\omega} \prod' \frac{1}{f_a(z, p)},$$

$$\prod' \frac{1}{f_a(z, p)} = (-1)^{i+j+k} (z_i - z_j) (z_j - z_k) (z_k - z_i) \prod_{a \neq i, j, k} \frac{1}{f_a(z; p)},$$

$$d\omega = (-1)^{p+q+r} \frac{dz_p dz_q dz_r}{(z_p - z_q) (z_q - z_r) (z_r - z_p)}.$$

Cachazo, He and Yuan, '13
The cyclic factor (or Parke-Taylor factor) is given by

$$C(\sigma, z) = \frac{1}{(z_{\sigma_1} - z_{\sigma_2})(z_{\sigma_2} - z_{\sigma_3}) \cdots (z_{\sigma_n} - z_{\sigma_1})}.$$ 

The cyclic factor **encodes the information on the cyclic order.**
The polarisation factor \( E(p, \varepsilon, z) \) encodes the information on the helicities of the external particles.

One possibility to define this factor is through a reduced Pfaffian.

(All definitions have to agree on the solutions of the scattering equations, but may differ away from this zero-dimensional sub-variety.)
The reduced Pfaffian

Define a \((2n) \times (2n)\) antisymmetric matrix \(\Psi(z, p, \varepsilon)\) through

\[
\Psi(z, p, \varepsilon) = \begin{pmatrix}
A & -C^T \\
C & B
\end{pmatrix}
\]

with

\[
A_{ab} = \begin{cases}
\frac{2p_a \cdot p_b}{z_a - z_b} & a \neq b, \\
0 & a = b,
\end{cases}
\quad
B_{ab} = \begin{cases}
\frac{2\varepsilon_a \cdot \varepsilon_b}{z_a - z_b} & a \neq b, \\
0 & a = b,
\end{cases}
\quad
C_{ab} = \begin{cases}
\frac{2\varepsilon_a \cdot p_b}{z_a - z_b} & a \neq b, \\
-\sum_{j=1, j \neq a}^{n} \frac{2\varepsilon_a \cdot p_j}{z_a - z_j} & a = b.
\end{cases}
\]

Denote by \(\Psi_{ij}^{ij}\) the \((2n - 2) \times (2n - 2)\)-matrix, where rows and columns \(i\) and \(j\) have been deleted \((1 \leq i < j \leq n)\).

The reduced Pfaffian \(E^{\text{Pfaff}}(z, p, \varepsilon)\) is defined by

\[
E^{\text{Pfaff}}(z, p, \varepsilon) = \frac{(-1)^{i+j}}{2(z_i - z_j)} \text{Pf} \ \Psi_{ij}^{ij}(z, p, \varepsilon).
\]

Cachazo, He and Yuan, '13
KLT relations
Independent primitive amplitudes

How many independent primitive amplitudes $A_n(\sigma, p, \varepsilon)$ are there for fixed momenta $p$ and polarisations $\varepsilon$?

- There are $n!$ external orderings.

- **Cyclic invariance** reduce the number to $(n - 1)!$.

- **Anti-symmetry** of the vertices reduce the number to $(n - 2)!$.
  
  Kleiss, Kuijf, 1989

- **Jacobi relations** reduce the number to $(n - 3)!$.
  
  Bern, Carrasco, Johansson, 2008

**Basis $B$** of independent amplitudes consists of $(n - 3)!$ elements.
Define \((n - 3)! \times (n - 3)!\)-dimensional matrix \(m_{\sigma \bar{\sigma}}\) for \(\sigma, \bar{\sigma} \in B\) by

\[
m_{\sigma \bar{\sigma}} = m_n(\sigma, \bar{\sigma}, p).
\]

The matrix \(m\) is invertible.

Define the KLT-matrix as the inverse of the matrix \(m\):

\[
S = m^{-1}
\]

Kawai, Lewellen, Tye, 1986,
Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove, 2010,
Cachazo, He and Yuan, 2013,
de la Cruz, Kniss, S.W., 2016
The KLT relations express the graviton amplitude $M_n(p, \varepsilon, \tilde{\varepsilon})$ through products of Yang-Mills amplitudes $A_n(\sigma, p, \varepsilon)$ and the KLT-matrix $S$:

$$M_n(p, \varepsilon, \tilde{\varepsilon}) = \sum_{\sigma, \tilde{\sigma} \in B} A_n(\sigma, p, \varepsilon) S_{\sigma \tilde{\sigma}} A_n(\tilde{\sigma}, p, \tilde{\varepsilon})$$

Graphically:
Positive geometries and canonical forms
Multivariate residues of differential forms

Let $X$ be a $m$-dimensional variety and $Y$ a co-dimension one sub-variety. Let us choose a coordinate system such that $Y$ is given locally by $z_1 = 0$. Assume that $\Omega$ has a pole of order 1 on $Y$:

$$\Omega = \frac{dz_1}{z_1} \wedge \psi + \theta.$$

The residue of $\Omega$ at $Y$ is defined by

$$\text{Res}_Y (\Omega) = \psi|_Y.$$

A pole of order 1 on $Y$ is called a logarithmic singularity on $Y$. 
Let $X$ be a $m$-dimensional (complex) variety and $X_{\geq 0}$ the positive part. A $m$-form $\Omega$ is called a canonical form if

1. For $m = 0$ one has $\Omega = \pm 1$.

2. The only singularities of $\Omega$ are on the boundary of $X_{\geq 0}$.

3. The singularities are logarithmic.

4. The residue of $\Omega$ on a boundary component is again the canonical form of a $(m - 1)$-dimensional positive geometry.

Arkani-Hamed, Bai, Lam, '17
Part II.5

Intersection theory
The CHY representation

The CHY half-integrands $C(\sigma, z)$ and $E(p, \varepsilon, z)$ transform under $\text{PSL}(2, \mathbb{C})$-transformations as

$$F(g \cdot z) = \left( \prod_{j=1}^{n} (cz_j + d)^2 \right) F(z)$$

Therefore, the $(n-3)$-forms

$$\Omega^{\text{cyclic}}(\sigma, z) = C(\sigma, z) \frac{d^n z}{d\omega}, \quad \Omega^{\text{pol}}(p, \varepsilon, z) = E(p, \varepsilon, z) \frac{d^n z}{d\omega}.$$ 

are $\text{PSL}(2, \mathbb{C})$-invariant.

Remark: We may add to $C(\sigma, z)$ and $E(p, \varepsilon, z)$ terms which vanish on the solutions of the scattering equations.
Consider a space $X$ of dimension $m$, equipped with a connection $\nabla = d + \eta$. The connection one-form $\eta$ is called the twist.

Elements of

$$H^m(X, \nabla) = \{ \varphi \mid \nabla \varphi = 0 \} / \{ \nabla \xi \}$$

are called twisted co-cycles.

The intersection number of two twisted co-cycles is defined by

$$(\varphi_1, \varphi_2) = \frac{1}{(2\pi i)^m} \int_X \iota(\varphi_1) \wedge \varphi_2,$$

where $\iota$ maps $\varphi_1$ to a twisted co-cycle in the same cohomology class but with compact support.

Cho, Matsumoto, ’95; Aomoto, Kita, ’94 (jap.), ’11 (engl.)
Intersection theory

Apply this to $X = \mathcal{M}_{0,n}$ and take

$$\eta = \sum_{i=1}^{n} f_i(z, p) \, dz_i.$$ 

Then (Mizera, ’17)

$$m_n (\sigma, \tilde{\sigma}, p) = i \left( \Omega^{\text{cyclic}} (\sigma, z), \Omega^{\text{cyclic}} (\tilde{\sigma}, z) \right),$$

$$A_n (\sigma, p, \epsilon) = i \left( \Omega^{\text{cyclic}} (\sigma, z), \Omega^{\text{pol}} (p, \epsilon, z) \right),$$

$$M_n (p, \epsilon, \tilde{\epsilon}) = i \left( \Omega^{\text{pol}} (p, \epsilon, z), \Omega^{\text{pol}} (p, \tilde{\epsilon}, z) \right).$$

Remark: We may still add to $\Omega^{\text{cyclic}}$ and $\Omega^{\text{pol}}$ terms which vanish on the solutions of the scattering equations.
Part III

Geometric interpretation of scattering amplitudes
Geometric interpretation of tree amplitudes

There exist two \((n - 3)\)-forms \(\Omega^{\text{cyclic}}(\sigma, z)\) and \(\Omega^{\text{pol}}(p, \varepsilon, z)\) on the compactified moduli space \(\overline{M}_{0,n}\) such that

1. The twisted intersection numbers give the amplitudes for the bi-adjoint scalar theory (cyclic, cyclic), Yang-Mills theory (cyclic, polarisation) and gravity (polarisation, polarisation).

2. The only singularities of the scattering forms are on the divisor \(\overline{M}_{0,n} \setminus M_{0,n}\).

3. The singularities are logarithmic.

4. The residues at the singularities factorise into two scattering forms of lower points.

L. de la Cruz, A. Kniss, S.W., '17
The scattering forms

The **cyclic scattering form** is defined by

\[
\Omega^{\text{cyclic}}(\sigma, z) = C(\sigma, z) \frac{d^n z}{d\omega}, \quad C(\sigma, z) = \frac{1}{(z\sigma_1 - z\sigma_2)(z\sigma_2 - z\sigma_3) \ldots (z\sigma_n - z\sigma_1)}.
\]

The **polarisation scattering form** is defined by

\[
\Omega^{\text{pol}}(p, \epsilon, z) = E(p, \epsilon, z) \frac{d^n z}{d\omega}, \quad E(p, \epsilon, z) = \sum_{\kappa \in \mathcal{S}_{n-2}^{(1,n)}} C(\kappa, z) \ N_{\text{comb}}(\kappa),
\]

where the sum is now over all permutations keeping \( \kappa_1 = 1 \) and \( \kappa_n = n \) fixed.

L. de la Cruz, A. Kniss, S.W., '17
The $n$-graviton amplitude is given by

$$M_n(p, \varepsilon, \tilde{\varepsilon}) = i(-1)^{n-3} \sum_{\text{trivalent graphs } G} \frac{N(G) \tilde{N}(G)}{D(G)}$$

colour-kinematics duality

$$= i \oint_{\mathcal{C}} d\Omega_{\text{CHY}} E(p, \varepsilon, z) E(p, \tilde{\varepsilon}, z)$$

CHY representation

$$= \sum_{\sigma, \tilde{\sigma} \in B} A_n(\sigma, p, \varepsilon) S_{\sigma \tilde{\sigma}} A_n(\tilde{\sigma}, p, \tilde{\varepsilon})$$

KLT relation

$$= i \left( \Omega_{\text{pol}}(p, \varepsilon, z), \Omega_{\text{pol}}(p, \tilde{\varepsilon}, z) \right)$$

intersection number
● Jacobi-like relations

● The scattering equations

● Positive geometries and canonical forms

● Intersection theory

● ... and some self-advertisement
Concluding controversial statement

There are interesting relations between the (tree-level) scattering amplitudes of Yang-Mills theory and gravity.

These relations are not manifest in the action as a coordinate space integral over a Lagrange density.

Should we not find and work with a formulation, which makes these structures manifest from the beginning?