Quantum field theory in curved spacetime

Assignment 5/Exam 1 – May 28

Please hand in this assignment before the tutorial at 11h15AM on May 28. In total, you need to obtain 40% of the combined points from this and the second exam.

Exercise 12: Particle creation in an expanding universe – 30pts.

Motivation: Back to square one. Let's compute the number of particles created in a more realistic scenario than before.

Consider a spatially flat universe which starts out changing adiabatically, then undergoes a rapid phase of expansion, to finally end up in another adiabatic phase. Such a universe is given by the scale factor

$$a^2(\eta) = a_1^2 + (a_2^2 - a_1^2)\frac{1+\tanh\frac{\eta}{\eta_0}}{2}, \tag{12.1}$$

with the dimensionless parameters a_1 and a_2, and the quantity η_0 which has units of time. Propagating in this universe, consider a conformally coupled, massive scalar field according to the action

$$S = \int \mathrm{d}^4x \left(\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2}\left(m^2 - \frac{R}{6}\right)\phi^2\right). \tag{12.2}$$

(a) Plot the scale factor. What do the parameters a_1, a_2 and η_0 stand for? Ignoring the flat asymptotic regions (large $|\eta|$), what part of the universe's history could the intermediate evolution be a toy model for?

(b) Show that the scale factor Eq. (12.1) results in the squared effective mass

$$m_{\text{eff}}^2 = m^2\left(a_1^2 + (a_2^2 - a_1^2)\frac{1+\tanh\frac{\eta}{\eta_0}}{2}\right) \tag{12.3}$$

for the scalar.

(c) Having applied the usual rescaling $\phi \to \chi = a\phi$, the field satisfies the usual mode equation

$$\chi_k'' + \left(k^2 + m_{\text{eff}}^2\right)\chi_k = 0. \tag{12.4}$$

Show that the mode equation can be solved by the following two linearly independent mode functions

$$v_k = \frac{e^{i\mathbf{kx} - i\omega_+\eta - i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\text{in}}}}\,{}_2F_1\left(1 + i\omega_-\eta_0, i\omega_-\eta_0; 1 - i\omega_{\text{in}}\eta_0; \frac{1+\tanh\frac{\eta}{\eta_0}}{2}\right), \tag{12.5}$$

$$u_k = \frac{e^{i\mathbf{kx} - i\omega_+\eta - i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\text{out}}}}\,{}_2F_1\left(1 + i\omega_-\eta_0, i\omega_-\eta_0; 1 + i\omega_{\text{out}}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right), \tag{12.6}$$

where we defined

$$\omega_{\text{in}}^2 \equiv k^2 + a_1^2m^2, \qquad \omega_{\text{out}}^2 \equiv k^2 + a_2^2m^2, \qquad \omega_\pm \equiv \frac{\omega_{\text{out}} \pm \omega_{\text{in}}}{2}, \tag{12.7}$$

and ${}_2F_1$ denotes a common type of hypergeometric function.

Hint: Try to recover the differential equation defining the hypergeometric function $f = {}_2F_1(a, b; c; z)$, namely

$$z(1-z)f'' + [c-(1+a+b)z]f' - abf = 0. \tag{12.8}$$

(d) Show that the mode functions asymptote to Minkowski-like positive-frequency solutions at early and late times

$$v_k \sim \frac{e^{i(\mathbf{kx}-\omega_{\rm in}\eta)}}{\sqrt{4\pi\omega_{\rm in}}}, \qquad \eta \to -\infty, \tag{12.9}$$

$$u_k \sim \frac{e^{i(\mathbf{kx}-\omega_{\rm out}\eta)}}{\sqrt{4\pi\omega_{\rm out}}}, \qquad \eta \to \infty. \tag{12.10}$$

Discuss why the mode equation has solutions with these asymptotics. Conclude that v_k defines a natural in-vacuum, and u_k a natural out-vacuum. Why?

Hint: The hypergeometric function has the limit $\lim_{z\to 0} f(a, b; c; z) = 1$ for all a, b, c.

Thus, the field can be expanded in modes as

$$\chi = \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \left(a_{\mathbf{k}} v_k + a^\dagger_{\mathbf{k}} v^*_k\right), \tag{12.11}$$

where $a_{\mathbf{k}}$ defines the in-vacuum via $a_{\mathbf{k}}|0_{\rm in}\rangle = 0$. We can write down a similar mode expansion, namely

$$\chi = \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \left(b_{\mathbf{k}} u_k + b^\dagger_{\mathbf{k}} u^*_k\right), \tag{12.12}$$

where $b_{\mathbf{k}}$ defines the out-vacuum via $b_{\mathbf{k}}|0_{\rm out}\rangle = 0$, and $b^\dagger_{\mathbf{k}}$ constructs particle states at late times. Clearly $v_k \neq u_k$. Thus, the two have to be related as

$$v_k = \alpha_k u_k + \beta_k u^*_{-k}, \tag{12.13}$$

with the Bogolyubov coefficients α_k, and β_k.

(e) Demonstrate that the Bogolyubov coefficients equal

$$\alpha_k = \sqrt{\frac{\omega_{\rm out}}{\omega_{\rm in}}} \frac{\Gamma(1-i\omega_{\rm in}\eta_0)\Gamma(-i\omega_{\rm out}\eta_0)}{\Gamma(-i\omega_+\eta_0)\Gamma(1-i\omega_+\eta_0)}, \tag{12.14}$$

$$\beta_k = \sqrt{\frac{\omega_{\rm out}}{\omega_{\rm in}}} \frac{\Gamma(1-i\omega_{\rm in}\eta_0)\Gamma(i\omega_{\rm out}\eta_0)}{\Gamma(i\omega_-\eta_0)\Gamma(1+i\omega_-\eta_0)}. \tag{12.15}$$

Hint: The hypergeometric function ${}_2F_1$ satisfies the identities

$$\begin{aligned}{}_2F_1(a, b; c; z) =& \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b; a+b-c+1; 1-z) \\ &+ (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} {}_2F_1(c-a, c-b; c-a-b+1; 1-z),\end{aligned} \tag{12.16}$$

$${}_2F_1(a, b; c; z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z). \tag{12.17}$$

(f) Assume that the field is in the vacuum state at early times. Show that the particle number density at late times equals

$$n_k = \frac{\sinh^2(\pi\omega_-\eta_0)}{\sinh(\pi\omega_{\rm in}\eta_0)\sinh(\pi\omega_{\rm out}\eta_0)}. \tag{12.18}$$

Exercise 13: Hawking radiation from non-singular black holes – 20pts.

Motivation: Black-hole solutions in GR are singular; it is expected that a more complete (quantum) theory of gravity can resolve this. We will consider whether regularity leaves any imprints in the Hawking temperature.

In this exercise, work with the Hayward metric. This is a metric that is not a solution to the Einstein equations (at least not for an energy-momentum tensor that satisfies the standard energy conditions). You can think of it as a phenomenological model for black holes beyond GR. The line-element in Schwarzschild-type coordinates is given by

$$ds^2 = f(r)\, dt^2 - f(r)^{-1} dr^2 - r^2 d\Omega_2^2, \tag{13.1}$$

with $f(r) = 1 - \frac{2G\,M\,r^2}{r^3+2G\,M\,\ell^2}$, with ℓ a parameter with units of length. All (non-derivative) curvature invariants remain finite in the limit $r \to 0$, as long as $\ell > 0$.

(a) What is the limit $\ell \to 0$?

(b) For spherically symmetric, static metrics in Schwarzschild-like coordinates, the event horizon is determined by the equation $g^{rr} = 0$. Find the location of the event horizon. If there is a qualitative (not just quantitative) difference to the Schwarzschild case, discuss it.

(c) Is the vector $\xi = \partial_t$ a Killing vector? If yes, is there a Killing horizon?

(d) Determine the surface gravity for a metric of the form Eq. (13.1) for an unspecified function $f(r)$; then insert $f(r)$ for the Hayward metric and discuss the difference to a Schwarzschild black hole.

(e) What are the implications of your previous results for the Hawking temperature of the Hayward black hole?