Quantum field theory in curved spacetime

Assignment 1 – Apr 28

Exercise 1: Quantum fields in an expanding universe

Motivation: In this first exercise, we'll follow how the vacuum state of a scalar evolves in a toy model of an expanding universe. Even though the setup is simple, it already reveals a key feature of quantum fields in curved spacetime: the vacuum isn't as empty as it seems.

Consider a real massive scalar field χ (minimally coupled) in an expanding universe. Its classical action is

$$S = \frac{1}{2}\int \mathrm{d}^4x \left(\chi'^2 - (\partial_i\chi)^2 - m_{\text{eff}}^2\chi^2\right), \tag{1.1}$$

where i denotes spatial indices and prime corresponds to derivative with respect to conformal time. The effective mass m_{eff}^2 is written as

$$m_{\text{eff}}^2 = m^2a^2 - \frac{a''}{a}, \tag{1.2}$$

with a, the scale factor. Assume that m_{eff}^2 is given by

$$m_{\text{eff}}^2(\eta) = \begin{cases} m_0^2, & \eta < 0 \text{ and } \eta > \eta_1, \\ -m_0^2, & 0 < \eta < \eta_1, \end{cases} \tag{1.3}$$

with m_0 a constant.

(**a**) Solve the equations of motion for χ.

(**b**) Construct the early ("in") and late ("out") time vacuum states.

(**c**) Prove that in the "out" region ($\eta > \eta_1$), the state $|0_{\text{in}}\rangle$ (the vacuum in the "in" region) contains particles. In other words, if we initially start in the vacuum the background evolution has created particles.

(**d**) Show that the mean particle number density in a mode $\mathbf{k}$ is given by

$$n_{\mathbf{k}} = \frac{m_0^4}{|k^4 - m_0^4|}\left|\sin\left(\eta_1\sqrt{k^2 - m_0^2}\right)\right|^2. \tag{1.4}$$

Sanity check: What happens in the limit $\eta_1 \to 0$? Why is this the result we expect?

(**e**) Discuss the regimes $k \gg m_0$ and $k \ll m_0$. What is the physical meaning of these limits?

(**a**) We express the scalar field in terms of its spatial Fourier transform

$$\chi_{\mathbf{k}}(t) = \int \mathrm{d}^3x \chi e^{i\mathbf{k}\mathbf{x}}, \tag{1.5}$$

which satisfies the equation of motion

$$\chi_{\mathbf{k}}'' + (m_{\text{eff}}^2 + \mathbf{k}^2)\chi_{\mathbf{k}} = 0. \tag{1.6}$$

We solve the equation of motion for the three cases $\eta < 0$, $0 < \eta < \eta_1$, $\eta > \eta_1$ individually, obtaining the mode expansions

$$\chi_{\mathbf{k}} = \begin{cases} a_{\mathbf{k}} e^{-i\omega_{k,+}\eta} + a_{-\mathbf{k}}^\dagger e^{i\omega_{k,+}\eta}\,, & \eta < 0\,, \\ b_{\mathbf{k}} e^{-i\omega_{k,-}\eta} + b_{-\mathbf{k}}^\dagger e^{i\omega_{k,-}\eta}\,, & 0 < \eta < \eta_1\,, \\ c_{\mathbf{k}} e^{-i\omega_{k,+}\eta} + c_{-\mathbf{k}}^\dagger e^{i\omega_{k,+}\eta}\,, & \eta > \eta_1\,, \end{cases} \tag{1.7}$$

with the amplitudes $a_{\mathbf{k}}$, $b_{\mathbf{k}}$ and $c_{\mathbf{k}}$ and the two dispersion relations

$$\omega_{k,\pm} = \sqrt{\mathbf{k}^2 \pm m_0^2}. \tag{1.8}$$

Note that $\omega_{k,-}$ is imaginary for the modes satisfying $m_0^2 > \mathbf{k}^2$. In order to obtain a valid solution, the scalar field has to be continuous and differentiable. This allows us to express the amplitudes $b_{\mathbf{k}}$ and $c_{\mathbf{k}}$ in terms of $a_{\mathbf{k}}$ by requiring continuity and differentiability at $\eta = 0$ and $\eta = \eta_1$. Making the ansatz

$$b_{\mathbf{k}} = \alpha_b a_{\mathbf{k}} + \beta_b a_{-\mathbf{k}}^\dagger, \tag{1.9}$$

we obtain the constraints

$$\alpha_b + \beta_b^* = \frac{\omega_{k,-}}{\omega_{k,+}}(-\alpha_b + \beta_b^*) = 1, \tag{1.10}$$

which have the solution

$$\alpha_b = \frac{\omega_{k,+} + \omega_{k,-}}{2\omega_{k,-}}, \qquad \beta_b = \frac{\omega_{k,+} - \omega_{k,-}}{2\omega_{k,-}}. \tag{1.11}$$

Similarly, with the ansatz

$$c_{\mathbf{k}} = \alpha_c b_{\mathbf{k}} + \beta_c b_{-\mathbf{k}}^\dagger, \tag{1.12}$$

we obtain the constraints

$$\alpha_c e^{-i\omega_{k,+}\eta_1} + \beta_c^* e^{i\omega_{k,+}\eta_1} = e^{-i\eta_1\omega_{k,-}}, \qquad \frac{\omega_{k,+}}{\omega_{k,-}}(-\alpha_c e^{-i\omega_{k,+}\eta_1} + \beta_c^* e^{i\omega_{k,+}\eta_1}) = e^{-i\omega_{k,-}\eta_1}, \tag{1.13}$$

which have the solution

$$\alpha_c = \frac{\omega_{k,+} + \omega_{k,-}}{\omega_{k,+}} e^{-i(\omega_{k,+} - \omega_{k,-})\eta_1}, \qquad \beta_c = \frac{\omega_{k,+} - \omega_{k,-}}{\omega_{k,+}} e^{i(\omega_{k,+} + \omega_{k,-})\eta_1}. \tag{1.14}$$

Altogether, we find

$$c_{\mathbf{k}} \equiv \alpha_{ac} a_{\mathbf{k}} + \beta_{ac} a_{-\mathbf{k}}^\dagger \tag{1.15}$$

$$= \frac{1}{4\omega_{k,+}\omega_{k,-}} \left((\omega_{k,+} + \omega_{k,-})^2 e^{-i(\omega_{k,-} - \omega_{k,+})\eta_1} - (\omega_{k,+} - \omega_{k,-})^2 e^{-i(\omega_{k,-} + \omega_{k,+})\eta_1} \right) a_{\mathbf{k}} \tag{1.16}$$

$$+ \frac{i}{2} \frac{\omega_{,+}^2 - \omega_{,-}^2}{\omega_{k,+}\omega_{k,-}} \sin(\eta_1 \omega_{k,-}) e^{i(\omega_{k,+})\eta_1} a_{-\mathbf{k}}^\dagger. \tag{1.17}$$

(**b**) When quantising the scalar, we promote the amplitudes to operators such that

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{k}' - \mathbf{k}), \qquad [c_{\mathbf{k}}, c_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{k}' - \mathbf{k}). \tag{1.18}$$

The vacuum states $|0_{\rm in}\rangle$ and $|0_{\rm out}\rangle$ (for $\eta < 0$ and $\eta > \eta_1$, respectively) vanish when the corresponding annihilation operators act on them, *i. e.* they satisfy

$$a_{\mathbf{k}}|0_{\rm in}\rangle = 0, \qquad\qquad c_{\mathbf{k}}|0_{\rm out}\rangle = 0, \tag{1.19}$$

for all $\mathbf{k}$.
Note: We are working in the Heisenberg picture. So if we start out in the in-vacuum, we also end up in the in-vacuum.

(**c**) To see whether there are particles in the in-vacuum at late times, we act on it with the annihilation operator at late times. If the result is nonzero, there are particles in that state. We obtain

$$c_{\mathbf{k}}|0_{\rm in}\rangle =(\alpha_{ac}a_{\mathbf{k}} + \beta_{ac}a_{\mathbf{k}})^\dagger|0_{\rm in}\rangle, \tag{1.20}$$
$$=\beta_{ac}a^\dagger_{-\mathbf{k}}|0_{\rm in}\rangle, \tag{1.21}$$
$$=\frac{i}{2}\frac{\omega^2_{,+} - \omega^2_{,-}}{\omega_{k,+}\omega_{k,-}}\sin(\eta_1\omega_{k,-})e^{i(\omega_{k,+})\eta_1}a^\dagger_{-\mathbf{k}}|0_{\rm in}\rangle, \tag{1.22}$$

where we used Eq. (1.17). This state is clearly non-zero.

(**d**) We define the number-density operator at late times as

$$n_{\rm out} = \frac{c^\dagger_{\mathbf{k}}c_{\mathbf{k}}}{V}, \tag{1.23}$$

with the volume of space V. In the in-vacuum state, the mean number density equals the expectation value of the number-density operator. Thus, we obtain

$$n_{\mathbf{k}} =\langle 0_{\rm in}|n_{\rm out}|0_{\rm in}\rangle \tag{1.24}$$
$$=V^{-1}\langle 0_{\rm in}|c^\dagger_{\mathbf{k}}c_{\mathbf{k}}|0_{\rm in}\rangle, \tag{1.25}$$
$$=V^{-1}||c_{\mathbf{k}}|0_{\rm in}\rangle||^2, \tag{1.26}$$
$$=\frac{(\omega^2_{k,+} - \omega^2_{k,-})^2}{4\omega_{k,+}\omega_{k,-}V}|\sin(\omega_{k,-}\eta_1)|^2||a^\dagger_{-\mathbf{k}}|0_{\rm in}\rangle||^2, \tag{1.27}$$
$$=\frac{m_0^4}{|\mathbf{k}^4 - m_0^4|}|\sin(\omega_{k,-}\eta_1)|^2\frac{\delta^{(3)}(0)}{V}. \tag{1.28}$$

Oops, $\delta^{(3)}(0)$ incoming. But don't despair! That's just the volume of space that we divide by anyway. This is why, in field theory on unbounded backgrounds, it is just more useful to consider densities. Thus, by a slight of hand $\delta^{(3)}(0)/V = 1$ and we obtain

$$n_{\mathbf{k}} = \frac{m_0^4}{|\mathbf{k}^4 - m_0^4|}|\sin(\omega_{k,-}\eta_1)|^2. \tag{1.29}$$

(**e**) The modes which satisfy $k \gg m_0$ have very large energies and thus probe very small distances. At small distances, spacetime is approximately Minkowskian. In our toy model, the change in the mass is of order m_0, which estimates the scale of background curvature we model. Thus, if $k \gg m_0$ the particles approximately see Minkowski spacetime, and particle creation is negligible. Indeed, we infer from Eq. (1.29) that $\lim_{k\to\infty} n_{\mathbf{k}} = 0$.

If, in turn, $k \ll m_0$ the wave length of the modes is much larger than background-curvature length scales. Therefore, particle creation is maximal. We see this in the number density

$$n_0 = \sinh^2(m_0\eta_1), \tag{1.30}$$

which grows exponentially with $m_0\eta_1$.

Exercise 2: Bogolyubov transformations

Motivation: We've seen that one person's vacuum can be filled with particles from another person's point of view. Now we derive general rules that relate the vacua of different observers.

Given a set of mode functions $v_k(\eta)$ (with conformal time η and $k = |\mathbf{k}|$), a scalar field on a cosmological background can be expanded as

$$\chi = \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}k^3}{(2\pi)^{3/2}} \left(v_k^* a_{\mathbf{k}} e^{i\mathbf{kx}} + v_k a_{\mathbf{k}}^\dagger e^{-i\mathbf{kx}} \right), \tag{2.1}$$

where

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = \delta^{(3)}(\mathbf{k}' - \mathbf{k}). \tag{2.2}$$

Let us define a new set of mode functions as a linear combination

$$u_k = \alpha_k v_k + \beta_k v_k^*. \tag{2.3}$$

The numbers α_k and β_k are called Bogolyubov coefficients. They are related as

$$|\alpha_k|^2 - |\beta_k|^2 = 1. \tag{2.4}$$

Given the new set of mode functions, we can equivalently expand the scalar as

$$\chi = \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}k^3}{(2\pi)^{3/2}} \left(u_k b_{\mathbf{k}} e^{i\mathbf{kx}} + u_k^* b_{\mathbf{k}}^\dagger e^{-i\mathbf{kx}} \right), \tag{2.5}$$

where again

$$[b_{\mathbf{k}}, b_{\mathbf{k}'}^\dagger] = \delta^{(3)}(\mathbf{k}' - \mathbf{k}). \tag{2.6}$$

Then, we can express the different classes of creation and anihilation operators as linear combinations, e. g.

$$b_{\mathbf{k}} = \alpha_k a_{\mathbf{k}} - \beta_k a_{-\mathbf{k}}^\dagger \tag{2.7}$$

In class, you have derived the average particle number density of modes associated with the operator $a^\dagger$ in the b-vacuum. Now, we go a step further and explicitly express the b-vacuum state in terms of creation and anihilation operators of the state a acting on the a-vacuum. The b-vacuum state for a pair of modes $(\mathbf{k}, -\mathbf{k})$ satisfies

$$b_{\mathbf{k}} |0_{\mathbf{k},-\mathbf{k}}^{(b)}\rangle = b_{-\mathbf{k}} |0_{\mathbf{k},-\mathbf{k}}^{(b)}\rangle = 0. \tag{2.8}$$

(**a**) Expand the b-vacuum in terms of a-particle states.

(**b**) Use the properties of $|0_{\mathbf{k},-\mathbf{k}}^{(b)}\rangle$ to obtain the expansion coefficients.

(**c**) Normalize the resulting state. You should obtain the result

$$|0_{\mathbf{k},-\mathbf{k}}^{(b)}\rangle = \frac{1}{|\alpha_k|} e^{\frac{\beta_k}{\alpha_k} a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger} |0_{\mathbf{k},-\mathbf{k}}^{(a)}\rangle. \tag{2.9}$$

(**d**) Write down the full b-vacuum state in terms of the mode-specific $|0_{\mathbf{k},-\mathbf{k}}^{(b)}\rangle$.

(**e**) Let's have a closer look at the expansion. What kind of state is the b-vacuum in terms of a-particle states?

(**a**) We can introduce a partition of unity in the Fock space describing modes of wave number $\pm k$ such that

$$\mathbb{1}_{\mathbf{k},-\mathbf{k}} = \frac{1}{|N|}\sum_{n,m=0}^{\infty} |m^{(a)}_{-\mathbf{k}}, n^{(a)}_{\mathbf{k}}\rangle\langle m^{(a)}_{-\mathbf{k}}, n^{(a)}_{\mathbf{k}}|, \tag{2.10}$$

with some normalizing factor $|N|$. Then, the vacuum state of particle b can be expanded as

$$|0^{(b)}_{\mathbf{k},-\mathbf{k}}\rangle = \frac{1}{|N|}\sum_{n,m=0}^{\infty} \langle m^{(a)}_{-\mathbf{k}}, n^{(a)}_{\mathbf{k}}|0^{(b)}_{\mathbf{k},-\mathbf{k}}\rangle |m^{(a)}_{-\mathbf{k}}, n^{(a)}_{\mathbf{k}}\rangle, \tag{2.11}$$

$$= \frac{1}{|N|}\sum_{n,m=0}^{\infty} c_{nm}|m^{(a)}_{-\mathbf{k}} n^{(a)}_{\mathbf{k}}\rangle. \tag{2.12}$$

(**b**) Eq. (2.8) tells us that the vacuum is invariant under parity transformations ($\mathbf{k} \to -\mathbf{k}$). This implies the constraint

$$c_{nm} = c_{mn}. \tag{2.13}$$

Besides, it has to be anihilated by the operator $b_{\mathbf{k}}$ which shifts Using Eq. (2.7), we obtain the relation

$$b_{\mathbf{k}}|0^{(b)}_{\mathbf{k},-\mathbf{k}}\rangle = \frac{1}{|N|}\sum_{n,m=0}^{\infty} c_{nm}(\alpha_k a_{\mathbf{k}} - \beta_k a^{\dagger}_{-\mathbf{k}})|m^{(a)}_{-\mathbf{k}} n^{(a)}_{\mathbf{k}}\rangle \tag{2.14}$$

$$= \frac{1}{|N|}\sum_{n,m=0}^{\infty} c_{nm}(\sqrt{n}\alpha_k|m^{(a)}_{-\mathbf{k}}(n-1)^{(a)}_{\mathbf{k}})\rangle - \sqrt{m+1}\beta_k|(m+1)^{(a)}_{-\mathbf{k}} n^{(a)}_{\mathbf{k}})\rangle \tag{2.15}$$

$$= \frac{1}{|N|}\sum_{n,m=0}^{\infty} (c_{n+1,m}\sqrt{n+1}\alpha_k - c_{n,m-1}\sqrt{m}\beta_k)|m^{(a)}_{-\mathbf{k}} n^{(a)}_{\mathbf{k}}\rangle \tag{2.16}$$

$$= 0. \tag{2.17}$$

Every coefficient of this linear combination has to vanish individually. Thus, we obtain the iterative relations

$$c_{n+1,m}\sqrt{n+1}\alpha_k - c_{n,m-1}\sqrt{m}\beta_k = 0. \tag{2.18}$$

Let's express everything in terms of $c_{0,0}$. Then, immediately $c_{n,0} = c_{0,n} = 0$ for all $n > 0$. But then, again, $c_{n,1} = c_{1,n} = 0$ for all $n > 1$ and so on. Thus, we find that

$$c_{nm} \propto \delta_{mn}. \tag{2.19}$$

For the diagonal elements, we obtain

$$c_{n+1,n+1} = \frac{\beta_k}{\alpha_k} c_{n,n}. \tag{2.20}$$

Thus, we obtain the relation

$$c_{n,n} = \left(\frac{\beta_k}{\alpha_k}\right)^n c_{0,0}. \tag{2.21}$$

Note that from the recurrence relation given in Eq. (2.18) alone one could think that $c_{0,0} = 0$, setting $n = -1$ and $m = 0$. However, $n = -1$ is not contained in the sum in Eq. (2.16) – thus, $n = -1$ is not applicable and $c_{0,0} \neq 0$.

Given our solution to the recurrence relation, we can expand the b-vacuum in the mode $\mathbf{k}$ as

$$|0^{(b)}_{\mathbf{k},-\mathbf{k}}\rangle = \frac{1}{|\bar{N}|}\sum_{n=0}^{\infty}\left(\frac{\beta_k}{\alpha_k}\right)^n |n^{(a)}_{-\mathbf{k}} n^{(a)}_{\mathbf{k}}\rangle \tag{2.22}$$

$$= \frac{1}{|\bar{N}|}\sum_{n=0}^{\infty}\left(\frac{\beta_k}{\alpha_k}\right)^n \frac{(a^\dagger_{\mathbf{k}})^n (a^\dagger_{-\mathbf{k}})^n}{n!}|0^{(a)}_{\mathbf{k},-\mathbf{k}}\rangle \tag{2.23}$$

$$= \frac{1}{|\bar{N}|} e^{\frac{\beta_k}{\alpha_k} a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}}}|0^{(a)}_{\mathbf{k},-\mathbf{k}}\rangle, \tag{2.24}$$

where we absorbed $c_{0,0}$ into $\bar{N}$ and discarded a global phase.

(**c**) The norm of our state reads

$$|||0^{(b)}_{\mathbf{k},-\mathbf{k}}\rangle||^2 = \frac{1}{|\bar{N}|^2}\sum_{n,m}\left(\frac{\beta_k}{\alpha_k}\right)^n \left(\frac{\beta^*_k}{\alpha^*_k}\right)^m \langle m^{(a)}_{-\mathbf{k}} m^{(a)}_{\mathbf{k}}|n^{(a)}_{-\mathbf{k}} n^{(a)}_{\mathbf{k}}\rangle \tag{2.25}$$

$$= \frac{1}{|\bar{N}|^2}\sum_{n}\left(\frac{|\beta_k|^2}{|\alpha_k|^2}\right)^n \tag{2.26}$$

$$= \frac{1}{|\bar{N}|^2}\frac{|\alpha_k|^2}{|\alpha|^2 - |\beta_k|^2} \tag{2.27}$$

$$= \frac{|\alpha_k|^2}{|\bar{N}|^2}. \tag{2.28}$$

Thus, we finally obtain

$$|0^{(b)}_{\mathbf{k},-\mathbf{k}}\rangle = \frac{1}{|\alpha_k|} e^{\frac{\beta_k}{\alpha_k} a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}}}|0^{(a)}_{\mathbf{k},-\mathbf{k}}\rangle. \tag{2.29}$$

Hooray.

(**d**) For different wave numbers, the states live in different Hilbert spaces. Thus, we can just take their tensor product for all values of $\mathbf{k}$

$$|0^{(b)}\rangle = \prod_{\mathbf{k}} \frac{1}{|\alpha_k|} e^{\frac{\beta_k}{\alpha_k} a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}}}|0^{(a)}_{\mathbf{k},-\mathbf{k}}\rangle. \tag{2.30}$$

(**e**) Expressed in terms of a-particle states, the b-vacuum is a two-mode squeezed coherent state. That's a mouthful. Let's break it down:

- That its coherent means that it saturates the uncertainty relation. It is, thus, as classical as it can get. Note that a squeezed coherent state is not necessarily a coherent state (an eigenstate of the anihilation operator), indeed the one we are dealing with is exactly such an example. That's a terminology trap right there!

- That its two-mode squeezed means that it comes in pairs – in this case with opposite wave numbers $(\mathbf{k}, -\mathbf{k})$. More generally, it comes in pairs with exactly opposite quantum numbers, *i. e.* particle-antiparticle pairs. This has to be the case because quantum numbers have to be conserved.

Exercise 3: Instantaneous vacuum

Motivation: Every mode function allows to construct a different vacuum. What could be a sensible definition of vacuum then? Let's find out!

Ordinarily, we define the vacuum as the lowest-energy state. In cosmology, however, the Hamiltonian is time dependent. Energy is not conserved. This creates particles. Thus, the lowest-energy state at one time (the *instantaneous vacuum*), may not be the lowest-energy state at a different time. Let's see, how this comes about.
As above consider a real massive scalar field, whose dynamics are characterized by the action given in Eq. (1.1). This results in the Hamiltonian

$$H = \frac{1}{2}\int_x \left(\pi^2 + (\partial_i\chi)^2 + m_{\text{eff}}^2\chi^2\right), \tag{3.1}$$

with the momentum conjugate π. Assume that the field possesses a mode expansion as in Eq. (2.1).

(a) Express the Hamiltonian in terms of creation and anihilation operators. You should obtain something of the form

$$H = \frac{1}{4}\int \mathrm{d}^3k \left[a_{\mathbf{k}}a_{-\mathbf{k}}F_{\mathbf{k}}^* + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger F_{\mathbf{k}} + (2a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \delta^{(3)}(0))E_{\mathbf{k}}\right] \tag{3.2}$$

for some $E_{\mathbf{k}}$, $F_{\mathbf{k}}$.

(b) Compute the mean energy density in the a-vacuum.

(c) Assuming that $\omega_k^2 = k^2 + m_{\text{eff}}^2 > 0$, find initial conditions for the mode function that minimize the mean energy density at conformal time η_0. (**Hint:** Normalize the mode functions.) What is the corresponding Hamiltonian at conformal time η_0? You should obtain that the Hamiltonian is diagonal in this case.

(d) Compute the initial conditions for the mode function after an infinitesimal time shift, *i. e.* at conformal time $\eta_0 + \delta\eta$. Compare these initial conditions to the ones derived in the previous exercise. How do we interpret this result? (**Hint:** Have in mind exercise 2.)

(e) Imagine that you could find a vacuum state which diagonalizes the Hamiltonian at all times. Which equation would the mode functions have to satisfy? Is this equation compatible with the equations of motion?

In specific situations, it can happen that the lowest-energy state at one time η_0 amounts to an infinite number density at a different time η_1, even if the geometry changes slowly compared to the time difference that is characteristic of the problem one would like to answer (*i. e.* adiabatically). This casts serious doubts on the physical interpretation of the instantaneous vacuum.
However, adiabatic evolution allows us to (at least approximately) define a different vacuum state with interesting properties: The adiabatic vacuum. If the energy density is changing slowly during the considered time interval, the equations of motion allow for the approximate solution[a]

$$v_k^{\text{WKB}}(\eta) = \frac{e^{i\int_{\eta_0}^{\eta}\omega_k(\eta')\mathrm{d}\eta'}}{\sqrt{\omega_k}}. \tag{3.3}$$

We can define the adiabatic vacuum $|0_{\text{ad}}(\eta_0)\rangle$ at a time η_0 by finding exact mode functions which satisfy the initial conditions

$$v_k(\eta_0) = v_k^{\text{WKB}}(\eta_0), \qquad v_k'(\eta_0) = v_k^{\text{WKB}\prime}(\eta_0), \tag{3.4}$$

and constructing the vacua relative to the corresponding anihilation operator.

(**f**) Quantify how adiabatic a general background evolution yielding $\omega_k(\eta)$ is. Which condition should an adiabatically evolving background satisfy if the quantum-field evolution is considered in a finite-time interval $\Delta\eta = \eta_1 - \eta_0$?

(**g**) Compute the energy density of the adiabatic vacuum in general. Is it minimal?

[a] This approximation is called WKB (Wentzel–Kramers–Brillouin) approximation, a standard method in quantum mechanics in general.

(**a**) Given the mode expansion in Eq. (2.1), the conjugate momentum reads

$$\pi(x) = \partial_\eta \chi = \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \left(v_k'^* a_{\mathbf{k}} e^{i\mathbf{kx}} + v_k' a_{\mathbf{k}}^\dagger e^{-i\mathbf{kx}} \right). \tag{3.5}$$

Thus, it's square contributes

$$\pi(x)^2 = \frac{1}{2} \int \frac{\mathrm{d}^3 k \mathrm{d}^3 k'}{(2\pi)^3} \left(v_k'^* a_{\mathbf{k}} e^{i\mathbf{kx}} + v_k' a_{\mathbf{k}}^\dagger e^{-i\mathbf{kx}} \right) \left(v_{k'}'^* a_{\mathbf{k}'} e^{i\mathbf{k'x}} + v_{k'}' a_{\mathbf{k}'}^\dagger e^{-i\mathbf{k'x}} \right), \tag{3.6}$$

$$= \frac{1}{2} \int \mathrm{d}^3 k \left(v_k'^{*2} a_{\mathbf{k}} a_{-\mathbf{k}} + |v_k'|^2 (a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}}) + v_k'^2 a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger \right), \tag{3.7}$$

Similarly, the gradient of the scalar has the square

$$(\nabla\chi)^2 = \frac{1}{2} \int \mathrm{d}^3 k k^2 \left(v_k^{*2} a_{\mathbf{k}} a_{-\mathbf{k}} + |v_k|^2 (a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}}) + v_k^2 a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger \right). \tag{3.8}$$

Then, the Hamiltonian reads

$$H = \frac{1}{4} \int \mathrm{d}^3 k \left[(v_k'^{*2} + \omega_k^2 v_k^{*2}) a_{\mathbf{k}} a_{-\mathbf{k}} + (v_k'^2 + \omega_k^2 v_k^2) a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + (|v_k'|^2 + \omega_k^2 |v_k|^2))(a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}}) \right], \tag{3.9}$$

$$= \frac{1}{4} \int \mathrm{d}^3 k \left[F_{\mathbf{k}}^* a_{\mathbf{k}} a_{-\mathbf{k}} + F_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + E_{\mathbf{k}} (2 a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \delta^{(3)}(0)) \right], \tag{3.10}$$

where we defined

$$F_{\mathbf{k}} = v_k'^2 + \omega_k^2 v_k^2, \qquad E_{\mathbf{k}} = |v_k'|^2 + \omega_k^2 |v_k|^2. \tag{3.11}$$

(**b**) The only nonvanishing contribution to the expectation value of the Hamiltonian in the a-vacuum reads

$$\langle 0^{(a)} | H | 0^{(a)} \rangle = \frac{\delta^{(3)}(0)}{4} \int \mathrm{d}^3 k E_{\mathbf{k}}. \tag{3.12}$$

The factor $\delta^{(3)}(0)$ amounts to the infinite volume of space. Therefore, we define the mean energy density as

$$\rho \equiv \frac{\langle 0^{(a)} | H | 0^{(a)} \rangle}{\delta^{(3)}(0)} = \frac{1}{4} \int \mathrm{d}^3 k E_{\mathbf{k}}. \tag{3.13}$$

(c) We have to minimize the energy density for each mode individually. Thus, we should minimize $E_{\mathbf{k}}(\eta_0)$. Note that the normalization condition for the mode functions reads

$$v_k' v_k^* - v_k'^* v_k = 2i. \tag{3.14}$$

We can shift the phase of the mode function by a constant $v_k \to e^{i\lambda} v_k$ without changing the physics. Such a shift implies $v_k' \to e^{i\lambda} v_k'$. By such a shift, we can make $v_k(\eta_0)$ real. Thus, the normalization condition, Eq. (3.14), at time η_0 reads

$$v_k(\eta_0) = \frac{1}{\mathrm{Im}(v_k')(\eta_0)}. \tag{3.15}$$

Thus, we have to minimize the quantity

$$E_{\mathbf{k}}(\eta_0) = [\mathrm{Re}(v_k')(\eta_0)]^2 + [\mathrm{Im}(v_k')(\eta_0)]^2 + \frac{\omega_k^2(\eta_0)}{[\mathrm{Im}(v_k')(\eta_0)]^2}, \tag{3.16}$$

with respect to both $x = \mathrm{Re}(v_k')(\eta_0)$ and $y = \mathrm{Im}(v_k')(\eta_0)$ individually. Thus, we have to minimize the functions x^2, yielding $x = 0$, and $y^2 + \omega_k^2(\eta_0)/y^2$, yielding $y = \sqrt{\omega_k(\eta_0)}$. We obtain the solution

$$v_k(\eta_0) = \frac{1}{\sqrt{\omega_k(\eta_0)}}, \qquad v_k'(\eta_0) = i\sqrt{\omega_k(\eta_0)} = i\omega_k(\eta_0)v_k(\eta_0). \tag{3.17}$$

Here, we had to assume that that $\omega_k^2(\eta_0) > 0$ – otherwise the manifestly real quantity $\mathrm{Im}(v_k')(\eta_0)$ would have been imaginary. This was expected, because the energy of modes with negative ω_k^2 is unbounded from below, thus not allowing for a minimal energy density.

Eq. (3.17) implies that $F_{\mathbf{k}}(\eta_0) = 0$, while

$$E_{\mathbf{k}}(\eta_0) = 2\omega_k(\eta_0). \tag{3.18}$$

As a result, the Hamiltonian at conformal time η_0 is diagonal (*i. e.* a function of the number operator $N_{\mathbf{k}} \equiv a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$ only), reading

$$H(\eta_0) = \int \mathrm{d}^3k\omega_k(\eta_0)\left[N_{\mathbf{k}} + \frac{\delta^{(3)}(0)}{2}\right]. \tag{3.19}$$

Note: The corresponding vacuum energy density

$$\rho(\eta_0) = \frac{1}{2}\int \mathrm{d}^3k\omega_k(\eta_0), \tag{3.20}$$

is divergent, and has to be renormalized. Simply subtracting it away, as one usually does it in flat space, does not work because it is time dependent.

(**d**) The initial conditions for the mode functions defining the instantaneous vacuum at conformal time $\eta_0 + \delta\eta$ read

$$u_k(\eta_0 + \delta\eta) = \frac{1}{\sqrt{\omega_k(\eta_0 + \delta\eta)}}, \qquad u_k'(\eta_0 + \delta\eta) = i\sqrt{\omega_k(\eta_0 + \delta\eta)}. \tag{3.21}$$

Expanding in the infinitesimal $\delta\eta$,

$$u_k(\eta_0) + u_k'(\eta_0)\delta\eta \simeq \frac{1}{\sqrt{\omega_k(\eta_0)}} - \frac{1}{2}\frac{\omega_k'(\eta_0)\delta\eta}{\omega_k^{3/2}(\eta_0)}, \quad u_k'(\eta_0) + u_k''(\eta_0)\delta\eta = i\left(\sqrt{\omega_k(\eta_0)} + \frac{1}{2}\frac{\omega_k'(\eta_0)\delta\eta}{\sqrt{\omega_k(\eta_0)}}\right). \tag{3.22}$$

Thus, the mode functions u and v can only be equal if

$$\omega_k'(\eta_0) = -2i\omega_k^2(\eta_0), \qquad u_k''(\eta_0) = \frac{1}{2}\frac{\omega_k'(\eta_0)}{\sqrt{\omega_k(\eta_0)}}. \tag{3.23}$$

The first of these two equalities provides an initial condition for the background at η_0. The second equality is an additional initial condition for the mode, which together with the equation of motion yields

$$u_k''(\eta_0) + \omega_k^2(\eta_0)u_k(\eta_0) = 2\left(\frac{i}{2}\omega_k'(\eta_0)\right)^{3/4} = 0. \tag{3.24}$$

Thus, u and v are equal iff $\omega_k'(\eta_0) \propto m_{\mathrm{eff}}^{2\prime}(\eta_0) = 0$, *i. e.* the background should not evolve at time η_0.

Thus, using the language from exercise 2, the operators $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ are related by a nontrivial Bogolyubov transformation, and the a-vacuum is filled with b-particles. In other words, even after an infinitesimal shift in conformal time, the state of minimal energy density is not the state of minimal energy density any more, unless the background is static during that time.

(**e**) In order to diagonalize the Hamiltonian, the mode function has to satisfy the differential equation

$$F_{\mathbf{k}} = v_k'^2 + \omega_k^2 v_k^2 = 0. \tag{3.25}$$

Taking a derivative with respect to conformal time, we obtain

$$v_k'' = i(\omega_k' v_k + \omega_k v_k') = (i\omega_k' - \omega_k^2) v_k. \tag{3.26}$$

At the same time, the equations of motion require that

$$v_k'' = -\omega_k^2 v_k. \tag{3.27}$$

Thus, the Hamiltonian can only be diagonalized by one mode function in a way consistent with the equations of motion if

$$\omega_k' = 0, \tag{3.28}$$

i. e. in a non-evolving background. In other words, in a cosmological setting it is impossible to diagonalize the Hamiltonian for all times.

(**f**) Being an energy, $\omega_k(\eta)$ has an associated characteristic frequency and with that an associated characteristic period

$$\Delta\eta = \frac{2\pi}{\omega_k(\eta)}. \tag{3.29}$$

For ω_k to be slowly changing, it then has to satisfy

$$\left| \frac{\omega(\eta + \Delta\eta) - \omega(\eta)}{\omega(\eta)} \right| \ll 1. \tag{3.30}$$

Expanding in $\Delta\eta$ (which we should be allowed to do if ω_k is slowly changing), we obtain

$$\left| \frac{\omega_k'(\eta)\Delta\eta}{\omega_k(\eta)} \right| = 2\pi \left| \frac{\omega_k'(\eta)}{\omega_k^2(\eta)} \right| \ll 1. \tag{3.31}$$

(**g**) The initial conditions for the mode functions defining the adiabatic vacuum (derived from Eq. (3.4)) are

$$v_k(\eta_0) = \frac{1}{\sqrt{\omega_k(\eta_0)}}, \qquad v_k'(\eta_0) = i\sqrt{\omega_k(\eta_0)} \left[1 + \frac{i}{2} \frac{\omega_k'}{\omega_k^2} \right]. \tag{3.32}$$

These deviate from the minimal-energy-density initial conditions by a term proportional to ω_k'/ω_k^2, which as we derived in the previous exercise is small for slowly changing backgrounds. Thus, we expect to be close to the minimal energy density. Indeed, we obtain

$$\rho = \frac{1}{4} \int \mathrm{d}^3 k E_k = \frac{1}{2} \int \mathrm{d}^3 k \omega_k \left(1 + \frac{1}{16} \left| \frac{\omega_k'}{\omega_k^2} \right|^2 \right). \tag{3.33}$$

Considering that the corrections are quadratic in a very small number, this is indeed very close to minimal energy density.

Quantum field theory in curved spacetime

Assignment 2 – May 5

Exercise 4: Conformally coupled scalar field

Motivation: In the lecture, we saw that nonminimally coupled scalar fields are not produced in an FLRW background if they are conformally coupled. But why those specific coupling values? Here, we'll work out precisely what conformal coupling means.

Consider a scalar field ϕ non-minimally coupled with gravity. It is described by the following action (**Note that in the original sheet this exercise was written in the "wrong" metric signature** $(+---)$**. To be consistent with the rest, it has been modified to** $(-+++)$)

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{m^2}{2}\phi^2 - \frac{\xi}{2}R\phi^2\right). \tag{4.1}$$

(**a**) Consider the conformal transformation

$$g_{\mu\nu}(x,t) \to \Omega^2(x,t)g_{\mu\nu}(x,t), \tag{4.2}$$
$$\phi(x,t) \to \Omega^{-1}(x,t)\phi(x,t). \tag{4.3}$$

Calculate the value of ξ under which the action is invariant under this conformal transformation (possibly up to a total divergence).

(**b**) Show that the energy-momentum tensor $T^{\mu\nu}$ is expressed as

$$\begin{aligned} T^{\mu\nu} &= \nabla^\mu\phi\nabla^\nu\phi - \frac{1}{2}g^{\mu\nu}\nabla^\rho\phi\nabla_\rho\phi - \frac{1}{2}g^{\mu\nu}m^2\phi^2 + \xi\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\phi^2 \\ &+ \xi\left[g^{\mu\nu}\nabla^\alpha\nabla_\alpha(\phi^2) - \nabla^\mu\nabla^\nu(\phi^2)\right]. \end{aligned} \tag{4.4}$$

(**c**) Show that $T^\nu{}_\nu = 0$ when $m = 0$ and $\xi = 1/6$.

(**a**) From here on on barred quantities denote the transformed metric such that $\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$. The inverse metric transforms as $\bar{g}^{\mu\nu} = \Omega^{-2}g^{\mu\nu}$ (recall that $\bar{g}^{\mu\nu}\bar{g}_{\nu\rho} = \delta^\mu_\rho$). As a result, the Christoffel symbol transforms as

$$\bar{\Gamma}^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\mu\nu} + \frac{g^{\rho\sigma}}{\Omega}\left(2g_{\sigma(\nu}\nabla_{\mu)}\Omega - g_{\mu\nu}\nabla_\sigma\Omega\right), \tag{4.5}$$

where we use the notation

$$g_{\rho(\nu}\nabla_{\mu)} = \frac{1}{2}\left(g_{\rho\nu}\nabla_\mu + g_{\rho\mu}\nabla_\nu\right). \tag{4.6}$$

Then, the Ricci tensor reads

$$\bar{R}_{\mu\nu} = \partial_\rho\bar{\Gamma}^\rho{}_{\mu\nu} - \partial_\nu\bar{\Gamma}^\rho{}_{\rho\mu} + \bar{\Gamma}^\rho{}_{\rho\sigma}\bar{\Gamma}^\sigma{}_{\mu\nu} - \bar{\Gamma}^\rho{}_{\mu\sigma}\bar{\Gamma}^\sigma{}_{\nu\rho}, \tag{4.7}$$
$$= R_{\mu\nu} - 2\frac{\nabla_\mu\nabla_\nu\Omega}{\Omega} - g_{\mu\nu}\frac{\Box\Omega}{\Omega} + 4\frac{\nabla_\mu\Omega\nabla_\nu\Omega}{\Omega^2} - g_{\mu\nu}\frac{g^{\rho\sigma}\nabla_\rho\Omega\nabla_\sigma\Omega}{\Omega^2}, \tag{4.8}$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. Finally, the Ricci scalar reads

$$\bar{R} = \frac{R}{\Omega^2} - 6\frac{\Box\Omega}{\Omega^3}. \tag{4.9}$$

At the same time, the determinant of the metric transforms as

$$\bar{g} \equiv \epsilon^{\mu_1\ldots\mu_4}\bar{g}_{1\mu_1}\ldots\bar{g}_{4\mu_4} = \Omega^8 g. \tag{4.10}$$

Together with the transformation of the scalar, we obtain

$$S = \int \mathrm{d}^4x\sqrt{-\bar{g}}\left(-\frac{1}{2}\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\phi}\bar{\nabla}_\nu\bar{\phi} - \frac{m^2}{2}\bar{\phi}^2 - \frac{\xi}{2}\bar{R}\bar{\phi}^2\right), \tag{4.11}$$

$$= \int \mathrm{d}^4x\sqrt{g}\left[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \phi\nabla_\mu\phi g^{\mu\nu}\frac{\nabla_\nu\Omega}{\Omega} - \frac{1}{2}\phi^2 g^{\mu\nu}\frac{\nabla_\mu\Omega\nabla_\nu\Omega}{\Omega^2} - \frac{m^2}{2}\phi^2\Omega^2 - \frac{\xi}{2}\phi^2\left(R - 6\frac{\Box\Omega}{\Omega}\right)\right], \tag{4.12}$$

$$= \int \mathrm{d}^4x\sqrt{g}\left[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{m^2}{2}\phi^2\Omega^2 - \frac{\xi}{2}\phi^2 R + 3\left(\xi - \frac{1}{6}\right)\phi^2\frac{\Box\Omega}{\Omega}\right], \tag{4.13}$$

where we used partial integration to obtain the equality

$$\int \mathrm{d}^4x\sqrt{-g}\frac{\phi^2}{2}g^{\mu\nu}\frac{\nabla_\mu\Omega\nabla_\nu\Omega}{\Omega^2} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{\phi^2}{2}\frac{\Box\Omega}{\Omega} + \phi\nabla_\mu\phi g^{\mu\nu}\frac{\nabla_\nu\Omega}{\Omega}\right]. \tag{4.14}$$

Thus, the action is invariant if $m = 0$ and $\xi = 1/6$.

(**b**) The energy-momentum tensor is defined as

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}. \tag{4.15}$$

We, thus, have to vary the action with respect to the inverse metric. Let us start with the metric determinant. We start with the trick

$$g = e^{\mathrm{tr}\log g^{\mu\nu}}. \tag{4.16}$$

Then, the variation becomes

$$\delta g = e^{\mathrm{tr}\log g^{\mu\nu}}\delta[\mathrm{tr}\log g^{\mu\nu}], \tag{4.17}$$

$$= g\,\mathrm{tr}[g_{\mu\nu}\delta g^{\mu\nu}], \tag{4.18}$$

$$= g g_{\mu\nu}\delta g^{\mu\nu}, \tag{4.19}$$

with the trace tr. Thus, the variation of the root of the negative determinant reads

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \tag{4.20}$$

As a result, the minimally coupled part becomes

$$T_{\mu\nu}|_{\xi=0} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\rho\phi\nabla_\rho\phi - \frac{1}{2}g_{\mu\nu}m^2\phi^2. \tag{4.21}$$

The nonminimal coupling is slightly more involved. We start as

$$\delta S_{\text{nonminimal}} = -\frac{1}{2}\int_x \xi\phi^2\left[g^{\rho\sigma}\delta R_{\rho\sigma}(x) + \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu}(x)\right], \tag{4.22}$$

where hereafter we abbreviate $\int_x = \int \mathrm{d}^4x\sqrt{-g}$. Here, we can use the identity (you can find it, for example, in Eq. 4.60 of S. Carroll's notes on relativity)

$$\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^\rho_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda{}_{\lambda\mu}. \tag{4.23}$$

Partially integrating, we obtain for the remaining part

$$-\frac{1}{2}\int_x \xi\phi^2 g^{\rho\sigma}\delta R_{\rho\sigma}(x) = \frac{1}{2}\int_x \xi g^{\mu\nu}\left(\nabla_\rho(\phi^2)\delta\Gamma^\rho{}_{\mu\nu} - \nabla_\nu(\phi^2)\delta\Gamma^\lambda{}_{\mu\lambda}\right). \tag{4.24}$$

To obtain the variation of the Christoffel symbols, we use the following shortcut: Construct (Riemann) normal coordinates around a point p. Then, the first derivatives of the metric at that point vanish. Thus, the variation of the Christoffel symbols in that coordinate system reads

$$\delta\Gamma^\rho{}_{\mu\nu}|_p = \frac{1}{2}g^{\rho\sigma}(2\partial_{(\mu}\delta g_{\nu)\sigma} - \partial_\sigma\delta g_{\mu\nu})|_p, \tag{4.25}$$

$$= \frac{1}{2}g^{\rho\sigma}(2\nabla_{(\mu}\delta g_{\nu)\sigma} - \nabla_\sigma\delta g_{\mu\nu})|_p, \tag{4.26}$$

where in the last step we used that in normal coordinates, the Christoffel symbols at p vanish so that $\nabla_\mu|_p = \partial_\mu$. Note now that we could have done this at any point p and that, contrary to $\Gamma^\rho{}_{\mu\nu}$, $\delta\Gamma^\rho{}_{\mu\nu}$ is a tensor (the difference of Christoffel symbols transforms as a tensor) such that Eq. (4.26) is a tensor equation. This implies that Eq. (4.26) holds not only at a point and in normal coordinates, but at all points and every system of coordinates, *i. e.*

$$\delta\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(2\nabla_{(\mu}\delta g_{\nu)\sigma} - \nabla_\sigma\delta g_{\mu\nu}). \tag{4.27}$$

The contracted form of the variation of the Christoffel symbol reads

$$\delta\Gamma^\nu{}_{\mu\nu} = \frac{1}{2}g^{\nu\rho}\nabla_\mu\delta g_{\nu\rho}. \tag{4.28}$$

Thus, we can rewrite the remaining part as

$$-\frac{1}{2}\int_x \xi\phi^2 g^{\rho\sigma}\delta R_{\rho\sigma}(x) = -\frac{1}{2}\int_x \frac{\xi}{2}g^{\mu\nu}\left(2\delta g_{\rho(\nu}\nabla_{\mu)}\nabla^\rho(\phi^2) - \delta g_{\mu\nu}\nabla_\rho\nabla^\rho(\phi^2) - g^{\nu\rho}\delta g_{\nu\rho}\nabla_\mu\nabla_\nu(\phi^2)\right), \tag{4.29}$$

$$= -\frac{1}{2}\int_x \xi\left(\nabla^\mu\nabla^\nu(\phi^2) - g^{\mu\nu}\nabla_\rho\nabla^\rho(\phi^2)\right)\delta g_{\mu\nu} \tag{4.30}$$

Finally, the variation of the metric can be obtained from

$$\delta(\delta^\mu_\nu) = \delta(g^{\mu\rho}g_{\nu\rho}) = \delta g^{\mu\rho}g_{\nu\rho} + g^{\mu\rho}\delta g_{\nu\rho} = 0 \tag{4.31}$$

such that

$$\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}. \tag{4.32}$$

Thus, we can express the variation of the Ricci tensor as

$$-\frac{1}{2}\int_x \xi\phi^2 g^{\rho\sigma}\delta R_{\rho\sigma}(x) = -\frac{1}{2}\int_x \xi\left(g_{\mu\nu}\nabla_\rho\nabla^\rho(\phi^2) - \nabla_\mu\nabla_\nu(\phi^2)\right)\delta g^{\mu\nu}. \tag{4.33}$$

After all of this tedious algebra, we finally obtain the result

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\rho\phi\nabla^\rho\phi - \frac{m^2}{2}g_{\mu\nu}\phi^2 + \xi\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\phi^2 + \xi\left[g_{\mu\nu}\nabla_\rho\nabla^\rho(\phi^2) - \nabla_\mu\nabla_\nu(\phi^2)\right]. \tag{4.34}$$

Let's go touch some grass.

(**c**) The trace of the stress-energy tensor reads

$$T^{\nu}_{\ \nu} = -\nabla^{\nu}\phi\nabla_{\nu}\phi - 2m^2\phi^2 - \xi R\phi^2 + 6\xi(\phi\Box\phi + \nabla^{\nu}\phi\nabla_{\nu}\phi), \tag{4.35}$$

$$= 6\left(\xi - \frac{1}{6}\right)\nabla^{\nu}\phi\nabla_{\nu}\phi + 2m^2\phi^2 + \xi\phi(6\Box\phi - R\phi). \tag{4.36}$$

But the scalar also satisfies its field equation

$$(\Box - \xi R - m^2)\phi = 0, \tag{4.37}$$

which we can plug in such that

$$T^{\nu}_{\ \nu} = 6\left(\xi - \frac{1}{6}\right)\nabla^{\nu}\phi\nabla_{\nu}\phi - 2m^2\phi^2 + \xi\phi\left[6\left(\xi - \frac{1}{6}\right)R\phi + 6m^2\phi\right], \tag{4.38}$$

$$= 6\left(\xi - \frac{1}{6}\right)\left(\nabla^{\nu}\phi\nabla_{\nu}\phi + \xi R\phi^2\right) + 2m^2\phi^2\left(3\xi - 1\right). \tag{4.39}$$

This clearly vanishes when $\xi = 1/6$ and $m = 0$.

Exercise 5: Electromagnetic fields on curved backgrounds

Motivation: Non-conformally coupled scalars are copiously produced in FLRW spacetimes. But how about photons? In other words, is the universe covered in "light" of horizon wavelength?

The Maxwell action on a curved background reads

$$S = -\frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}, \tag{5.1}$$

with the field-strength tensor

$$F_{\mu\nu} = \nabla_{\mu}A_{\nu} - \nabla_{\nu}A_{\mu}, \tag{5.2}$$

and the gauge field A_μ. Instead of going through the whole derivation of particle creation again, we take a shortcut.

(**a**) **Try to do this sub-exercise before reading the remaining ones.**

Work smarter, not harder. Before doing any calculation, think it through: Should Maxwell theory be Weyl invariant in curved spacetime? Why? What does this tell us about photon production in an FLRW background?

Let's now work through the details step by step.

(**b**) The gauge field transforms trivially under Weyl transformations, *i. e.* $A_\mu \to A_\mu$. Calculate how the Maxwell action transforms under Weyl rescalings. Is it invariant?

(**c**) If we rescale the FLRW metric to remove the scale factor, how do Maxwell's equations change? What does this mean for the electromagnetic vacuum in an FLRW background? Is the universe covered in "light" of horizon wave length?

Finally, let's explore how Weyl invariance shows up in the structure of the energy-momentum tensor.

(**d**) Compute the energy-momentum tensor $T^{\mu\nu}$ of the gauge field and show that $T^{\mu}_{\ \mu} = 0$.

(**e**) **Bonus exercise:** What does the tracelessness of the energy-momentum tensor have to do with Weyl invariance? (**Hint:** How does the matter Lagrangian change under a small Weyl rescaling? What would this imply for $T^{\mu}_{\ \mu}$?)

(**a**) Weyl invariance is the invariance under local scale transformations. Thus, if the theory is supposed to be the same at all scales, it cannot introduce absolute scales like dimensionful coupling constants. In other words, the only possible terms which can contribute to a Weyl invariant theory are those with dimensionless couplings. For electromagnetism, there are no minimal-coupling terms with couplings of vanishing dimension which are also gauge invariant (verify that no contractions of one Riemann tensor and one electromagnetic field strength could be viable). Thus, there is no wiggle room for a minimal coupling which may have to be added to render the theory locally Weyl invariant as for the scalar.

This does not imply that the theory is Weyl invariant yet. We can get there keeping in mind that electromagnetism has no scale and is therefore scale invariant in flat space. At the same time, the Maxwell action is made up of two contracted field strength tensors (which are defined with indices down). Their contraction requires two inverse metric tensors, whose behaviour under scale transformations balances exactly the one from the metric determinant. In other words, in flat space, the gauge field transforms trivially under scale transformations.

Recall that in exercise 4 the derivatives in the kinetic term of the scalar spoiled local Weyl invariance by introducing derivatives of the conformal factor. This cannot happen for the electromagnetic field (whose kinetic term is exactly the Maxwell action) because the field strength transforms trivially. Thus, the Maxwell action has to be locally Weyl invariant.

If we take seriously what we learned in the lecture, this indicates that the electromagnetic field does not "see" the cosmological evolution. This would indicate that there can be no photon production in cosmology.

(**b**) We know that $A_\mu \to A_\mu$. Besides, the covariant derivatives in field strength receive no gravitational contributions because

$$F_{\mu\nu} = 2\nabla_{[\mu}A_{\nu]} = 2\partial_{[\mu}A_{\nu]} - \Gamma^{\rho}{}_{[\mu\nu]} = 2\partial_{[\mu}A_{\nu]}, \tag{5.3}$$

where we used that the Levi-Civita connection is torsionless (aka that $\Gamma^{\rho}{}_{\mu\nu}$ is symmetric in (μ,ν)). As a result, we find that $F_{\mu\nu} \to F_{\mu\nu}$. At the same time $g^{\mu\nu} \to \Omega^{-2}g^{\mu\nu}$ and $g \to \Omega^8 g$. So the action

$$S = -\frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} = -\frac{1}{4}\int \mathrm{d}^4x\sqrt{-\bar{g}}\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{5.4}$$

is indeed invariant.

(**c**) Maxwell's equations in a curved background read

$$\nabla^{\nu}F_{\mu\nu} = 0. \tag{5.5}$$

For the covariant derivative, we have

$$\nabla^{\nu}F_{\mu\nu} = \partial^{\nu}F_{\mu\nu} - g^{\nu\sigma}\Gamma^{\rho}{}_{\mu\sigma}F_{\rho\nu} - g^{\nu\sigma}\Gamma^{\rho}{}_{\sigma\nu}F_{\mu\rho}. \tag{5.6}$$

Recall that the Christoffel symbols transform under Weyl transformations as given in Eq. (4.5), implying

$$\bar{\Gamma}^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\mu\nu} = \frac{2\delta^{\rho}_{(\mu}\nabla_{\nu)}\Omega - g_{\mu\nu}\nabla^{\rho}\Omega}{\Omega}. \tag{5.7}$$

$$\delta\left(\nabla^\nu F_{\mu\nu}\right) = \bar{\nabla}^\nu \bar{F}_{\mu\nu} - \nabla^\nu F_{\mu\nu}, \tag{5.8}$$
$$= -\frac{2\nabla_{(\sigma}\Omega F_{\mu)}{}^{\sigma} - F_{\rho\mu}\nabla^\rho\Omega}{\Omega} + \frac{2F_{\mu\rho}\nabla^\rho\Omega}{\Omega}, \tag{5.9}$$
$$= \frac{-F_{\mu\sigma}\nabla^\sigma\Omega + F_{\rho\mu}\nabla^\rho\Omega + 2F_{\mu\rho}\nabla^\rho\Omega}{\Omega}, \tag{5.10}$$
$$= 0, \tag{5.11}$$

where we used the antisymmetry of the field strength tensor. Thus, as expected, they remain unchanged.

Thus, the mode equation for photons is equivalent to the one in flat space. Then, there is a unique vacuum. The universe is therefore not filled with light.

(**d**) The energy-momentum tensor is defined in Eq. (4.15). We have to vary the Maxwell action with respect to the inverse metric. We thus have

$$\delta S = -\frac{1}{4}\int \mathrm{d}^4x\,\delta\left(\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\right), \tag{5.12}$$
$$= -\frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}\left[\left(2F_{\mu\rho}F^{\rho}_{\ \nu} - \frac{1}{2}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)\delta g^{\mu\nu}\right]. \tag{5.13}$$

There is no contribution from the variation of the field strength because due to Eq. (5.3) it is independent of the metric. Then, the energy-momentum tensor reads

$$T_{\mu\nu} = F_{\mu\rho}F^{\rho}_{\ \nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}. \tag{5.14}$$

Contracting the two indices, we obtain the trace

$$T^{\mu}_{\ \mu} = 0. \tag{5.15}$$

(**e**) Consider some matter action $\mathcal{L}_{\text{matter}}(g_{\mu\nu}, \Phi^a)$, with some set of matter fields Φ_a, where the index a enumerates the fields. These can be vectors, spinors, scalars, apples ... If we apply an infinitesimal Weyl transformation such that $\Omega^2 = 1 + \delta\omega$, the action changes as

$$\delta S = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}T_{\mu\nu}\delta g^{\mu\nu} + \frac{\delta\mathcal{L}_{\text{matter}}}{\delta\Phi^a}\delta\Phi^a + \text{total derivatives}\right). \tag{5.16}$$

Applying the equations of motion, removing total derivatives and plugging in $\delta g^{\mu\nu} = -g^{\mu\nu}\delta\omega$, we obtain the transformation

$$\delta S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}T^{\mu}_{\ \mu}\delta\omega. \tag{5.17}$$

Hence, the change in the action vanishes iff $T^{\mu}_{\ \mu} = 0$. In plain English, a theory is Weyl invariant if the energy-momentum tensor has vanishing trace.

Exercise 6: Impact of general nonminimal coupling on particle production

Motivation: Last week, we found that minimally coupled scalars are generically produced in FLRW backgrounds. This does not happen for conformally coupled scalars. Here we estimate what happens for general nonminimal coupling.

Consider a nonminimally coupled massive scalar whose action is given by Eq. (4.1).

(**a**) Compute the equation of motion for the scalar in an FLRW background, and redefine the field $\phi \to \chi = a(\eta)\phi$ such that the friction term ($\sim \phi'$) disappears. You should obtain that the nonminimal coupling gives you an additional contribution to the effective mass.

Sanity check: What happens in the limit $\xi \to 1/6$, $m \to 0$?

(**b**) Assume that the background is changing slowly and consider modes with small wavelength. Find out when these two assumptions are actually equivalent.

(**c**) Start in the adiabatic vacuum at some conformal time $\eta = \eta_0$, namely $|0_{\text{ad},\eta_0}\rangle$, and look at the resulting state at a time $\eta = \eta_0 + \Delta\eta$. Try to get at some qualitative properties of the average particle-number density $\langle 0_{\text{ad},\eta_0}|n_k|0_{\text{ad},\eta_0}\rangle(\eta_0 + \Delta\eta)$ without calculating it. Sketch how you expect the particle-number density to depend on ξ. (**Hint:** Keep in mind that particles are not produced if the background is static, and that in terms of some complex time-dependent Bogolyubov parameter $\beta_k(\eta)$

$$\langle 0_{\text{ad},\eta_0}|n_k|0_{\text{ad},\eta_0}\rangle(\eta) = |\beta_k(\eta)|^2. \tag{6.1}$$

You can find inspiration in exercise 1.)

The scalar satisfies the equation of motion

$$(\Box - m^2 - \xi R)\phi = 0. \tag{6.2}$$

Specifying to a flat FLRW spacetime in the conformal slicing (with conformal time η as time) and a field

$$\phi = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \phi_{\mathbf{k}}(\eta) e^{-i\mathbf{k}\mathbf{x}}, \tag{6.3}$$

the equation of motion takes the form

$$\phi_{\mathbf{k}}'' + 2\frac{a'}{a}\phi' + \left(k^2 + m^2a^2 + \xi R\right)\phi_{\mathbf{k}} = 0. \tag{6.4}$$

Defining a new field $\chi \equiv \phi a$, and correspondingly $\chi_{\mathbf{k}} = \phi_{\mathbf{k}} a$, we obtain field equation

$$\chi_{\mathbf{k}}'' + \left[k^2 + a^2m^2 + 6a^2\left(\xi - \frac{1}{6}\right)R\right]\chi_{\mathbf{k}} \equiv \chi_{\mathbf{k}}'' + \omega_k^2(\eta)\chi_{\mathbf{k}} = 0, \tag{6.5}$$

where we used that in spatially flat FLRW $R = 6a''/a^3$. Therefore, we find a modification to the effective mass, which, including nonminimal coupling, now reads

$$m_{\text{eff}}^2 = a^2\left[m^2 + 6\left(\xi - \frac{1}{6}\right)R\right]. \tag{6.6}$$

When $\xi = 1/6$, $m = 0$, the effective mass vanishes and the field equation for χ becomes that of a massless scalar in Minkowski spacetime.

(**b**) If the background is changing slowly, we have (see exercise 3)

$$\frac{\omega_k'}{\omega_k^2} \ll 1. \tag{6.7}$$

For modes with small wavelength $k^2 \gg m_{\text{eff}}^2$ during the whole evolution considered such that

$$\omega_k \simeq k + \frac{m_{\text{eff}}^2}{2k}. \tag{6.8}$$

Thus, for its time derivative, we obtain

$$\frac{\omega_k'}{\omega_k^2} \simeq \frac{(m_{\rm eff}^2)'}{2k^3}. \tag{6.9}$$

Thus, for the adiabatic approximation not to apply to small-wavelength modes, the scale factor has to satisfy

$$a^2(m^2+(6\xi-1)R) \ll k^2, \qquad 2\mathcal{H}a^2(m^2+(6\xi-1)R)+a^2(6\xi-1)R' \geq k^3 \tag{6.10}$$

over the whole evolution (here $\mathcal{H}=a'/a$). Thus, (neglecting unexpected cancellations) either $\mathcal{H} \gg k$ or $a^2R' \sim \mathcal{H}''+2\mathcal{H}'\mathcal{H} \geq k^3/a^2$, while at the same time $a^2R \sim \mathcal{H}'+\mathcal{H}^2 \ll k^2$. In other words, some derivatives of the scale factor have to be very large, while others have to be very small. Unless this is the case (which is rarely so, especially when evolving over longer times), large modes experience a slowly varying background.

(**c**) Let's start with the hint: The expected number density will be the squared norm of some complex Bogolyubov coefficient β_k. We have to estimate that coefficient. There is no particle creation if there is no time evolution. Thus, β_k at first order has to be a function of ω_k'/ω_k^2, the first kind of correction in the adiabatic approximation. Besides, it has to have an oscillating component which stems from the mixing of positive- and negative-frequency modes. According to the WKB-approximation, oscillating phases generically have arguments proportional to $\int \omega_k(\eta')\mathrm{d}\eta'$. Inspired by Eq. (1.4), we take the oscillating function to be a sinus. Besides, the particle density is the integrated number of created particles (they don't just disappear from one moment to the other) – by dimensional analysis the integral has to be balanced by an additional power of ω_k. Thus, we arrive at the estimate for the Bogolyubov coefficient

$$\beta_k = a\int_{\eta_0}^{\eta_0+\Delta\eta} \mathrm{d}\bar\eta \frac{\omega_k'(\bar\eta)}{\omega_k(\bar\eta)} \sin\left[b\int_{\eta_0}^{\bar\eta}\mathrm{d}\tilde\eta\,\omega_k(\tilde\eta)\right], \tag{6.11}$$

for a complex coefficient a and a real coefficient b, both of which are expected to be of order 1 in absolute value. Note how close this estimate is to Eq. (1.4) even though the evolution there is not adiabatic (the box-like behaviour produces delta-functions in ω_k' which definitely break the adiabatic approximation). At large k, we obtain

$$\beta_k \simeq a\int_{\eta_0}^{\eta_0+\Delta\eta} \mathrm{d}\bar\eta \frac{m_{\rm eff}^{2\prime}(\bar\eta)}{k^2} \sin\left[b\int_{\eta_0}^{\bar\eta}\mathrm{d}\tilde\eta\,\omega_k(\tilde\eta)\right]. \tag{6.12}$$

Say $\Delta\eta$ is small enough such that the adiabatic approximation holds integrated over the whole evolution (see Eq. (3.30)). Then, we can take the effective mass out of the integral and obtain

$$\beta_k \simeq a\frac{m_{\rm eff}^{2\prime}(\eta_0)}{k^2}\int_{\eta_0}^{\eta_0+\Delta\eta}\mathrm{d}\bar\eta \sin\left[b(\bar\eta-\eta_0)\omega_k(\tilde\eta)\right], \tag{6.13}$$

$$\simeq a\frac{m_{\rm eff}^{2\prime}(\eta_0)}{k^3}\left[1-\cos(b\Delta\eta\,\omega_k(\eta_0))\right]. \tag{6.14}$$

The number density then reads

$$\langle 0_{{\rm ad},\eta_0}|n_{\mathbf{k}}|0_{{\rm ad},\eta_0}\rangle = 4|a|^2\frac{|m_{\rm eff}^{2\prime}|^2}{k^4}\sin^4(b\Delta\eta\,\omega_k(\eta_0)/2). \tag{6.15}$$

Note that, here, the oscillating function was a choice. Depending on the specific situation, it can be more complicated. However, the qualitative behaviour (oscillating, proportional to $|m_{\rm eff}^{2\prime}|^2/k^4$) is rather robust, unless $a=0$ which would imply that the corrections are of higher order. In particular, at a fixed moment in time, the scaling with the nonminimal-coupling parameter is like

$$\langle 0_{{\rm ad},\eta_0}|n_{\mathbf{k}}|0_{{\rm ad},\eta_0}\rangle \sim |6\xi-1+\text{const linear in } m^2|^2. \tag{6.16}$$

Thus, the number density scales quadratically with ξ. Of course, if $m=0$, the particle-number density vanishes for $\xi=1/6$. As an example we plot the scaling $|6\xi-1+6m^2|$ in fig. 1.

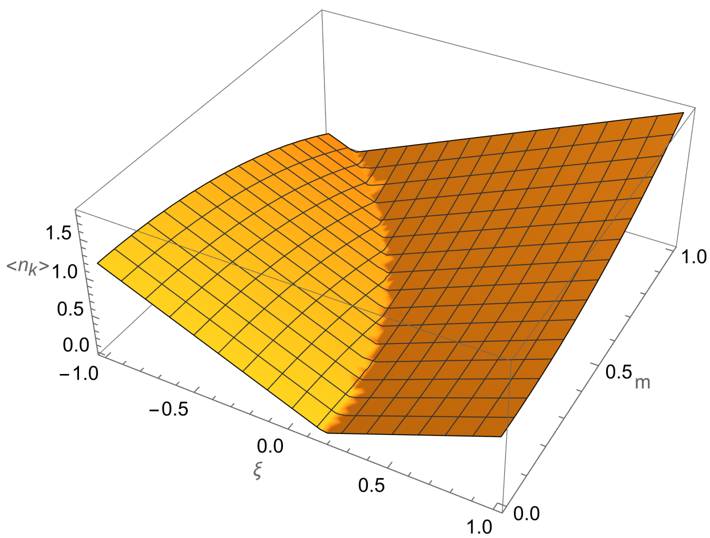

Figure 1: Exemplary scaling of number density with nonminimal-coupling parameter and mass. The exact value of $\langle n_{\mathbf{k}} \rangle$ as it is plotted here has no physical meaning because we did not derive an exact result.

Quantum field theory in curved spacetime

Assignment 3 – May 12

Exercise 7: Uniformly accelerated observers aka Rindler space

Motivation: Before we explore how the Minkowski vacuum appears to uniformly accelerated observers, we first need to understand the dynamics of accelerated motion in special relativity. As we'll find out, this brings up some of the exciting concepts usually reserved for general relativity.

First things first: For this and the following exercise, we do not need any general relativity. In the end, we'll just use a weird coordinate system to parametrize Minkowski spacetime. Don't believe the detractors who say special relativity can't describe non-inertial motion!
For simplicity, let's start in two-dimensional Minkowski space described in terms of Cartesian coordinates such that

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{7.1}$$

We want to describe a timelike observer moving along a uniformly accelerated trajectory. Parameterize the curve by the proper time of the observer τ. We denote their four-velocity, *i. e.* the tangent vector along the curve, as u^μ with norm $u^\mu u_\mu = -1$. We can, thus, define the proper acceleration as

$$a^\mu = u^\nu \nabla_\nu u^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu{}_{\nu\rho} u^\nu u^\rho, \tag{7.2}$$

which we readily identify as the left-hand side of the geodesic equation. As expected, if $a^\mu \neq 0$, motion is not geodesic/inertial.

(a) Show that the proper acceleration is spacelike. (**Hint:** Work in Cartesian coordinates)

(b) Assume that the acceleration is uniform. Then, the norm of the proper acceleration is constant. Denote it as $a^\mu a_\mu \equiv \mathfrak{a}^2 = \text{const}$. Construct the unique four-velocity that yields the correct norm for both the velocity and the acceleration. (**Hint:** Relative to an observer at rest, constant acceleration is like a boost with time-dependent boost parameter.)

(c) Find a parametrization of the curve ($\gamma(\tau) = (t(\tau), x(\tau))$, and show that uniformly accelerated observers move on hyperbolae in spacetime, *i. e.* $x^2 - t^2 = \mathfrak{a}^{-2}$. (**Hint:** For the second part, try to eliminate any explicit dependence on τ.)

(d) Find a coordinate system (η, ρ) such that the parabolic motion is realized by setting $\rho = \mathfrak{a}^{-1} = \text{const}$. You should obtain

$$\mathrm{d}s^2 = -\rho^2 \mathrm{d}\eta^2 + \mathrm{d}\rho^2. \tag{7.3}$$

Show that the new coordinate system amounts to the rest frame of the accelerated particle. (**Hint:** Think of an analogue of polar coordinates.)

Sanity check: Make sure that $u^\mu u_\mu = -1$ continues to be the case in the new coordinate system.

(**e**) For QFT (see next exercise), it is useful to coordinatize the space using the Rindler proper time $\tau = a^{-1}\eta$, and the position coordinate $\xi = \mathfrak{a}^{-1}\log\mathfrak{a}\rho$. Show that the resulting metric reads

$$\mathrm{d}s^2 = e^{2\mathfrak{a}\xi}(-\mathrm{d}\tau^2 + \mathrm{d}\xi^2). \tag{7.4}$$

Why could the metric be particularly interesting in this shape?

(**f**) Draw the Rindler trajectories in a Cartesian coordinate system for different values of $\mathfrak{a}$. What happens in the limit $\mathfrak{a} \to \infty$? Draw the corresponding limiting surface in your diagram as well as light-cones (light rays=45°-lines) emanating from points for which $t > x$. Can light from these points reach accelerated observers? How do we interpret this result?

Bonus question: Do the Rindler coordinates cover the whole Minkowski space?

(**g**) How do we generalize to four dimensions? A short argument should be sufficient.

(**a**) In Cartesian coordinates, the Christoffel symbols vanish. Thus, we have

$$a^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}. \tag{7.5}$$

If we take the derivative with respect to τ of the norm of the four-velocity, we obtain

$$\frac{\mathrm{d}(u^\mu u_\mu)}{\mathrm{d}\tau} = 2u_\mu \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = 2u^\mu a_\mu = 0. \tag{7.6}$$

Thus, the proper acceleration is normal to the four-velocity. Since the four-velocity is timelike, the proper acceleration has to be spacelike.

(**b**) A boost with rapidity η applied to a rest-frame four-velocity ($u^\mu = (1, 0)$) reads

$$u^\mu = (\cosh\eta, \sinh\eta). \tag{7.7}$$

One can check that indeed $u^\mu u_\mu = -\cosh^2\eta + \sinh^2\eta = -1$. Let's make $\eta = \eta(\tau)$ time dependent to indicate the acceleration. Then, the proper acceleration reads

$$a^\mu = \dot{\eta}(\sinh\eta, \cosh\eta). \tag{7.8}$$

Its norm reads

$$a^\mu a_\mu = \dot{\eta}^2(\cosh^2\eta - \sinh^2\eta) = \dot{\eta}^2 = \mathfrak{a}^2. \tag{7.9}$$

In other words, up to an irrelevant integration constant $\eta = \mathfrak{a}\tau$. Thus, the four-velocity reads

$$u^\mu = (\cosh\mathfrak{a}\tau, \sinh\mathfrak{a}\tau). \tag{7.10}$$

(**c**) Integration of the four-velocity yields $\gamma = (t(\tau), x(\tau)) = \mathfrak{a}^{-1}(\sinh\mathfrak{a}\tau, \cosh\mathfrak{a}\tau)$. We can eliminate any dependence on τ by considering the combination

$$x(\tau)^2 - t(\tau)^2 = \mathfrak{a}^{-2}. \tag{7.11}$$

This confirms that the motion traces a hyperbola. This is the Minkowski-space analogue of a circle: a curve of constant proper distance from the origin.

(**d**) By analogy with spherical coordinates, we define the hyperbolic coordinates $x'^\mu = (\eta, \rho)$, also called Rindler coordinates,[1]

$$t = \rho\sinh\eta, \qquad x = \rho\cosh\eta. \tag{7.12}$$

[1]The variable name η is chosen on purpose by a slight abuse of notation.

Then, uniformly accelerated motion simply amounts to $\rho(\tau) = \mathfrak{a}^{-1} = \text{const}$. We will derive the parametrization of $\eta(t)$ below.

The Jacobian of the transformation from Cartesian coordinates to Rindler coordinates reads

$$\frac{\partial x^\mu}{\partial x'^\nu} = \begin{pmatrix} \rho\cosh\eta & \rho\sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}, \qquad \frac{\partial x'^\mu}{\partial x^\nu} = \begin{pmatrix} \rho^{-1}\cosh\eta & -\sinh\eta \\ -\rho^{-1}\sinh\eta & \cosh\eta \end{pmatrix}. \tag{7.13}$$

Then, the metric reads

$$\begin{aligned} \mathrm{d}s^2 =& g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu & (7.14)\\ =& g_{\mu\nu}\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma}\mathrm{d}x'^\rho \mathrm{d}x'^\sigma & (7.15)\\ =& g'_{\mu\nu}\mathrm{d}x'^\mu \mathrm{d}x'^\nu & (7.16)\\ =& -\rho^2\mathrm{d}\eta^2 + \mathrm{d}\rho^2. & (7.17) \end{aligned}$$

In Rindler coordinates, the four-velocity of the uniformly accelerated observer reads

$$\begin{aligned} u =& u^\mu\partial_\mu, & (7.18)\\ =& u^\mu\frac{\partial x'^\nu}{\partial x^\mu}\partial'_\nu|_{\rho=\mathfrak{a}^{-1},\eta=\eta(\tau)}, & (7.19)\\ =& \rho^{-1}\left(\cosh\mathfrak{a}\tau\cosh\eta - \sinh\mathfrak{a}\tau\sinh\eta\right)\partial_\eta \\ & + \left(\cosh\mathfrak{a}\tau\sinh\eta - \sinh\mathfrak{a}\tau\cosh\eta\right)\partial_\rho|_{\rho=\mathfrak{a}^{-1},\eta=\eta(\tau)}. & (7.20) \end{aligned}$$

We recover the expected form of the velocity if $\eta(\tau) = \mathfrak{a}\tau$ such that

$$u = \mathfrak{a}\partial_\eta. \tag{7.21}$$

Indeed, the velocity has no component along the spatial direction. In other words, this coordinate system tracks the proper time of (this could have also been seen from $\eta = \mathfrak{a}\tau$) and defines the rest frame comoving with the uniformly accelerated observer.

The norm of the four-velocity then becomes

$$u^\mu u^\nu g'_{\mu\nu}|_{\rho=\mathfrak{a}^{-1},\eta=\mathfrak{a}\tau} = -\mathfrak{a}^2\rho^2|_{\rho=\mathfrak{a}^{-1}} = -1. \tag{7.22}$$

(**e**) The differentials after the coordinate transformation read

$$\mathrm{d}\eta = \mathfrak{a}\mathrm{d}\tau, \qquad \mathrm{d}\rho = e^{\mathfrak{a}\xi}\mathrm{d}\xi. \tag{7.23}$$

Thus, the metric clearly reads

$$\mathrm{d}s^2 = e^{2\mathfrak{a}\xi}(\mathrm{d}\tau^2 - \mathrm{d}xi^2). \tag{7.24}$$

In this form, the metric is conformally flat. This is useful especially when considering Weyl invariant theories because those satisfy Cartesian-like equations of motion also for Rindler coordinates.

(**f**) I plot the trajectories of different Rindler observers in fig. 2. In the limit $\mathfrak{a} \to \infty$, the trajectories asymptotically approach the light cone emanating from the origin which satisfies $x = t$. As light cones do not cross in Minkowski spacetime, no light can reach the accelerated observers from points where $t > x$. Such a surface is called a horizon. This resembles the event horizon of a black hole, just that (in more than two dimensions) it's infinitely large. Thus whatever happens to accelerating observers is analogous to what happens to observers close to a black hole.

Having a closer look at the metric in Rindler coordinates (Eq. (7.17)), its determinant vanishes at $\rho = 0$, *i. e.* at the horizon. This implies that there is a singularity there. As there is no singularity in Minkowski spacetime in Cartesian coordinates, this has to be a coordinate singularity. Indeed, this is

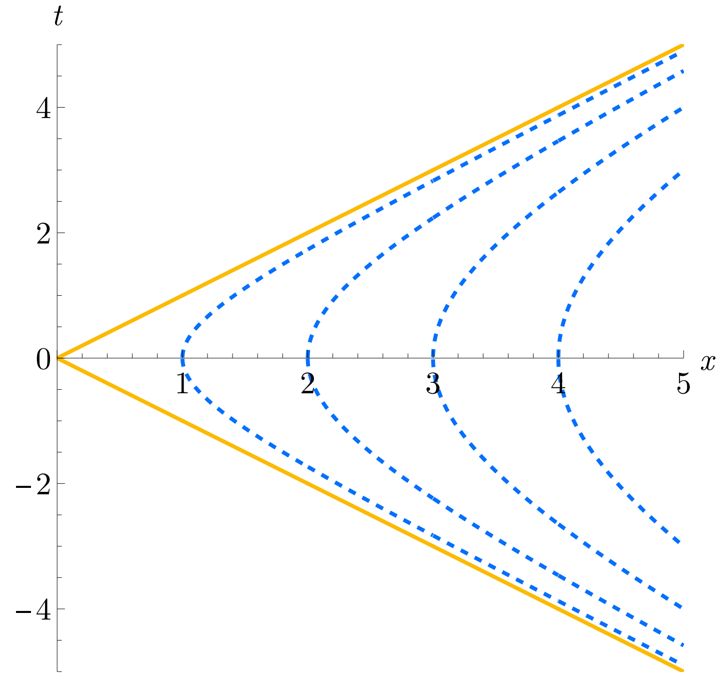

Figure 2: Trajectories of Rindler observers in Cartesian coordinates in two-dimensional Minkowski spacetime. The trajectories for $\mathfrak{a} = 1/n$ for $n = 1, 2, 3, 4$ are represented by blue dashed lines, while the yellow line is approached in the limit $\mathfrak{a} \to \infty$.

the same coordinate singularity you have in the origin in polar coordinates. It also indicates that you cannot describe Minkowski spacetime beyond the horizon in terms of Rindler coordinates as we defined them. While it is possible to extend the coordinates to negative ρ such that the mirror image of the region covered in fig. 2 can be described, the whole light-cone emanating from the origin requires a modified coordinate system. In the rest of the sheet we will concentrate on the so-called Rindler patch, *i. e.* the part $\rho > 0$.

(**g**) The uniformly accelerated observer is uniformly accelerated along one spatial direction. Without loss of generality, we can assume this axis to be the z-axis (with Cartesian coordinates (t, x, y, z)) such that the metric assumes the form

$$\mathrm{d}s^2 = -\rho^2 \mathrm{d}\eta^2 + \mathrm{d}\rho^2 + \mathrm{d}x^2 + \mathrm{d}y^2. \tag{7.25}$$

Exercise 8: Unruh effect

Motivation: Particle creation is not just an effect of curved spacetime. Accelerated observers in flat spacetime are embedded into a thermal bath of particle-antiparticle pairs. Let's find out how.

We want to quantize a massless Klein-Gordon field in two-dimensional Minkowski spacetime sliced by the set of Rindler coordinates (τ, ξ), see Eq. (7.4) for the metric. The action for the massless scalar reads

$$S = -\frac{1}{2} \int \mathrm{d}x \sqrt{-g} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi. \tag{8.1}$$

The corresponding equation of motion for the scalar reads

$$\Box \phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = 0. \tag{8.2}$$

We will quantize the solutions to this equation. This task is analogous to quantization in two-dimensional FLRW spacetime – in FLRW you have a time-dependent conformal factor in the metric. Now you have a position-dependent conformal factor in the metric.

(a) Express the equation of motion in Rindler coordinates (τ, ξ). The result should look analogous to the equation of motion in Cartesian coordinates. Why is it so simple?

(b) Solve the equation of motion.

Next we need to define what particles and antiparticles are. By convention, we write down the mode expansion in terms of orthonormal solutions as

$$\phi = \int_0^\infty \mathrm{d}\omega \left(a_\omega v_\omega + a_\omega^\dagger v_\omega^* \right), \tag{8.3}$$

where the creation and annihilation operators satisfy the usual commutation relations, and v_ω is a set of orthonormal solutions of the equations of motion. We generically choose $\omega > 0$ because throughout this exercise, we only consider right-moving solutions. This is simpler, and including left-movers wouldn't change the result.

For those who haven't done GR yet, the following may be a lot. If all this lingo makes no sense to you, just skip to sub-exercise (d). On general curved backgrounds (or backgrounds described by weird curvilinear coordinates), the space of solutions to the Klein-Gordon equation possesses an inner product on hypersurfaces of constant time Σ (whatever crazy time one may choose to work with). Given unitary time evolution along the normal vector to the surfaces n^μ (which is, of course, timelike and normalized), this inner product is conserved, thus allowing to find orthonormal solutions for all times. The Klein-Gordon inner product for $d+1$-dimensional backgrounds reads

$$(v_1, v_2) = -i \int_\Sigma \mathrm{d}^d x \sqrt{h} \left(v_1^* n^\mu \nabla_\mu v_2 - v_2^* n^\mu \nabla_\mu v_1 \right), \tag{8.4}$$

where the integral is over a surface of constant time, and h denotes the determinant of the induced metric.

(c) **Try to do this exercise before reading on:** What is the dimensionality of the hypersurface and what are h and n^μ for $1+1$-dimensional Minkowski spacetime sliced by Rindler time? Construct the Klein-Gordon inner product.

Orthonormal solutions v_ω satisfy the conditions

$$(v_\omega, v_{\omega'}) = \delta(\omega - \omega'), \qquad (v_\omega, v_{\omega'}^*) = 0. \tag{8.5}$$

(d) The Klein-Gordon inner product you should have obtained in sub-exercise (**c**) reads

$$(v_1, v_2) = -i \int_{-\infty}^\infty \mathrm{d}\xi (v_1^* \partial_\tau v_2 - v_2 \partial_\tau v_1^*). \tag{8.6}$$

Show that the (by convention) right-moving negative-frequency solutions

$$v_\omega = \frac{e^{i\omega(\tau - \xi)}}{\sqrt{4\pi\omega}} \tag{8.7}$$

are orthonormal (recall that $\omega > 0$). Thus, we can express the quantized field in terms of its mode expansion

$$\phi = \int_0^\infty \frac{\mathrm{d}\omega}{\sqrt{4\pi\omega}} \left[a_\omega e^{i\omega(\tau-\xi)} + a_\omega^\dagger e^{-i\omega(\tau-\xi)} \right]. \tag{8.8}$$

The Klein-Gordon inner product helps us with more than just constructing the mode expansion. It can also give us the overlap between modes of different mode expansions.
If you remember our last exercise sheet, distinct bases of mode functions are related by Bogolyubov coefficients. The right-moving, negative-frequency Cartesian mode functions read

$$u_k = \frac{e^{ik(t-x)}}{\sqrt{4\pi k}}, \tag{8.9}$$

for some $k > 0$ (again we only consider right-moving modes) such that the Cartesian mode expansion reads

$$\phi = \int_0^\infty \mathrm{d}k \left(b_k u_k + b_k^\dagger u_k^* \right), \tag{8.10}$$

with creation and annihilation operators b_k, $b_k^\dagger$. The inertial vacuum (*i. e.* the vacuum from the Cartesian mode expansion) $|0_{\text{inert}}\rangle$ is defined such that it is annihilated by b_k.
The inertial mode functions can be expressed in terms of the Rindler mode functions as

$$u_k = \int_0^\infty \mathrm{d}\omega \left(\alpha_{k\omega} v_\omega + \beta_{k\omega} v_\omega^* \right), \tag{8.11}$$

for some Bogolyubov coefficients $\alpha_{k\omega}$, $\beta_{k\omega}$. Make sure you understand why this works. Thus, generically they mix positive- and negative-frequency Rindler modes: The inertial vacuum is not necessarily empty for non-inertial observers. As we learned last week, we can express the one vacuum in terms of states of a different basis by computing the Bogolyubov coefficients. Let's see how this works exactly.

(e) Using the orthogonality relations in Eq. (8.5), show that

$$\alpha_{k\omega} = (v_\omega, u_k), \qquad \beta_{k\omega} = (v_\omega^*, u_k). \tag{8.12}$$

(f) Last week, we found in an analogous problem that the vacuum state in one basis may be populated with particles in a different one, and that the particle-number density depends on the Bogolyubov coefficients. Therefore, show that

$$\alpha_{k\omega} = \frac{\omega}{2\pi\sqrt{\omega k \mathfrak{a}}} \mathfrak{a}^{\frac{i\omega}{\mathfrak{a}}} k^{-\frac{i\omega}{\mathfrak{a}}} e^{\frac{\pi\omega}{2\mathfrak{a}}} \Gamma\left(\frac{i\omega}{\mathfrak{a}}\right), \tag{8.13}$$

$$\beta_{k\omega} = -\frac{\omega}{2\pi\sqrt{\omega k \mathfrak{a}}} \mathfrak{a}^{-\frac{i\omega}{\mathfrak{a}}} k^{\frac{i\omega}{\mathfrak{a}}} e^{-\frac{\pi\omega}{2\mathfrak{a}}} \Gamma\left(-\frac{i\omega}{\mathfrak{a}}\right). \tag{8.14}$$

(**Hint:** Keep in mind that ϕ is a scalar field and transforms as such. You may use that the Laplace transform of a power is related to the Gamma function as

$$\int_0^\infty z^{s-1} e^{-\lambda z} \mathrm{d}z = \lambda^{-s}\Gamma(s), \tag{8.15}$$

and analytically continue to complex s and λ.)

Work smarter, not harder. Do you have to compute both $\alpha_{k\omega}$ and $\beta_{k\omega}$ individually or are they somehow related?

(**g**) Show that the number density in the inertial vacuum can be expressed as (**Note that this expression was ordered wrongly in the original sheet. This mistake was propagating through the exercise.**)

$$n_\omega = V^{-1}\langle 0_{\text{inert}}|a^\dagger_\omega a_\omega|0_{\text{inert}}\rangle = \int_0^\infty \mathrm{d}k|\beta_{k\omega}|^2, \tag{8.16}$$

with the volume of the spacelike slice V, which as seen in earlier exercises, is required for regularization.

(**h**) The integral in Eq. (8.16) is divergent and would have to be regularized. Instead of evaluating the integral explicitly, we use a trick. First show that

$$n_\omega = -1 + \int_0^\infty \mathrm{d}k|\alpha_{k\omega}|^2. \tag{8.17}$$

Next, having a closer look at Eqs. (8.13) and (8.14), express $\beta_{k\omega}$ in terms of $\alpha_{k\omega}$. This should allow you to compute n_ω without integrating to obtain

$$n_\omega = \frac{1}{e^{\frac{2\pi\omega}{\mathfrak{a}}} - 1}. \tag{8.18}$$

Interpret this result: What type of distribution is this?

(**a**) In Rindler coordinates, the equation of motion reads

$$(\partial_\tau^2 - \partial_\xi^2)\phi = 0. \tag{8.19}$$

This equation is so simple just because the massless scalar is Weyl invariant in two dimensions. Thus, we can transform away the conformal factor without implications for the scalar. This is fully analogous to the conformally coupled scalar and the Maxwell field in four dimensions on the last sheet.

(**b**) Solutions to the equation of motion Eq. (8.19) are equivalent to the Cartesian solution. Thus, the most general solution reads

$$\phi = Ae^{i\omega(\tau+\xi)} + Be^{i\omega(\tau-\xi)} + Ce^{-i\omega(\tau+\xi)} + De^{-i\omega(\tau-\xi)}, \tag{8.20}$$

for some complex constants A, B, C and D. That ϕ is a real scalar imposes the constraints

$$C = A^*, \qquad D = B^*, \tag{8.21}$$

such that we obtain the general solution

$$\phi = Ae^{i\omega(\tau+\xi)} + Be^{i\omega(\tau-\xi)} + \text{h. c.} \tag{8.22}$$

Note here that the direction of motion is encoded in the sign of ξ such that generically $\omega \geq 0$.

(**c**) The hypersurface is one-dimensional, *i. e.* a line. As shown in exercise 7 (**d**), the normal equals the Rindler velocity $n = u = \rho^{-1}\partial_\eta = e^{-\mathfrak{a}\xi}\partial_\tau$, and the induced "metric" on the line is $\mathrm{d}s^2_{(1)} = e^{2\mathfrak{a}\xi}\mathrm{d}\xi^2$, so $h = e^{2\mathfrak{a}\xi}$. Thus, the factors from the determinant of the induced metric and the normal vector cancel, and the Klein-Gordon inner product reads

$$(\phi_1, \phi_2) = -i\int_{-\infty}^{\infty} \mathrm{d}\xi(\phi_1^*\partial_\tau\phi_2 - \phi_2\partial_\tau\phi_1^*). \tag{8.23}$$

(**d**) For the negative-frequency solutions Eq. (8.7), we obtain the inner products

$$(v_\omega, v_{\omega'}) = \frac{\omega+\omega'}{4\pi\sqrt{\omega\omega'}} e^{-i(\omega-\omega')\tau} \int_{-\infty}^{\infty} \mathrm{d}\xi e^{i(\omega-\omega')\xi}, \tag{8.24}$$

$$= \frac{\omega'+\omega}{2\sqrt{\omega\omega'}} e^{-i(\omega-\omega')\tau} \delta(\omega-\omega'), \tag{8.25}$$

$$= \delta(\omega-\omega'), \tag{8.26}$$

$$(v_\omega, v^*_{\omega'}) = \frac{\omega+\omega'}{4\pi\sqrt{\omega\omega'}} e^{-i(\omega+\omega')\tau} \int_{-\infty}^{\infty} \mathrm{d}\xi e^{i(\omega+\omega')\xi}, \tag{8.27}$$

$$= \frac{\omega'+\omega}{2\sqrt{\omega\omega'}} e^{-i(\omega+\omega')\tau} \delta(\omega+\omega'), \tag{8.28}$$

$$= 0 \tag{8.29}$$

as long as $\omega, \omega' > 0$.

(**e**) We start on the right-hand side of Eq. (8.12) and show that it equals the left-hand side:

$$(v_\omega, u_k) = \int_0^\infty \mathrm{d}\omega' \left[\alpha_{k\omega'}(v_\omega, v_{\omega'}) + \beta_{k\omega'}(v_\omega, v^*_{\omega'})\right], \tag{8.30}$$

$$= \int_0^\infty \mathrm{d}\omega' \alpha_{k\omega'} \delta(\omega-\omega'), \tag{8.31}$$

$$= \alpha_{k\omega}. \tag{8.32}$$

Analogously, we obtain

$$(v^*_\omega, u_k) = \int_0^\infty \mathrm{d}\omega' \left[\alpha_{k\omega'}(v^*_\omega, v_{\omega'}) + \beta_{k\omega'}(v^*_\omega, v^*_{\omega'})\right], \tag{8.33}$$

$$= \int_0^\infty \mathrm{d}\omega' \beta_{k\omega'} \delta(\omega-\omega'), \tag{8.34}$$

$$= \beta_{k\omega}. \tag{8.35}$$

(**f**) The Cartesian negative-frequency solution in Rindler coordinates reads

$$u_k = \frac{e^{i\mathfrak{a}^{-1}ke^{\mathfrak{a}\xi}(\sinh \mathfrak{a}\tau - \cosh \mathfrak{a}\tau)}}{\sqrt{4\pi k}} \tag{8.36}$$

$$= \frac{e^{-i\mathfrak{a}^{-1}ke^{-\mathfrak{a}(\tau-\xi)}}}{\sqrt{4\pi k}}. \tag{8.37}$$

Thus, their overlap with the Rindler negative-frequency solutions reads

$$(v_\omega, u_k) = \frac{1}{4\pi\sqrt{\omega k}} \int_{-\infty}^{\infty} \mathrm{d}\xi (\omega + ke^{\mathfrak{a}(\xi-\tau)}) e^{-i\left(\omega(\tau-\xi) + \frac{k}{\mathfrak{a}} e^{-\mathfrak{a}(\tau-\xi)}\right)}, \tag{8.38}$$

$$= \frac{1}{4\pi\sqrt{\omega k}} \int_{-\infty}^{\infty} \mathrm{d}V (\omega + ke^{-\mathfrak{a}V}) e^{-i\left(\omega V + \frac{k}{\mathfrak{a}} e^{-\mathfrak{a}V}\right)}, \tag{8.39}$$

where $V = \tau - \xi$. We now introduce a new variable

$$z = ke^{-\mathfrak{a}V}. \tag{8.40}$$

Then, the measure becomes

$$-\frac{\mathrm{d}z}{\mathfrak{a}z} = \mathrm{d}V. \tag{8.41}$$

Thus, we massaged the integral into the shape

$$(v_\omega, u_k) = \frac{1}{4\pi\sqrt{\omega k}} \int_0^\infty \frac{\mathrm{d}z}{\mathfrak{a}z}(\omega + z)\left(\frac{z}{k}\right)^{i\omega/\mathfrak{a}} e^{-\frac{iz}{\mathfrak{a}}}, \tag{8.42}$$

$$= \frac{k^{-\frac{i\omega}{\mathfrak{a}}}}{4\pi\sqrt{\omega k}\mathfrak{a}} \int_0^\infty \mathrm{d}z \left[z^{\frac{i\omega}{\mathfrak{a}}}\left(1 + \frac{\omega}{z}\right)\right] e^{-\frac{iz}{\mathfrak{a}}}. \tag{8.43}$$

Here, we apply the definition of the Γ-function as Laplace transform of a power law

$$\int_0^\infty z^{s-1} e^{-\lambda z} \mathrm{d}z = \lambda^{-s}\Gamma(s), \tag{8.44}$$

which analytically continued to $\lambda = i/\mathfrak{a}$ and for $s = i\omega/\mathfrak{a}$ yields

$$\int_0^\infty z^{\frac{i\omega}{\mathfrak{a}}-1} e^{-\frac{iz}{\mathfrak{a}}} \mathrm{d}z = \left(\frac{i}{\mathfrak{a}}\right)^{-\frac{i\omega}{\mathfrak{a}}} \Gamma\left(\frac{i\omega}{\mathfrak{a}}\right) = \mathfrak{a}^{\frac{i\omega}{\mathfrak{a}}} e^{\frac{\pi\omega}{2\mathfrak{a}}} \Gamma\left(\frac{i\omega}{\mathfrak{a}}\right). \tag{8.45}$$

Thus, the Bogolyubov coefficient becomes

$$\alpha_{k\omega} = \frac{1}{4\pi\sqrt{\omega k}\mathfrak{a}} \mathfrak{a}^{\frac{i\omega}{\mathfrak{a}}} k^{-\frac{i\omega}{\mathfrak{a}}} e^{\frac{\pi\omega}{2\mathfrak{a}}} \left[\omega\Gamma\left(\frac{i\omega}{\mathfrak{a}}\right) - i\mathfrak{a}\Gamma\left(1 + \frac{i\omega}{\mathfrak{a}}\right)\right], \tag{8.46}$$

$$= \frac{\omega}{2\pi\sqrt{\omega k}\mathfrak{a}} \mathfrak{a}^{\frac{i\omega}{\mathfrak{a}}} k^{-\frac{i\omega}{\mathfrak{a}}} e^{\frac{\pi\omega}{2\mathfrak{a}}} \Gamma\left(\frac{i\omega}{\mathfrak{a}}\right). \tag{8.47}$$

The Bogolyubov coefficient $\beta_{k\omega}$ is derived from $v_\omega^* = iv_{-\omega}$. Thus, keeping in mind that v_ω^* appears on the left entry of the Klein-Gordon inner product in Eq. (8.12) and is therefore complex conjugated, we immediately obtain

$$\beta_{k\omega} = -i\alpha_{k-\omega}, \tag{8.48}$$

$$= -\frac{\omega}{2\pi\sqrt{\omega k}\mathfrak{a}} \mathfrak{a}^{-\frac{i\omega}{\mathfrak{a}}} k^{\frac{i\omega}{\mathfrak{a}}} e^{-\frac{\pi\omega}{2\mathfrak{a}}} \Gamma\left(-\frac{i\omega}{\mathfrak{a}}\right). \tag{8.49}$$

(**g**) First we have to express the operators a_ω, $a_\omega^\dagger$ in terms of the operators b_k, $b_k^\dagger$. We find this relation by expressing

$$\phi = \int_0^\infty \mathrm{d}k \left(b_k u_k + b_k^\dagger u_k^*\right), \tag{8.50}$$

$$= \int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}\omega \left[b_k(\alpha_{k\omega} v_\omega + \beta_{k\omega} v_\omega^*) + b_k^\dagger(\alpha_{k\omega}^* v_\omega^* + \beta_{k\omega}^* v_\omega)\right], \tag{8.51}$$

$$= \int_0^\infty \mathrm{d}\omega \int_0^\infty \mathrm{d}k \left[(\alpha_{k\omega} b_k + \beta_{k\omega}^* b_k^\dagger) v_\omega + (\beta_{k\omega} b_k + \alpha_{k\omega}^* b_k^\dagger) v_\omega^*\right]. \tag{8.52}$$

It follows that

$$a_\omega = \int_0^\infty \left(\alpha_{k\omega} b_k + \beta_{k\omega}^* b_k^\dagger\right) \mathrm{d}k. \tag{8.53}$$

Thus, we obtain for the particle-number density

$$n_\omega = V^{-1} \langle 0_{\text{inert}} | a_\omega^\dagger a_\omega | 0_{\text{inert}} \rangle \tag{8.54}$$

$$= V^{-1} \int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}k' \beta_{k\omega}^* \beta_{k'\omega}^* \langle 0_{\text{inert}} | b_k b_{k'}^\dagger | 0_{\text{inert}} \rangle, \tag{8.55}$$

$$= V^{-1} \int_0^\infty \mathrm{d}k |\beta_{k\omega}|^2. \tag{8.56}$$

(**h**) We could have equivalently expressed the number density as

$$n_\omega = V^{-1}\langle 0_{\rm inert}| - \delta(0) + a_\omega a_\omega^\dagger |0_{\rm inert}\rangle, \tag{8.57}$$

$$= V^{-1}\left(-\delta(0) + \int_0^\infty {\rm d}k \int_0^\infty {\rm d}k' \alpha_{k\omega}^* \alpha_{k'\omega} \langle 0_{\rm inert}| b_k b_{k'}^\dagger |0_{\rm inert}\rangle\right), \tag{8.58}$$

$$= V^{-1}\left(-\delta(0) + \int_0^\infty {\rm d}k |\alpha_{k\omega}|^2\right). \tag{8.59}$$

You should be used to the pesky $\delta(0)$s now. Cancel infinity with infinity as if there was no tomorrow, wave some hands, and we obtain

$$n_\omega = -1 + V^{-1}\int_0^\infty {\rm d}k |\alpha_{k\omega}|^2. \tag{8.60}$$

This is fine.

We see that the Bogolyubov coefficients are related as

$$\alpha_{k\omega} = -e^{\frac{\pi\omega}{\mathfrak{a}}} \beta_{k\omega}^*. \tag{8.61}$$

Thus, we can express the particle number as

$$n_\omega = -1 + e^{\frac{2\pi\omega}{\mathfrak{a}}} V^{-1}\int_0^\infty {\rm d}k |\alpha_{k\omega}|^2, \tag{8.62}$$

$$= -1 + e^{\frac{2\pi\omega}{\mathfrak{a}}} n_\omega. \tag{8.63}$$

If we solve for the particle-number density, we obtain (trumpets please!)

$$n_\omega = \frac{1}{e^{\frac{2\pi\omega}{\mathfrak{a}}} - 1}. \tag{8.64}$$

This is a Planckian distribution. Thus, the field is in a thermal state – a uniformly accelerated observer is surrounded by a thermal bath of temperature $T = \mathfrak{a}/2\pi$ (in units in which $k_{\rm B} = 1$). The vacuum, thus, really has a temperature to non-inertial observers.

Extra material 1: Rindler approximation to horizons

The Rindler horizon can be understood as the first approximation of the geometry experienced by an observer hovering above a general horizon. Here's how:
Consider a general static spherically symmetric spacetime in Schwarzschild-like coordinates (t, r, ϕ, θ) such that

$${\rm d}s^2 = -f(r){\rm d}t^2 + \frac{{\rm d}r^2}{f(r)} + r^2\left({\rm d}\theta^2 + \sin^2\theta {\rm d}\phi^2\right), \tag{9.1}$$

for some function $f(r)$. Let's assume that there is some $r = r_{\rm h}$ at which $f(r_{\rm h}) = 0$, $f'(r_{\rm h}) \neq 0$, implying that there is some (coordinate) singularity indicating a non-extremal horizon. Considering observers at $r > r_{\rm h}$, we can approximate the metric close to the horizon by choosing coordinates $\epsilon = r - r_{\rm h}$ and expanding in ϵ to leading order such that

$${\rm d}s^2 = -f'(r_{\rm h})\epsilon {\rm d}t^2 + \frac{{\rm d}\epsilon^2}{f'(r_{\rm h})\epsilon} + r_{\rm h}^2\left({\rm d}\theta^2 + \sin^2\theta {\rm d}\phi^2\right). \tag{9.2}$$

Next, we introduce a new radial coordinate ρ which trivializes the radial part of the metric (*i. e.* the proper distance from the horizon) such that ${\rm d}\rho = {\rm d}\epsilon/\sqrt{f'(r_{\rm h})\epsilon}$, *i. e.* $\rho = 2\sqrt{\epsilon/f'(r_{\rm h})}$. As a result, we obtain

$$\mathrm{d}s^2 = -\frac{f'(r_\mathrm{h})^2}{4}\rho^2\mathrm{d}t^2 + \mathrm{d}\rho^2 + r_\mathrm{h}^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right). \tag{9.3}$$

Here, the (t,r)-part amounts exactly to the two-dimensional Rindler metric in terms of the proper time (in the language of exercise 7: τ) of the uniformly accelerated observer. The acceleration reads

$$\mathfrak{a} = \frac{f'(r_\mathrm{h})}{2} = \kappa, \tag{9.4}$$

where κ is usually called the surface gravity of the horizon (if gravity was still a force, this would be the force experienced on the horizon, therefore the terminology).

Schwarzschild-like time is usually defined with respect to a static observer (who is inertial only in asymptotically flat spacetimes in the limit $r \to \infty$). Here, we expanded close to the horizon, so the static observer has to be statically hovering above the horizon. This requires the radial acceleration Eq. (9.4).

In a nutshell, this means that

1. Any spacetime which is static, spherically symmetric and has a non-extremal horizon is approximated by Rindler space in the near-horizon region.

2. The Unruh effect may also apply to other kinds of horizons (chrchrm Hawking effect chrchrm). Indeed, consider the vacuum as seen by an inertial observer falling into a Schwarzschild black hole. If we plug in $f = 1 - 2M/r$, we obtain the temperature the hovering observer will experience, namely

$$T = \frac{1}{8\pi M}. \tag{9.5}$$

   We will see in a bit that this is exactly the Hawking temperature of a Schwarzschild black hole. Indeed, this is a way to derive the Hawking temperature of a general spherically symmetric black hole as

$$T = \frac{\kappa}{2\pi} = \frac{f'(r_\mathrm{h})}{4\pi}. \tag{9.6}$$

3. If there is no other length scale in the model $\kappa \propto r_\mathrm{h}^{-1}$ (amounting to wavelengths of horizon size), so the temperature is generically very low for large (e. g. astrophysical) black holes.

Exercise 9: Inflation

Motivation: In the lecture we were introduced to early-universe inflation. For those who have not seen inflation in a different course yet, this exercise should be like a very short primer. For more info, see this review.

This exercise is divided into a motivational part and a computational part.

Motivational part: Why would we want a period of accelerated expansion in the very early universe? Let's estimate. We know that the sky is made up of $N \sim 10^4$ patches which have never been in causal contact during the history of the universe if we run back ordinary FLRW evolution only with radiation and matter. If they haven't been in causal contact, we would expect them to be uncorrelated so their average temperature fluctuation $x \equiv \delta T/T_\mathrm{CMB}$ should be random, say Gaussian distributed around 0.

The important bit about the Gaussian is its standard deviation. Since x is dimensionless and there's no physical scale to set its size, a natural guess is that its standard deviation is $\Delta x \sim \mathcal{O}(1)$. In CMB measurements, correlations of x across the sky are encoded in the coefficients of the multipole expansion of the power spectrum $a_{\ell m}$. All of these coefficients have been measured to satisfy

$$|a_{\ell m}| \leq 10^{-5}. \tag{9.7}$$

Roughly, each of the first N multipole coefficients captures independent information from each causally disconnected patch. Thus, we can interpret the inequality Eq. (9.7) as measurements of x across the N independent patches.

(**a**) Compute the probability that all of the first N multipoles satisfy the inequality Eq. (9.7) if they are all Gaussian distributed around 0 with standard deviation $\Delta a_{\ell m} = 1$. You should obtain something overwhelmingly tiny.

(**b**) This has been one of the main original arguments in favour of introducing inflation: The probability of all this correlation being there randomly (the temperature fluctuations being so small everywhere) is incomprehensibly small, so the patches must have been in contact after all. Inflation brings the disconnected regions into causal contact in the past. Try to find weaknesses of this argument.

Next we get to the technical part. The usual way to go in inflation is to propose a model usually containing GR plus additional fields, which lead to accelerated expansion. This accelerated expansion implies that many areas which appear not to be causally connected now actually were during inflation. Let's start with a single minimally coupled scalar field ϕ, the inflaton – the simplest and most common type of model. The action reads

$$S = \int \mathrm{d}^x \sqrt{-g} \left(-\frac{m_\mathrm{P}^2}{2} R - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) \right), \tag{9.8}$$

with the Planck mass m_P, the Ricci scalar R and a potential $V(\phi)$. Inflation is usually set in an FLRW background in comoving coordinates such that

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \mathrm{d}\vec{x}^2, \tag{9.9}$$

with the scale factor $a(t)$. Then, as you have already derived, the scalar satisfies the equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{9.10}$$

with the Hubble parameter $H = \dot{a}/a$. Besides, the cosmological principle requires the scalar, too, to only depend on time, *i. e.* $\vec{\nabla}\phi = 0$.
Slow-roll inflation occurs, when the potential energy dominates over the kinetic energy – then, the scalar does not really change and the scalar contribution to the action is approximately constant. Thus, like a cosmological constant it leads to de Sitter-like exponential expansion for some time in the very early universe. Let's make this statement more precise.

(**c**) Compute the stress energy tensor of the scalar. For a perfect fluid, the stress energy tensor equals

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}, \tag{9.11}$$

with the energy density ρ and the pressure p and the four-velocity of the fluid $u = u^\mu \partial_\mu$ which in comoving coordinates equals $u = \partial_t$. Show that

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{9.12}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{9.13}$$

The Friedmann equations read

$$H^2 = \frac{\rho}{3m_\mathrm{P}^2}, \qquad \frac{\ddot{a}}{a} = -\frac{\rho + 3p}{2m_\mathrm{P}^2}. \tag{9.14}$$

Thus, we obtain accelerated expansion if the equation-of-state parameter of the perfect fluid w satisfies

$$w = \frac{p}{\rho} < -\frac{1}{3}. \tag{9.15}$$

Let's make precise what it means to be close to de Sitter expansion.

(d) Show that we can express the second Friedmann equation as

$$\frac{\ddot{a}}{a} = H^2(1-\epsilon), \tag{9.16}$$

where

$$\epsilon \equiv -\frac{\dot{H}}{H^2}. \tag{9.17}$$

(e) Express ϵ in terms of w to obtain

$$\epsilon = \frac{3}{2}\left(w+1\right). \tag{9.18}$$

Verify that accelerated expansion occurs for $\epsilon < 1$. (**Hint:** Perfect fluids satisfy the continuity equation $\dot{\rho} = -3H(\rho + p)$.)

The dimensionless ϵ is the first slow-roll parameter. If $\epsilon = 0$, the universe is undergoing exact de Sitter expansion (then, H is constant because $w = -1$ which amounts to a cosmological constant). So for near-de Sitter expansion, we need $\epsilon \ll 1$.

(f) Show that $\epsilon \ll 1$ implies that $\dot{\phi}^2/2 \ll V(\phi)$, *i. e.* indeed the kinetic energy is much larger than the potential energy.

Besides, for the field to roll slowly enough, the scalar should not accelerate too strongly (ϵ should not change very fast), a behaviour which is captured by the second slow-roll parameter **(In an earlier version of the assignment, the − sign was missing in the definition below.)**

$$\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}. \tag{9.19}$$

In the slow-roll regime, $\eta \ll 1$. In the slow-roll regime, the two slow-roll parameters can be expressed in terms solely in terms of the potential and its derivatives.

(g) Demonstrate that in the slow-roll regime, **(In an earlier version of the assignment, the Planck masses were missing in the below equation)**

$$\epsilon_V \equiv \frac{m_\mathrm{P}^2}{2}\left(\frac{V'}{V}\right)^2 \simeq \epsilon, \tag{9.20}$$

$$\eta_V \equiv m_\mathrm{P}^2 \frac{V''}{V} \simeq \eta + \epsilon. \tag{9.21}$$

In the slow-roll regime, it's sufficient to monitor ϵ_V and η_V.
Inflation ends when $\epsilon = 1$. As the previous expansion was quasi exponential, it is useful to adapt our measure of time accordingly. Therefore, we introduce the "e-fold" such that

$$\mathrm{d}N = -H\mathrm{d}t, \tag{9.22}$$

normalized such that at the end of inflation $N = 0$. Thus, considering that $a \sim e^{Ht}$, at some N, the universe was smaller by a factor $\sim e^{-N}$ than at the end of inflation.
Let's get to observables. Everything that's within the horizon at the end of inflation is generically observable to us and may leave an imprint on the CMB. This amounts to fluctuations generated up to some $N = N_\star$, where depending on the model $40 \leq N_\star \leq 60$. Observables are generically evaluated at $N_\star$ because that's when the seeds of primordial fluctuations were sown. The two most well-known CMB-observables are[a]

- Δ_s^2 denotes the amplitude of spectrum of scalar fluctuations which has to be normalized as $\Delta_\mathrm{s}^2 \sim 10^{-9}$. For slow-roll inflation it reads

$$\Delta_\mathrm{s}^2 \simeq \frac{1}{24\pi^2}\frac{V}{m_\mathrm{P}^4}\epsilon_V^{-1}|_{N=N_\star}. \tag{9.23}$$

- r is the tensor-to-scalar ratio, *i. e.* the ratio of the amplitudes of tensor fluctuations (stemming from GR's spin-two part) and scalar fluctuations (stemming from the inflaton and GR's spin-zero part). Expressed in terms of the potential-derived slow-roll parameters it reads

$$r = 16\epsilon_V^\star. \tag{9.24}$$

  Hereafter the superscript $\star$ means that the respective quantity is evaluated at the effective onset of inflation $N_\star$.

- n_s is the scalar spectral index which you should have already seen in the lecture. It governs the tilt of the power spectrum of scalar fluctuations towards red. Expressed in terms of the potential-derived slow-roll parameters it reads **(In an earlier version of the assignment there was a wrong sign in the equation below.)**

$$n_\mathrm{s} = 1 + 2\eta_V^\star - 6\epsilon_V^\star. \tag{9.25}$$

Now, we compute the observables for a simple example. Consider an inflaton with quadratic potential

$$V = \frac{m^2}{2}\phi^2. \tag{9.26}$$

Assume that $N_\star = 60$, *i. e.* Inflation occurs, effectively, for 60 e-folds.

(h) Compute the value of the field at the end of inflation. From this derive $\phi_\star$, *i. e.* the field value at $N = N_\star$.

(i) Fix the mass m such that $\Delta_\mathrm{s} \sim 10^{-9}$.

(j) Compute r and n_s. Compare with constraints on r from Planck data and constraints on n_s from the latest ACT data. Is the model viable?

[a] You can find the full derivation of these quantities in the review mentioned above.

(**a**) The probability to obtain a value of 10^{-5} for the absolute value of a parameter that is Gauss distributed around 0 with standard deviation 1 reads

$$p = \frac{1}{\sqrt{2\pi}} \int_{-10^{-5}}^{10^{-5}} e^{-\frac{x^2}{2}} = \mathrm{Erf}(10^{-5}) \sim 10^{-5}. \tag{9.27}$$

The total probability for this to happen for 10^4 parameters reads

$$p_{\text{tot}} = p^{10^4} = (10^{-5})^{10^4}, \tag{9.28}$$

which is really a remarkably small number.

(**b**) In the preceding exercise, we assumed what we would expect would be a unbiased distribution over completely uncorrelated patches in the sky. There are a number of problems with such arguments:

- **What is an unbiased distribution?** By using a Gaussian, we assume the patches to be entirely uncorrelated. But even if they had not been in causal contact in the past, they could still be correlated somehow. We've imposed a flat prior over possibilities without strong justification. In other words, a flat probability measure is still a probability measure.

- **Is this a problem of theory or initial conditions?** We can find a theory like inflation, which explains the small temperature fluctuations dynamically. Yet, the laws of physics are formulated in terms of differential equations. Differential equations propagate initial data — they don't generate it. So inflation just shifts the fine-tuning problem to earlier times.

- **Why exactly $\Delta x = 1$?** It's natural to expect dimensionless quantities to be of order 1, but that's a guess, not a rule. If we'd assumed $\Delta x \sim 10^{-11}$, the probability of the observed fluctuations would be close to one. The argument is extremely sensitive to arbitrary assumptions.

- **Why inflation?** The problem is to explain correlations between regions that seem causally disconnected. Inflation is one way to do that, but not the only way. Any mechanism that establishes correlations — even without causal contact — could in principle do the job. Given how little we know about the early universe, this argument allows for a lot of wiggle room.

That this argument (as well as other arguments which originally motivated inflation) is questionable does not mean that inflation in and of itself is equally questionable. Inflationary models predominantly make the prediction of a slightly reddish spectrum (scalar spectral index N_{s} close to but below one), which has been measured to be the case, see exercise (**j**). Yet, other very-early-universe models also make this prediction, so this is rather a statement on how small the amount of information is we have about the very early universe.

(**c**) The stress energy tensor reads

$$T_{\mu\nu} \equiv -2\frac{\delta S}{\delta g^{\mu\nu}} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\rho\phi\nabla^\rho\phi - g_{\mu\nu}V. \tag{9.29}$$

As the scalar is just a function of time, we obtain

$$T_{00} = \frac{\dot{\phi}^2}{2} + V, \tag{9.30}$$

$$T_{ij} = a^2\left(\frac{\dot{\phi}^2}{2} - V\right). \tag{9.31}$$

Thus, considering that $u = \partial_t$ in comoving coordinates, we obtain the density and pressure

$$\rho = \frac{\dot{\phi}^2}{2} + V, \tag{9.32}$$

$$p = \frac{\dot{\phi}^2}{2} - V. \tag{9.33}$$

(**d**) We can write the derivative of the Hubble parameter as

$$\dot{H} = \frac{\ddot{a}}{a} - H^2. \tag{9.34}$$

Regrouping, we obtain

$$\frac{\ddot{a}}{a} = \left(\frac{\dot{H}}{H^2} + 1\right) H^2 = H^2(1 - \epsilon). \tag{9.35}$$

(**e**) We need the Friedmann equations to express ϵ in terms of the equation-of-state parameter. The time derivative of the first Friedmann equation yields

$$\dot{H} = \frac{\dot{\rho}}{6m_{\mathrm{P}}^2 H}. \tag{9.36}$$

Plugging in the continuity equation

$$\dot{H} = -\frac{\rho + p}{2m_{\mathrm{P}}^2}. \tag{9.37}$$

Then, we can plug into the definition of ϵ :

$$\epsilon = \frac{3(\rho + p)}{2\rho} = \frac{3}{2}(w + 1). \tag{9.38}$$

Indeed, when $\epsilon < 1$, $w < -1/3$ so the expansion is accelerated.

(**f**) According to Eq. (9.18) the regime $0 < \epsilon \ll 1$ amounts to the regime $-1 < w \ll 1/3$. Plugging in Eqs. (9.12) and (9.13), we obtain

$$-1 \leq \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V} = \frac{\frac{\frac{1}{2}\dot{\phi}^2}{V} - 1}{\frac{\frac{1}{2}\dot{\phi}^2}{V} + 1} \ll -\frac{1}{3}. \tag{9.39}$$

The value -1 is realized for $\dot{\phi} = 0$. Thus, deviations from $w = -1$ are small if

$$\frac{\frac{1}{2}\dot{\phi}^2}{V} \ll 1. \tag{9.40}$$

To make this more explicit, we can expand

$$w = -1 + \frac{\dot{\phi}^2}{V}. \tag{9.41}$$

(**g**) In the slow-roll regime, we can approximate ϵ as

$$\epsilon = \frac{3(p + \rho)}{2\rho} = \frac{3\dot{\phi}^2}{2(\frac{\dot{\phi}^2}{2} + V)} \simeq \frac{3\dot{\phi}^2}{2V}, \tag{9.42}$$

The equation of motion for the scalar approximately equals

$$3H\dot{\phi} + V' \simeq 0, \tag{9.43}$$

while the first Friedmann equation reads

$$H^2 \simeq \frac{V}{3m_\mathrm{P}^2}. \tag{9.44}$$

Plugging in both equations, we obtain

$$\epsilon \simeq \frac{3V'^2}{2H^2V} \simeq \frac{m_\mathrm{P}^2}{2}\left(\frac{V'}{V}\right)^2 \equiv \epsilon_V. \tag{9.45}$$

Additionally, the second slow-roll parameter becomes

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}}. \tag{9.46}$$

Taking a time-derivative of the scalar-field equation, *i. e.* Eq. (9.43),

$$3\dot{H}\dot{\phi} + 3H\ddot{\phi} + V''\dot{\phi} \simeq 0. \tag{9.47}$$

Thus, we can replace $\ddot{\phi}$ to obtain

$$\eta \simeq \frac{\dot{H} + \frac{V''}{3}}{H^2} \simeq -\epsilon + m_\mathrm{P}^2\frac{V''}{V} \equiv -\epsilon + \eta_V. \tag{9.48}$$

Thus, we obtain

$$\eta_V = \eta + \epsilon. \tag{9.49}$$

(**h**) At the end of inflation, we have

$$\epsilon_V(\phi_\mathrm{end}) = \frac{m_\mathrm{P}^2}{2}\left(\frac{V'(\phi_\mathrm{end})}{V(\phi_\mathrm{end})}\right)^2 = \frac{2m_\mathrm{P}^2}{\phi_\mathrm{end}^2} = 1. \tag{9.50}$$

Thus, the field value at which inflation ends equals $\phi_\mathrm{end} = \sqrt{2}m_\mathrm{P}$. Using the definition of the e-fold

$$N(\phi) = \int_{t_\mathrm{end}}^{t} H\mathrm{d}t. \tag{9.51}$$

Yet, during slow-roll inflation, we can express H as a function of ϕ and change the integration to $\mathrm{d}t = \dot{\phi}^{-1}\mathrm{d}\phi \simeq -3H\mathrm{d}\phi/V'$.

Therefore, the integration of time becomes

$$N(\phi) = \int_{\phi_\mathrm{end}}^{\phi} \frac{3H^2}{V'}\mathrm{d}\phi \simeq \int_{\sqrt{2}m_\mathrm{P}}^{\phi} \frac{V}{m_\mathrm{P}^2V'}\mathrm{d}\phi = \int_{\sqrt{2}m_\mathrm{P}}^{\phi} \frac{\phi}{2m_\mathrm{P}^2}\mathrm{d}\phi = \frac{1}{4}\left(\frac{\phi^2}{m_\mathrm{P}^2} - 2\right). \tag{9.52}$$

Thus, we can express $\phi_\star$ in terms of $N_\star$ as

$$\phi_\star = \sqrt{4N_\star + 2}m_\mathrm{P}. \tag{9.53}$$

At $N_\star = 60$ we obtain $\phi_\star = \sqrt{242}m_\mathrm{P} \simeq 16m_\mathrm{P}$.

(**i**) The first slow-roll parameter for the quadratic potential is given in Eq. (9.50). Thus, the amplitude of the scalar fluctuations reads

$$\Delta_\mathrm{s}^2 \simeq \frac{m^2\phi_\star^4}{96\pi^2m_\mathrm{P}^6} = \frac{242^2}{96\pi^2}\frac{m^2}{m_\mathrm{P}^2}. \tag{9.54}$$

Given that $242^2/96\pi^2 \simeq 62 \sim \mathcal{O}(10)$, and $\Delta_s^2 \sim 10^{-9}$, we obtain the mass

$$m \sim 10^{-5} m_{\rm P} \sim 10^{14}{\rm GeV}. \tag{9.55}$$

Considering that H amounts to the energy scale of inflation, and $H \sim m$, inflation occurs at around this scale.

(**j**) We can immediately compute the tensor-to-scalar ratio

$$r = 32 \frac{m_{\rm P}^2}{\phi_\star^2} = \frac{32}{242} \simeq 0.1. \tag{9.56}$$

For the scalar spectral index we need the second slow-roll parameter

$$\eta_V(\phi_\star) = \frac{2m_{\rm P}^2}{\phi_\star^2} = \frac{1}{121} \simeq 0.008 = \epsilon_V(\phi_\star). \tag{9.57}$$

Thus, the scalar spectral index equals

$$n_{\rm s} \simeq 0.967. \tag{9.58}$$

Let's compare to data. The latest ACT data predicts $n_{\rm s} = 0.9709 \pm 0.0038$. Thus, the quadratic model is well within error tolerance. Error bars amount to 1σ. A little bit more than 1σ does not imply a tension. The value of r from Planck data, namely $r < 0.044$ at 3σ, cannot be accommodated in the quadratic model for inflation, however. Indeed, quadratic inflation is in 4σ-tension with the Planck data (and actually in even higher tension with newer data), and, therefore, ruled out.

Quantum field theory in curved spacetime

Assignment 4 – May 21

Exercise 10: Kerr black hole

Motivation: Real black holes spin. That twist adds a whole new layer of physics. Let's dive in.

A rotating black hole cannot be static and spherically symmetric anymore. Instead it is axially symmetric and stationary, *i. e.* it changes in time but this change is time-translation invariant. This is the resulting metric:

$$\mathrm{d}s^2 = \left(1 - \frac{2GMr}{\rho^2}\right)\mathrm{d}t^2 + \frac{4GMar\sin^2\theta}{\rho^2}\mathrm{d}t\mathrm{d}\phi - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\theta^2 - \left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]\frac{\sin^2\theta}{\rho^2}\mathrm{d}\phi^2, \tag{10.1}$$

where the constants M and $a = J/GM$ (with the angular momentum J) parametrize the geometry, while $\Delta = r^2 - 2GMr + a^2$ and $\rho^2 = r^2 + a^2\cos^2\theta$.
In the following, we learn some of the key properties of this metric. The required computations can become involved due to the complexity of the metric itself, so it may be advised to use symbolic equation manipulation software like Mathematica or SageMath.

(a) Take the limit $a \to 0$. Which geometry do you obtain? Do the same for the limit $M \to 0$. (**Hint:** You may have to apply coordinate transformations after the limit to properly identify the resulting geometry. Another way to characterize a geometry (which may be simpler) is to compute the corresponding curvature tensor. While ordinarily it is necessary to compute curvature invariants, in this case the curvature tensor is sufficient.)

(b) Depending on the constant parameters, the metric has zero, one or two horizons. If they exist, compute their radii (say r_+ and r_- for $r_+ > r_-$), and the parameter ranges corresponding to the three cases. Which of these regimes are expected to be physical? (**Hint:** For the Kerr metric, horizons and their properties can be read off the radial part of the metric.)

(c) As for the Schwarzschild geometry, the singularities at the horizons are mere coordinate singularities. Introducing a Tortoise coordinate

$$\mathrm{d}r^* = \frac{r^2+a^2}{\Delta}\mathrm{d}r, \tag{10.2}$$

we can transform the time and angular coordinates to obtain

$$\mathrm{d}v = \mathrm{d}t + \mathrm{d}r^*, \qquad \mathrm{d}\tilde{\phi} = \mathrm{d}\phi - \frac{a}{\Delta}\mathrm{d}r. \tag{10.3}$$

Here the transformation $\phi \to \tilde{\phi}$ is required so that the basis vectors ∂_r and ∂_ϕ remain linearly independent close to the horizon. Show that the metric in the resulting set of ingoing Eddington-Finkelstein coordinates $(v, r, \theta, \tilde{\phi})$ is regular at the outer horizon.

(**d**) A horizon-generating Killing vector is a Killing vector which is null on the horizon. Given such a horizon-generating Killing vector χ, we can then define the horizon as the submanifold constrained by the condition $\chi^2 = 0$. The surface gravity κ is defined such that

$$\chi^\nu \nabla_\nu \chi_\mu|_{r=r_+} = \kappa \chi_\nu|_{r=r_+}. \tag{10.4}$$

What's the left-hand side of this equation? What does it mean that the right-hand side is non-zero?

Contrary to the Schwarzschild case, the horizon generating Killing vector for the outer horizon is not simply ∂_t. Instead, it reads $\chi = \partial_t + \Omega_H \partial_\phi$ with $\Omega_H = a/(a^2 + r_+^2)$. Demonstrate that the surface gravity at the outer horizon equals

$$\kappa = \frac{r_+ - r_-}{2(r_+^2 + a^2)}. \tag{10.5}$$

This amounts to the acceleration required to hover on top of the outer horizon. What does this mean for the vacuum state defined by an observer inertially falling into the hole? (**Hint:** Turn the vector equation into a scalar equation by projecting on a vector. Note that on the horizon $\partial_\phi|_{r=r+} = \partial_{\tilde{\phi}}|_{r=r+}$.)

(**e**) Compute the surface gravity in the limit $a \to GM$. Borrowing an analogy with the third law of thermodynamics, what do we learn about trying to spin up a black hole until it is extremal? Indeed one can prove the answer to this problem in full generality in general relativity – the result is called the third law of black hole mechanics.

(**a**) In the limit $a \to 0$

$$\Delta = r^2 - 2GMr, \qquad \rho^2 = r^2. \tag{10.6}$$

Thus, the metric becomes

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 - \frac{1}{1-\frac{2GM}{r}}\mathrm{d}r^2 \tag{10.7}$$

$$- r^2 \mathrm{d}\theta^2 - r^2 \sin^2\theta \mathrm{d}\phi^2, \tag{10.8}$$

which is the Schwarzschild metric. This is in line with the fact that a represents the angular momentum – a Schwarzschild black hole is the non-rotating version of a Kerr black hole.

In the limit $M \to 0$

$$\Delta = r^2 + a^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta. \tag{10.9}$$

As a result, the metric reads

$$\begin{aligned}\mathrm{d}s^2 =& \mathrm{d}t^2 - \frac{r^2 + a^2\cos^2\theta}{r^2+a^2}\mathrm{d}r^2 - \left(r^2 + a^2\cos^2\theta\right)\mathrm{d}\theta^2 \\ &- \left[(r^2+a^2)^2 - a^2(r^2+a^2)\sin^2\theta\right]\frac{\sin^2\theta}{r^2+a^2\cos^2\theta}\mathrm{d}\phi^2.\end{aligned} \tag{10.10}$$

This can be simplified because

$$\frac{(r^2+a^2)^2 - a^2(r^2+a^2)\sin^2\theta}{r^2+a^2\cos^2\theta} = (r^2+a^2)\frac{r^2+a^2(1-\sin^2\theta)}{r^2+a^2\cos^2\theta} = r^2+a^2. \tag{10.11}$$

Thus, the metric reads

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}\mathrm{d}r^2 - \left(r^2 + a^2\cos^2\theta\right)\mathrm{d}\theta^2 - \left(r^2 + a^2\right)\sin^2\theta\mathrm{d}\phi^2. \tag{10.12}$$

This is just Minkowski space in disguise – if by disguise you mean ellipsoidal coordinates. Applying the coordinate transformation to Cartesian coordinates

$$x = \sqrt{r^2 + a^2}\sin\theta\cos\phi, \qquad y = \sqrt{r^2 + a^2}\sin\theta\sin\phi \qquad , z = r\cos\theta, \tag{10.13}$$

we obtain

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}\vec{x}^2. \tag{10.14}$$

(**b**) At horizons in stationary geometries, the radial component of the metric diverges. This happens when $\Delta = 0$. This quadratic equation has the solutions

$$r_\pm = GM \pm \sqrt{(GM)^2 - a^2}. \tag{10.15}$$

Thus, there are 0,1 or 2 real roots if $a > GM$, $a = GM$ or $a < GM$, respectively.

As there is still a ring-shaped singularity (better: "ringularity") inside the hole, the case $a > GM$ is a naked singularity, which violates the cosmic censorship conjecture and would allow access to arbitrarily-large-curvature regions – an effect we appear not to see in observations.

If $a = GM$, the Kerr black hole is extremal, *i. e.* the two horizons are equal. Compared to the black holes we see in the sky, this is still excessively fast. Actually, there is a theorem that in GR you can not spin up a black hole up to $a = 1$, just as in thermodynamics you can't lower the temperature to 0 with a finite number of steps (chrchrm Hawking effect chrchrm).

The physical regime is thus $a < GM$. All the observed black holes fall into this category.

(**c**) Applying the coordinate transformation given in Eqs. (10.2) and (10.3), after some algebra we obtain the metric

$$\begin{aligned}\mathrm{d}s^2 = &\left(1 - \frac{2GMr}{\rho^2}\right)\mathrm{d}v^2 + \frac{4GMar}{\rho^2}\mathrm{d}v\mathrm{d}\phi - 2\mathrm{d}v\mathrm{d}r + 2a\sin^2\theta\mathrm{d}r\mathrm{d}\phi \\ &- \rho^2\mathrm{d}\theta^2 - \left[(r^2 + a^2)^2 - a^2\Delta\sin^2\theta\right]\frac{\sin^2\theta}{\rho^2}\mathrm{d}\phi^2\end{aligned} \tag{10.16}$$

As there is no Δ in any denominator, the metric is regular at $\Delta = 0$.

(**d**) The left-hand side of Eq. (10.4) is the left-hand side of the geodesic equation. Thus, the right-hand side being non-zero captures the acceleration required to stay put, *i. e.* hover over the horizon.

Killing vectors satisfy the Killing equation

$$\nabla_{(\mu}\chi_{\nu)} = 0. \tag{10.17}$$

Therefore, we can express Eq. (10.4) as

$$\chi^\nu\nabla^\mu\chi_\nu|_{r=r_+} = \frac{1}{2}\nabla_\mu(\chi^2)|_{r=r+} = -\kappa\chi_\mu|_{r=r_+}, \tag{10.18}$$

where $\chi^2 = \chi^\mu\chi_\mu$. Projecting on some vector ξ which is not normal to χ on the horizon, we obtain

$$\kappa = -\left.\frac{\xi^\mu\nabla_\mu(\chi^2)}{2\xi^\mu\chi_\mu}\right|_{r=r_+}. \tag{10.19}$$

In Eddington-Finkelstein coordinates, we can choose $\xi = \partial_r$ to obtain

$$\kappa = - \left.\frac{\partial_r(\chi^2)}{2\chi_r}\right|_{r=r_+} = \frac{1}{2}\left(\frac{1}{GM} - \frac{GM}{a} + \frac{\sqrt{(GM)^2 - a^2}}{a^2}\right) = \frac{r_+ - r_-}{2(r_+^2 + a^2)}. \tag{10.20}$$

Thus, we obtained the surface gravity of the Kerr black hole. As we learned last week, for static horizons, the surface gravity of a horizon is proportional to its Hawking temperature. Thus, if a QFT is in the inertial vacuum, *i. e.* the vacuum defined by an inertial observer falling into the black hole, an observer hovering above a Kerr black hole is surrounded by a thermal bath of temperature

$$T = \frac{r_+ - r_-}{4\pi(r_+^2 + a^2)}, \tag{10.21}$$

which in the limit $a \to 0$ recovers the temperature of the Schwarzschild black hole as expected.

(**e**) In the limit $a \to GM$, the surface gravity vanishes. Thus, extremal black holes have temperature $T = 0$. The third law of thermodynamics tells us that it takes an infinite number of steps to lower the temperature to 0. Thus, it is impossible to spin up a black hole to $a = GM$ in a finite number of steps. The third law of black-hole mechanics is indeed that one can't lower the surface gravity of a black hole to 0 by a finite number of steps.

Exercise 11: Ergoregion and Penrose process

Motivation: Rotating black holes might just be the most powerful energy source in the universe, real sci-fi stuff. Here's how.

As you derived in the last exercise, contrary to the Schwarzschild black hole, the timelike Killing vector K is not null on the outer horizon. This does not mean that there are no surfaces, where it satisfies $K^2 = 0$.

(**a**) **Do this exercise before reading on.** Verify that the two vectors $K = \partial_t$ and $R = \partial_\phi$ are Killing vectors of the Kerr metric. Killing vectors are defined such that their covariant derivative is antisymmetric. What physical quantities are conserved due to these symmetries, and how are they related to the black hole parameters M and a?

(**b**) **Do this exercise before reading on:** Find the hypersurface ∂E at which the Killing vector K is null. Verify that it is generically outside the horizon, *i. e.* $r_{\partial E}(\theta) \geq r_+$, where $r_{\partial E}$ parametrizes ∂E.

(**c**) The region bounded by ∂E is called ergoregion. Draw a constant-time hypersurface of the outer horizon and the positive-r branch of the ergosurface ($r_{\partial E}$) as seen from the side, say for $\phi = \pi$, for large a (e. g. $a = 0.9$).

(**d**) The ergoregion is a very special place. There, the norm of the Killing vector K is spacelike. On this basis, argue that timelike observers *cannot* stand still in the ergoregion. This effect is called frame dragging. Which direction do they have to move in? (**Hint:** Note that ∂E is not a horizon. One can leave it.)

(**e**) Consider a particle of mass m which has momentum $p = mu$ with timelike four-velocity u. For simplicity, assume that the particle is only moving on hypersurfaces of constant θ and r, *i. e.* $p = p^t\partial_t + p^\phi\partial_\phi$, and into the future, *i. e.* $p^t > 0$. The particle's energy is defined as $E = K^\mu p_\mu$, its angular momentum as $L = -R^\mu p_\mu$. Are the energy and the angular momentum always positive in the ergoregion outside the horizon, *i. e.* for $r_+ < r_0 < r_{\partial E}$?

(f) Consider that you send an object into the ergoregion, it breaks up into two equal-mass pieces, one of which comes back out. Start out with the original object at infinity such that the conserved total energy equals $E_{\text{tot}} = E_{\text{in}} + E_{\text{out}} = \mu$. Assume the pieces, both of mass m break apart inside the ergoregion, one particle falls into the black hole and one particle escapes to infinity. Use energy conservation to argue that the energy of the outgoing particle can satisfy $E_{\text{out}} > \mu$. Where is the energy extracted from?

(g) Let's maximize the efficiency of this process, *i. e.* the quantity

$$\eta = \frac{E_{\text{out}}}{E_{\text{tot}}}, \tag{11.1}$$

where E_{in} denotes the energy of the infalling particle. Evidently, an efficiency larger than one implies that one gets more energy out of the black hole than went in. Assume that:

- The object breaks apart somewhere outside the outer horizon, *i. e.* at some $r_0 > r_+$.
- The object breaks apart at a turning point in radial motion (*i. e.* $\dot{r} = 0$). At this point during the evolution, on any object of mass m, energy E and angular momentum L, the geodesic equation enforces the constraint

$$\frac{r^2 E}{m} + \frac{2GM}{r}\left(\frac{aE}{m} - \frac{L}{m}\right)^2 + \left(a^2\frac{E^2}{m^2} - \frac{L^2}{m^2}\right) - \Delta = 0. \tag{11.2}$$

- The black hole is extremal, *i. e.* $a = GM$.
- Energy and angular momentum are conserved during the process. This follows from K and R being Killing vectors.

Compute E_{in}, E_{out} and $\eta(m, r_0)$ and maximize η for allowed values of r_0 and m. (**Hint:** An object can be more than the sum of its decay products.)

(h) The Penrose process decreases both the angular momentum J and the mass M of the black hole. Over all, however, J decreases more than M such that $a = J/GM$ decreases. This works until the black hole stops rotating. The mass of a black hole that has been spun down to $a = 0$ by the Penrose process is called the irreducible mass M_{irr}.

Fun fact aka assume without proof: You may have heard of the area theorem stating that the area of black holes cannot decrease in GR. A perfectly administered Penrose process is actually optimal: It does not change the area of the outer horizon.

Show that the area of the Kerr horizon equals

$$A = 4\pi(r_+^2 + a^2). \tag{11.3}$$

Use the fun fact to compute the irreducible mass.

(i) The black hole in the centre of the Galaxy has mass $M \simeq 4 \times 10^6 M_\odot$, where $M_\odot$ denotes the mass of the sun. Imagine that it would be extremal (its actual spin parameter is around $a \in [0.1, .5]$ but never mind reality). Compute how much energy could be extracted from the black hole. You should obtain that it's around 29% of the original black-hole mass. For comparison, nuclear fusion converts 0.7% of matter into energy, and the mass of visible matter in the milky way equals around $\sim 5 \times 10^{10} M_\odot$. Connect the dots. True sci-fi stuff!

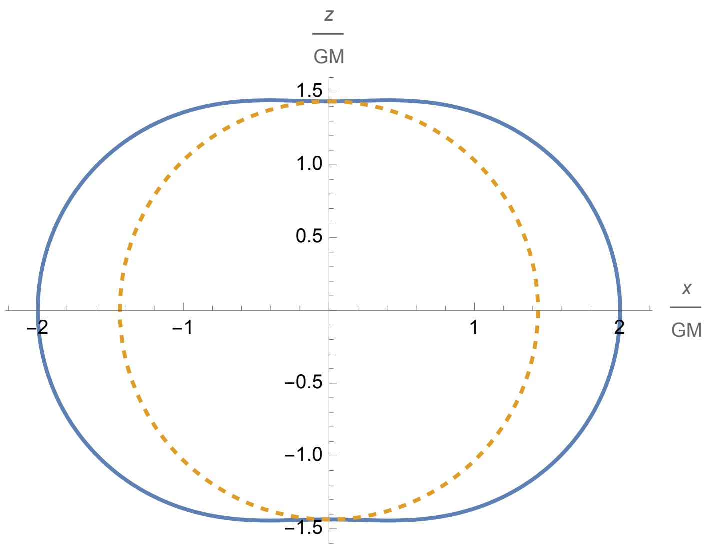

Figure 3: Side view on the ergoregion of a spacelike hypersurface ($t = \text{const}$) of a Kerr black hole with $a = 0.9GM$ given in

(**a**) Killing vector fields satisfy the Killing equation Eq. (13.6) We do this in all generality. Assume that the metric is independent of a coordinate ξ, *i. e.* $\partial_\xi g_{\mu\nu} = 0$. Then, the vector $X = \partial_\xi$ satisfies

$$\nabla_\mu X_\nu = g_{\nu\sigma}\Gamma^\sigma_{\mu\rho}X^\rho = g_{\nu\sigma}\Gamma^\sigma_{\mu\xi} = \frac{1}{2}\left(\partial_\xi g_{\mu\nu} + \partial_\mu g_{\xi\nu} - \partial_\nu g_{\mu\xi}\right). \tag{11.4}$$

However, the metric is independent of ξ, so

$$\nabla_\mu X_\nu = \partial_{[\mu} g_{\nu]\xi} = \nabla_{[\mu} X_{\nu]}. \tag{11.5}$$

Thus, the vector X is a Killing vector. The metric is independent of both t and ϕ. Therefore, K and R are Killing vectors.

(**b**) In Boyer-Lindquist coordinates the norm of the Killing vector K is just $K^\mu K_\mu = g_{tt}$. Thus, K is null when

$$r^2 - 2GMr + a^2\cos^2\theta = 0. \tag{11.6}$$

The most positive root of this equation yields the parametric solution

$$r_{\partial E} = GM + \sqrt{(GM)^2 - a^2\cos^2\theta}, \tag{11.7}$$

which defines the boundary of the ergoregion. Clearly, $r_{\partial E}|_{\theta=0,\pi} = r_+$, but generally $r_{\partial E} \geq r_+$. Thus, the hypersurface where K becomes null lies outside the horizon.

(**c**) I plot the ergoregion for $a = 0.9GM$ in fig. 3.

(**d**) Inside the ergoregion, the Killing vector K becomes spacelike. Since observers must follow timelike worldlines, their 4-velocity cannot be proportional to K. In Boyer–Lindquist coordinates, viz. Eq. (10.1), there is exactly one contribution to the metric which is timelike inside the ergoregion, namely the cross term proportional to $\mathrm{d}t\mathrm{d}\phi$. Thus, the only way to construct a future-directed timelike 4-velocity is to include a nonzero $\mathrm{d}\phi$ component. This means the observer must move azimuthally—i.e., co-rotate with the black hole. This effect is known as frame dragging.

(**e**) Let's compute energy and angular momentum in Boyer-Lindquist coordinates

$$E = g_{tt}p^t + g_{t\phi}p^\phi, \tag{11.8}$$

$$L = -g_{\phi\phi}p^\phi - g_{\phi t}p^t. \tag{11.9}$$

In the physical regime $a < GM$, both K and R are spacelike inside the ergoregion, *i. e.* $g_{tt} < 0$, and $g_{\phi\phi} < 0$. At the same time, the momentum of a timelike observer is timelike, *i. e.* (considering for simplicity an observer with $p^\theta = p^r = 0$)

$$p_\mu p^\mu = g_{tt}(p^t)^2 + 2g_{t\phi}p^t p^\phi + g_{\phi\phi}(p^\phi)^2 = m^2. \tag{11.10}$$

As $g_{tt}, g_{\phi\phi} < 0$ while $g_{t\phi} > 0$, we naturally obtain $p^\phi > 0$, which is just the answer to exercise (**f**).

Given the mass-shell constraint, we can express the energy and the angular momentum as

$$E = \frac{m^2 - g_{\phi\phi}(p^\phi)^2 - g_{t\phi}p^t p^\phi}{p^t} = \frac{m^2 - p^\phi(|g_{t\phi}|p^t - |g_{\phi\phi}|p^\phi)}{p^t}, \tag{11.11}$$

$$L = -\frac{m^2 - g_{tt}(p^t)^2 - g_{t\phi}p^t p^\phi}{p^\phi} = -\frac{m^2 - p^t(|g_{t\phi}|p^t - |g_{tt}|p^\phi)}{p^t}. \tag{11.12}$$

Thus, if p^ϕ and m^2 are chosen sufficiently small while at the same time satisfying Eq. (11.10), the energy is negative, and so can be the angular momentum. A concrete example is $a = GM/2$, $r = 1.9GM$, $\theta = \pi/2$, $p^t = 34.3m$ and $p^\phi = .24/G$ yielding $E = -0.05M$ and $L = -0.82GM^2$.

(**f**) If the piece falling into the black hole is allowed to acquire negative energies, *i. e.* $E_{\text{in}} < 0$, the outgoing piece can have more energy than the whole object had before falling in because

$$E_{\text{out}} = E_{\text{tot}} - E_{\text{in}} = \mu + |E_{\text{in}}| > \mu. \tag{11.13}$$

Thus, energy has been extracted from somewhere. The only place this energy can come from is the black hole itself, which will lose rotational energy and mass by this process to match the negative energy and the negative angular momentum of the infalling particle. Note that due to the area theorem, namely that black hole horizon areas don't decrease, this loss has to be such that the change in $a = J/M$ is negative.

(**g**) First, we solve the constraint from the geodesic equation at r_0 for the total angular momentum obtaining

$$L_{\text{tot}} = \frac{2aGM \pm \sqrt{2GMr_0\Delta}}{2GM - r_0}\mu. \tag{11.14}$$

As this is also the place where the object breaks into two pieces, the geodesic equation equivalently applies to those two. Thus, for both particles we obtain the angular momenta

$$L_{\text{in/out}} = \frac{2aGME_{\text{in/out}} \pm \sqrt{r_0\Delta[E^2_{\text{in/out}}r_0^2 + m^2(2GM - r_0)]}}{2GM - r_0}. \tag{11.15}$$

At the same time, we know that $E_{\text{tot}} = \mu$. Thus, we can use energy conservation and angular-momentum conservation to determine the individual energies $E_{\text{in/out}}$. Thus, imposing

$$E_{\text{tot}} = E_{\text{in}} + E_{\text{out}}, \qquad L_{\text{tot}} = L_{\text{in}} + L_{\text{out}}, \tag{11.16}$$

we obtain (choosing the roots $E_{\text{in}} < E_{\text{out}}$)

$$E_{\text{in/out}} = \frac{\mu}{2}\left[1 \mp \sqrt{\left(1 - \frac{4m^2}{\mu^2}\right)\frac{2GM}{r_0}}\right]. \tag{11.17}$$

Thus, we obtain the efficiency

$$\eta = \frac{1}{2}\left[1 + \sqrt{\left(1 - \frac{4m^2}{\mu^2}\right)\frac{2GM}{r_0}}\right]. \tag{11.18}$$

The efficiency is maximal for minimal r_0, *i. e.* when hovering just above the horizon, and minimal m, *i. e.* the object decays into massless particles like photons. For $a = GM$, $m = 0$, and $r_0 = r_+ = GM$, we obtain

$$\eta = \frac{1}{2}\left[1+\sqrt{2}\right] \simeq 120\%. \tag{11.19}$$

Thus, we get out 120% of the energy we sent into the black hole.

(**h**) First we need to compute the area of the horizon. The horizon is a surface of constant time ($\mathrm{d}t = 0$) and radius $r = r_+$ (thus, also $\mathrm{d}r = 0$). Thus, the induced metric on the horizon reads

$$\mathrm{d}s^2_{(2)} = -\rho^2(r_+)\mathrm{d}\theta^2 - \left[(r_+^2 + a^2)^2 - a^2\Delta\sin^2\theta\right]\frac{\sin^2\theta}{\rho^2(r_+)}\mathrm{d}\phi^2. \tag{11.20}$$

The area is then defined as

$$A \equiv \int_{\mathrm{H}} \sqrt{h}\mathrm{d}^2x, \tag{11.21}$$

where h is the determinant of the induced metric on the horizon surface. Thus, we have to compute

$$A = (a^2 + r_+^2)\int_0^\pi\int_0^{2\pi}\sin\theta\mathrm{d}\phi\mathrm{d}\theta = 4\pi(a^2 + r_+^2). \tag{11.22}$$

The irreducible mass amounts to the black hole not rotating any more such that

$$A = 16\pi(GM_{\mathrm{irr}})^2. \tag{11.23}$$

As the black-hole area does not change, this equals the area before all of its rotational energy has been extracted. Thus, we obtain

$$M_{\mathrm{irr}} = \frac{1}{2G}\sqrt{a^2 + r_+^2} = \sqrt{\frac{M^2 + M\sqrt{M - a/G}}{2}}. \tag{11.24}$$

In particular, for extremal black holes the irreducible mass becomes minimal, yielding

$$M_{\mathrm{irr}} = \frac{M}{\sqrt{2}}. \tag{11.25}$$

(**i**) The amount of energy that could be extracted is

$$M - M_{\mathrm{irr}} = \left(1 - \frac{1}{\sqrt{2}}\right)M \simeq 0.29M. \tag{11.26}$$

This amounts to 29% of the original mass, which is an extremely large amount.

Throughout their lifetime, all stars in the milky way generate 0.7% of their mass in radiation from nuclear fusion. The mass of all stars in the galaxy being around $5 \times 10^{10} M_\odot$, all in all this amounts to a radiated energy of $E_{\mathrm{stars}} \sim 4 \times 10^8 M_\odot$. If the black hole at the centre of the Milky Way was extremal, we could extract $E_{\mathrm{Penrose}} \sim 10^6 M_\odot$, which is just about 400 times smaller. Besides, that black hole is rather small as supermassive black holes go. This is probably the strongest energy source in the universe we know of.

Quantum field theory in curved spacetime

Assignment 5/Exam 1 – May 28

Please hand in this assignment before the tutorial at 11h15AM on May 28. In total, you need to obtain 40% of the combined points from this and the second exam.

Exercise 12: Particle creation in an expanding universe – 30pts.

Motivation: Back to square one. Let's compute the number of particles created in a more realistic scenario than before.

Consider a spatially flat universe which starts out changing adiabatically, then undergoes a rapid phase of expansion, to finally end up in another adiabatic phase. Such a universe is given by the scale factor

$$a^2(\eta) = a_1^2 + (a_2^2 - a_1^2)\frac{1 + \tanh\frac{\eta}{\eta_0}}{2}, \tag{12.1}$$

with the dimensionless parameters a_1 and a_2, and the quantity η_0 which has units of time. Propagating in this universe, consider a conformally coupled, massive scalar field according to the action

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2}\left(m^2 - \frac{R}{6}\right)\phi^2\right). \tag{12.2}$$

(a) Plot the scale factor. What do the parameters a_1, a_2 and η_0 stand for? Ignoring the flat asymptotic regions (large $|\eta|$), what part of the universe's history could the intermediate evolution be a toy model for?

(b) Show that the scale factor Eq. (12.1) results in the squared effective mass

$$m_{\text{eff}}^2 = m^2\left(a_1^2 + (a_2^2 - a_1^2)\frac{1 + \tanh\frac{\eta}{\eta_0}}{2}\right) \tag{12.3}$$

for the scalar.

(c) Having applied the usual rescaling $\phi \to \chi = a\phi$, the field satisfies the usual mode equation

$$\chi_k'' + \left(k^2 + m_{\text{eff}}^2\right)\chi_k = 0. \tag{12.4}$$

Show that the mode equation can be solved by the following two linearly independent mode functions

$$v_k = \frac{e^{i\mathbf{kx} - i\omega_+\eta - i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\text{in}}}}\,{}_2F_1\left(1 + i\omega_-\eta_0, i\omega_-\eta_0; 1 - i\omega_{\text{in}}\eta_0; \frac{1 + \tanh\frac{\eta}{\eta_0}}{2}\right), \tag{12.5}$$

$$u_k = \frac{e^{i\mathbf{kx} - i\omega_+\eta - i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\text{out}}}}\,{}_2F_1\left(1 + i\omega_-\eta_0, i\omega_-\eta_0; 1 + i\omega_{\text{out}}\eta_0; \frac{1 - \tanh\frac{\eta}{\eta_0}}{2}\right), \tag{12.6}$$

where we defined

$$\omega_{\text{in}}^2 \equiv k^2 + a_1^2 m^2, \qquad \omega_{\text{out}}^2 \equiv k^2 + a_2^2 m^2, \qquad \omega_\pm \equiv \frac{\omega_{\text{out}} \pm \omega_{\text{in}}}{2}, \tag{12.7}$$

and ${}_2F_1$ denotes a common type of hypergeometric function.

Hint: Try to recover the differential equation defining the hypergeometric function $f = {}_2F_1(a,b;c;z)$, namely

$$z(1-z)f'' + [c-(1+a+b)z]f' - abf = 0. \tag{12.8}$$

(d) Show that the mode functions asymptote to Minkowski-like positive-frequency solutions at early and late times

$$v_k \sim \frac{e^{i(\mathbf{kx}-\omega_{\text{in}}\eta)}}{\sqrt{4\pi\omega_{\text{in}}}}, \qquad \eta \to -\infty, \tag{12.9}$$

$$u_k \sim \frac{e^{i(\mathbf{kx}-\omega_{\text{out}}\eta)}}{\sqrt{4\pi\omega_{\text{out}}}}, \qquad \eta \to \infty. \tag{12.10}$$

Discuss why the mode equation has solutions with these asymptotics. Conclude that v_k defines a natural in-vacuum, and u_k a natural out-vacuum. Why?

Hint: The hypergeometric function has the limit $\lim_{z\to 0} f(a,b;c;z) = 1$ for all a, b, c.

Thus, the field can be expanded in modes as

$$\chi = \frac{1}{\sqrt{2}}\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\left(a_{\mathbf{k}}v_k + a^\dagger_{\mathbf{k}}v^*_k\right), \tag{12.11}$$

where $a_{\mathbf{k}}$ defines the in-vacuum via $a_{\mathbf{k}}|0_{\text{in}}\rangle = 0$. We can write down a similar mode expansion, namely

$$\chi = \frac{1}{\sqrt{2}}\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\left(b_{\mathbf{k}}u_k + b^\dagger_{\mathbf{k}}u^*_k\right), \tag{12.12}$$

where $b_{\mathbf{k}}$ defines the out-vacuum via $b_{\mathbf{k}}|0_{\text{out}}\rangle = 0$, and $b^\dagger_{\mathbf{k}}$ constructs particle states at late times. Clearly $v_k \neq u_k$. Thus, the two have to be related as

$$v_k = \alpha_k u_k + \beta_k u^*_{-k}, \tag{12.13}$$

with the Bogolyubov coefficients α_k, and β_k.

(e) Demonstrate that the Bogolyubov coefficients equal

$$\alpha_k = \sqrt{\frac{\omega_{\text{out}}}{\omega_{\text{in}}}}\frac{\Gamma(1-i\omega_{\text{in}}\eta_0)\Gamma(-i\omega_{\text{out}}\eta_0)}{\Gamma(-i\omega_+\eta_0)\Gamma(1-i\omega_+\eta_0)}, \tag{12.14}$$

$$\beta_k = \sqrt{\frac{\omega_{\text{out}}}{\omega_{\text{in}}}}\frac{\Gamma(1-i\omega_{\text{in}}\eta_0)\Gamma(i\omega_{\text{out}}\eta_0)}{\Gamma(i\omega_-\eta_0)\Gamma(1+i\omega_-\eta_0)}. \tag{12.15}$$

Hint: The hypergeometric function ${}_2F_1$ satisfies the identities

$$\begin{aligned}{}_2F_1(a,b;c;z) =& \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a,b;a+b-c+1;1-z)\\ &+(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}{}_2F_1(c-a,c-b;c-a-b+1;1-z),\end{aligned} \tag{12.16}$$

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b;c;z). \tag{12.17}$$

(f) Assume that the field is in the vacuum state at early times. Show that the particle number density at late times equals

$$n_k = \frac{\sinh^2(\pi\omega_-\eta_0)}{\sinh(\pi\omega_{\text{in}}\eta_0)\sinh(\pi\omega_{\text{out}}\eta_0)}. \tag{12.18}$$

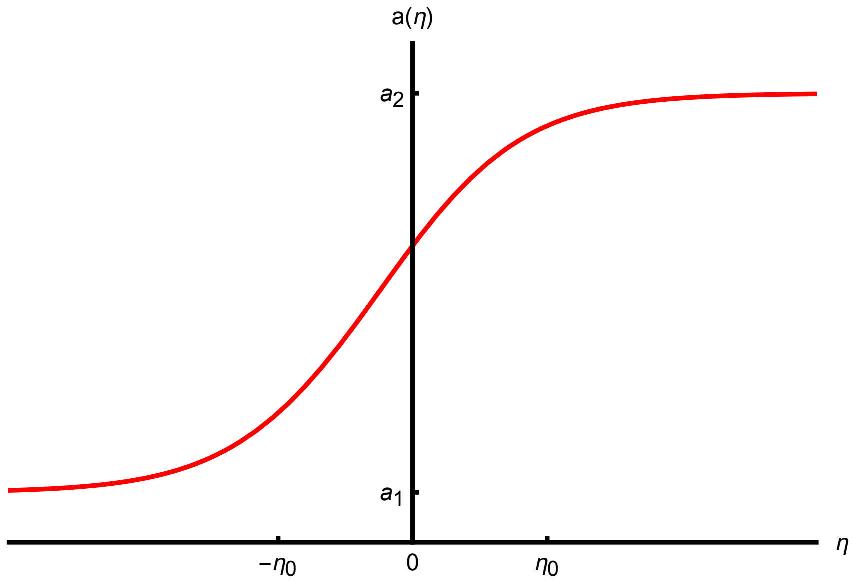

Figure 4: Scale factor given in Eq. (12.1).

(**a**) I plot the scale factor in fig. 4. It starts out flat at $a = a_1$ and ends up flat at $a = a_2$. The intermediate evolution smoothly interpolates between the two points. The parameter η_0 is the characteristic time scale over which the transition from a_1 to a_2 occurs.

To interpret the intermediate evolution, we have to keep in mind that the scale factor is given in terms of conformal time, not cosmological time. To see what is physically happening, we compute the acceleration of the scale factor in physical time t (derivatives with respect to t are denoted by overdots)

$$\ddot{a} = \frac{a''}{a^2} - \frac{a'^2}{a^3}. \tag{12.19}$$

I plot the acceleration of the scale factor during the intermediate evolution in fig. 5. In a nutshell, the intermediate evolution features accelerated expansion followed by deceleration, resembling cosmic inflation and subsequent radiation-/matter-dominated phases.

(**b**) As we computed in exercise 6, the effective mass of a nonminimally coupled scalar in an FLRW background reads

$$m_{\text{eff}}^2 = a^2 \left[m^2 + 6 \left(\xi - \frac{1}{6} \right) R \right], \tag{12.20}$$

where ξ is the nonminimal coupling. For the present case, *i. e.* $\xi = 1/6$, we obtain simply

$$m_{\text{eff}}^2 = a^2 m^2 = m^2 \left[a_1^2 + (a_2^2 - a_1^2) \frac{1 + \tanh \frac{\eta}{\eta_0}}{2} \right]. \tag{12.21}$$

(**c**) We make the ansatz

$$v_k = \frac{e^{i\mathbf{kx} - i\omega_+ \eta - i\omega_- \eta_0 \log(2 \cosh \frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\text{in}}}} f(\eta). \tag{12.22}$$

As a result, f has to satisfy the differential equation

$$f'' - 2i \left(\omega_+ + \omega_- \tanh \frac{\eta}{\eta_0} \right) f' + \frac{i + \eta_0 \omega_-}{\eta_0} \frac{\omega_-}{\sinh^2 \frac{\eta}{\eta_0}} f = 0. \tag{12.23}$$

If we now introduce the new variable

$$z = \frac{1 + \tanh \frac{\eta}{\eta_0}}{2}, \tag{12.24}$$

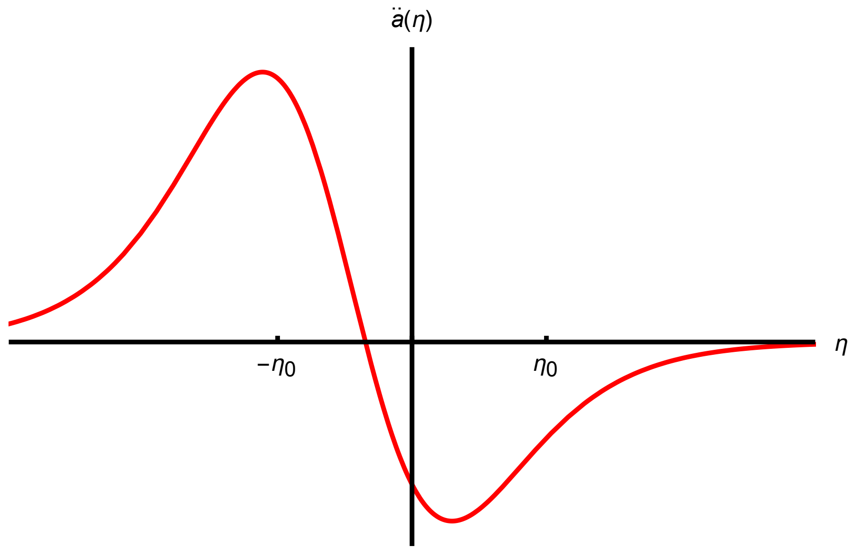

Figure 5: Scale factor acceleration with respect to cosmic time as a function of conformal time as given in Eq. (12.19).

$f(z)$ satisfies Eq. (12.8) with

$$a = 1 + i\omega_-\eta_0, \qquad b = i\omega_-\eta_0, \qquad c = 1 - i\omega_{\text{in}}\eta_0. \tag{12.25}$$

Similarly, for u_k, we make the ansatz

$$u_k = \frac{e^{i\mathbf{kx} - i\omega_+\eta - i\omega_-\eta_0 \log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\text{out}}}} g(\eta), \tag{12.26}$$

yielding the differential equation

$$g'' - 2i\left(\omega_+ + \omega_- \tanh\frac{\eta}{\eta_0}\right) g' + \frac{i + \eta_0\omega_-}{\eta_0} \frac{\omega_-}{\sinh^2\frac{\eta}{\eta_0}} g = 0. \tag{12.27}$$

If we now introduce the new variable

$$w = \frac{1 - \tanh\frac{\eta}{\eta_0}}{2}, \tag{12.28}$$

$g(w)$ satisfies Eq. (12.8) with

$$a = 1 + i\omega_-\eta_0, \qquad b = i\omega_-\eta_0, \qquad c = 1 + i\omega_{\text{out}}\eta_0. \tag{12.29}$$

(**d**) In the limit $\eta \to -\infty$, the variable z goes to

$$\lim_{\eta\to-\infty} z = \lim_{\eta\to-\infty} \frac{1 + \tanh\frac{\eta}{\eta_0}}{2} = 0. \tag{12.30}$$

Similarly, in the limit $\eta \to \infty$, the variable w goes to

$$\lim_{\eta\to\infty} w = \lim_{\eta\to\infty} \frac{1 - \tanh\frac{\eta}{\eta_0}}{2} = 0. \tag{12.31}$$

Then, the hypergeometric function in the mode functions v and u equals

$${}_2F_1(a, b; c; 0) = 1. \tag{12.32}$$

Thus, for large negative η the mode function v_k goes like

$$v_k \sim \frac{e^{i\mathbf{kx}-i\omega_+\eta-i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\text{in}}}}, \tag{12.33}$$

$$= \frac{e^{i\mathbf{kx}-i\omega_+\eta-i\omega_-\eta_0\log\left(e^{\frac{\eta}{\eta_0}}+e^{-\frac{\eta}{\eta_0}}\right)}}{\sqrt{4\pi\omega_{\text{in}}}}, \tag{12.34}$$

$$\sim \frac{e^{i\mathbf{kx}-i(\omega_+-\omega_-)\eta}}{\sqrt{4\pi\omega_{\text{in}}}}, \tag{12.35}$$

$$= \frac{e^{i\mathbf{kx}-i\omega_{\text{in}}\eta}}{\sqrt{4\pi\omega_{\text{in}}}}. \tag{12.36}$$

Similarly, for large positive η the mode function u_k goes like

$$u_k \sim \frac{e^{i\mathbf{kx}-i\omega_+\eta-i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\text{out}}}}, \tag{12.37}$$

$$= \frac{e^{i\mathbf{kx}-i\omega_+\eta-i\omega_-\eta_0\log\left(e^{\frac{\eta}{\eta_0}}+e^{-\frac{\eta}{\eta_0}}\right)}}{\sqrt{4\pi\omega_{\text{out}}}}, \tag{12.38}$$

$$\sim \frac{e^{i\mathbf{kx}-i(\omega_++\omega_-)\eta}}{\sqrt{4\pi\omega_{\text{out}}}}, \tag{12.39}$$

$$= \frac{e^{i\mathbf{kx}-i\omega_{\text{out}}\eta}}{\sqrt{4\pi\omega_{\text{out}}}}. \tag{12.40}$$

The mode equation has solutions which asymptote to plane waves in Minkowski spacetime with constant scale factors a_1 as $\eta \to -\infty$ and a_2 as $\eta \to \infty$. It allows for these solutions exactly because for large negative and large positive η the spacetime itself asymptotes to Minkowski spacetime up to an overall constant rescaling by a_1 and a_2, respectively. Therefore, v is suited to describe the asymptotic Minkowski vacuum at early times, while u is suited to construct the asymptotic Minkowski vacuum at late times.

(**e**) Expressing v_k in terms of u_k by Eq. (12.13), we have

$$\begin{aligned} v_k =& e^{i\mathbf{kx}}\left[\alpha_k \frac{e^{-i\omega_+\eta-i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\text{out}}}}\,{}_2F_1\left(1+i\omega_-\eta_0, i\omega_-\eta_0; 1+i\omega_{\text{out}}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)\right. \\ &\left.+\beta_k \frac{e^{i\omega_+\eta+i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\text{out}}}}\,{}_2F_1^*\left(1+i\omega_-\eta_0, i\omega_-\eta_0; 1+i\omega_{\text{out}}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)\right]. \end{aligned} \tag{12.41}$$

Conjugating the differential equation defining the hypergeometric, *i. e.* Eq. (12.8) function, we obtain

$$z^*(1-z^*)f^{*\prime\prime} + [c^* - (1+a^*+b^*)z^*]f^{*\prime} - a^*b^*f^* = 0. \tag{12.42}$$

Clearly, this differential equation defines the same hypergeometric function with conjugated arguments, *i. e.*

$${}_2F_1^*\left(a,b;c;z\right) = {}_2F_1\left(a^*,b^*;c^*;z^*\right). \tag{12.43}$$

Thus, we obtain the equality

$$\begin{aligned} v_k =& e^{i\mathbf{kx}}\left[\alpha_k \frac{e^{-i\omega_+\eta-i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\text{in}}}}\,{}_2F_1\left(1+i\omega_-\eta_0, i\omega_-\eta_0; 1+i\omega_{\text{out}}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)\right. \\ &\left.+\beta_k \frac{e^{i\omega_+\eta+i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\text{in}}}}\,{}_2F_1\left(1-i\omega_-\eta_0, -i\omega_-\eta_0; 1-i\omega_{\text{out}}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)\right]. \end{aligned} \tag{12.44}$$

Let's now see whether we can get the left-hand side into the same shape. We use the first hint (Eq. (12.16)) to re-express the hypergeometric function in v_k as

$$v_k = \frac{e^{i\mathbf{k}\mathbf{x}-i\omega_+\eta-i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\text{in}}}}\left[\frac{\Gamma(1-i\omega_{\text{in}}\eta_0)\Gamma(-i\omega_{\text{out}}\eta_0)}{\Gamma(-i\omega_+\eta_0)\Gamma(1-i\omega_+\eta_0)}{}_2F_1\left(1+i\omega_-\eta_0, i\omega_-\eta_0; 1+i\omega_{\text{out}}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)\right.$$
$$\left. + \left(\frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)^{-i\omega_{\text{out}}\eta_0}\frac{\Gamma(1-i\omega_{\text{in}}\eta_0)\Gamma(i\omega_{\text{out}}\eta_0)}{\Gamma(1+i\omega_-\eta_0)\Gamma(i\omega_-\eta_0)}{}_2F_1\left(-i\omega_+\eta_0, 1-i\omega_+\eta_0; 1-i\omega_{\text{out}}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)\right]. \tag{12.45}$$

The first term is already in the right shape. Regarding the second term, note that we can rewrite

$$\left(\frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)^{-i\omega_{\text{out}}\eta_0} = e^{-i\omega_{\text{out}}\eta_0\log\left(\frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)}, \tag{12.46}$$

$$= e^{-i\omega_{\text{out}}\eta_0\log\left(\frac{e^{-\frac{\eta}{\eta_0}}}{2\cosh\frac{\eta}{\eta_0}}\right)}, \tag{12.47}$$

$$= e^{i\omega_{\text{out}}\left[\eta+\eta_0\log\left(2\cosh\frac{\eta}{\eta_0}\right)\right]}. \tag{12.48}$$

To flip the phase factor in the second term of Eq. (12.45), we reformulate

$$\left(\frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)^{-i\omega_{\text{out}}\eta_0} = e^{2i\omega_+\eta+2i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}e^{-i(2\omega_+-\omega_{\text{out}})\eta-i(2\omega_--\omega_{\text{out}})\eta_0\log(2\cosh\frac{\eta}{\eta_0})}, \tag{12.49}$$

$$= e^{2i\omega_+\eta+2i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}e^{i\omega_{\text{in}}[-\eta+\eta_0\log(2\cosh\frac{\eta}{\eta_0})]}, \tag{12.50}$$

$$= e^{2i\omega_+\eta+2i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}e^{-i\omega_{\text{in}}\eta_0\log\left[\frac{e^{\frac{\eta}{\eta_0}}}{2\cosh\frac{\eta}{\eta_0}}\right]}, \tag{12.51}$$

$$= e^{2i\omega_+\eta+2i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}\left(\frac{1+\tanh\frac{\eta}{\eta_0}}{2}\right)^{-i\omega_{\text{in}}\eta_0}. \tag{12.52}$$

Thus, we can rewrite the second term in Eq. (12.45) as

$$v_k - \alpha_k u_k = \frac{e^{i\mathbf{k}\mathbf{x}+i\omega_+\eta+i\omega_-\eta_0\log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\text{in}}}}\frac{\Gamma(1-i\omega_{\text{in}}\eta_0)\Gamma(i\omega_{\text{out}}\eta_0)}{\Gamma(1+i\omega_-\eta_0)\Gamma(i\omega_-\eta_0)}$$
$$\times\left(\frac{1+\tanh\frac{\eta}{\eta_0}}{2}\right)^{-i\omega_{\text{in}}\eta_0}{}_2F_1\left(-i\omega_+\eta_0, 1-i\omega_+\eta_0; 1-i\omega_{\text{out}}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right). \tag{12.53}$$

Now we use the second hint (Eq. (12.17)) to express

$$\left(\frac{1+\tanh\frac{\eta}{\eta_0}}{2}\right)^{-i\omega_{\text{in}}\eta_0}{}_2F_1\left(-i\omega_+\eta_0, 1-i\omega_+\eta_0; 1-i\omega_{\text{out}}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right) \tag{12.54}$$

$$= {}_2F_1\left(1-i\omega_-\eta_0, 1-i\omega_-\eta_0; 1-i\omega_{\text{out}}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right). \tag{12.55}$$

Thus, after all the mode function can be expressed as

$$v_k = e^{i\mathbf{k}\mathbf{x}}\left[\frac{e^{-i\omega_+\eta - i\omega_-\eta_0 \log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\rm in}}}\frac{\Gamma(1-i\omega_{\rm in}\eta_0)\Gamma(-i\omega_{\rm out}\eta_0)}{\Gamma(-i\omega_+\eta_0)\Gamma(1-i\omega_+\eta_0)}{}_2F_1\left(1+i\omega_-\eta_0, i\omega_-\eta_0; 1+i\omega_{\rm out}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)\right.$$
$$\left.+\frac{e^{i\omega_+\eta + i\omega_-\eta_0 \log(2\cosh\frac{\eta}{\eta_0})}}{\sqrt{4\pi\omega_{\rm in}}}\frac{\Gamma(1-i\omega_{\rm in}\eta_0)\Gamma(i\omega_{\rm out}\eta_0)}{\Gamma(1+i\omega_-\eta_0)\Gamma(i\omega_-\eta_0)}{}_2F_1\left(1-i\omega_-\eta_0, -i\omega_-\eta_0; 1-i\omega_{\rm out}\eta_0; \frac{1-\tanh\frac{\eta}{\eta_0}}{2}\right)\right]. \tag{12.56}$$

This is exactly the shape of Eq. (12.44). Therefore, we can read off the Bogolyubov coefficients

$$\alpha_k = \sqrt{\frac{\omega_{\rm out}}{\omega_{\rm in}}}\frac{\Gamma(1-i\omega_{\rm in}\eta_0)\Gamma(-i\omega_{\rm out}\eta_0)}{\Gamma(-i\omega_+\eta_0)\Gamma(1-i\omega_+\eta_0)}, \tag{12.57}$$

$$\beta_k = \sqrt{\frac{\omega_{\rm out}}{\omega_{\rm in}}}\frac{\Gamma(1-i\omega_{\rm in}\eta_0)\Gamma(i\omega_{\rm out}\eta_0)}{\Gamma(i\omega_-\eta_0)\Gamma(1+i\omega_-\eta_0)}. \tag{12.58}$$

(**f**) As usual, we define the number density in the in-vacuum as

$$n_{\mathbf{k}} \equiv V^{-1}\langle 0_{\rm in}|b^\dagger_{\mathbf{k}} b_{\mathbf{k}}|0_{\rm in}\rangle \tag{12.59}$$

$$= |\beta_k|^2, \tag{12.60}$$

where we used a simplified version of Eq. (8.16). Plugging in Eq. (12.58), we obtain the number density

$$n_{\mathbf{k}} = \frac{\omega_{\rm out}}{\omega_{\rm in}}\left|\frac{\Gamma(1-i\omega_{\rm in}\eta_0)\Gamma(i\omega_{\rm out}\eta_0)}{\Gamma(i\omega_-\eta_0)\Gamma(1+i\omega_-\eta_0)}\right|^2, \tag{12.61}$$

$$= \frac{\sinh^2(\pi\omega_-\eta_0)}{\sinh(\pi\omega_{\rm in}\eta_0)\sinh(\pi\omega_{\rm out}\eta_0)}, \tag{12.62}$$

where we introduced the spatial volume V, and used the fact that

$$|\Gamma(bi)|^2 = \frac{\pi}{b\sinh(\pi b)}, \qquad |\Gamma(1+bi)|^2 = \frac{\pi b}{\sinh(\pi b)}, \tag{12.63}$$

for real b.

Exercise 13: Hawking radiation from non-singular black holes – 20pts.

Motivation: Black-hole solutions in GR are singular; it is expected that a more complete (quantum) theory of gravity can resolve this. We will consider whether regularity leaves any imprints in the Hawking temperature.

In this exercise, work with the Hayward metric. This is a metric that is not a solution to the Einstein equations (at least not for an energy-momentum tensor that satisfies the standard energy conditions). You can think of it as a phenomenological model for black holes beyond GR. The line-element in Schwarzschild-type coordinates is given by

$$ds^2 = f(r)\,dt^2 - f(r)^{-1}dr^2 - r^2 d\Omega_2^2, \tag{13.1}$$

with $f(r) = 1 - \frac{2G\,M\,r^2}{r^3+2G\,M\,\ell^2}$, with ℓ a parameter with units of length. All (non-derivative) curvature invariants remain finite in the limit $r \to 0$, as long as $\ell > 0$.

(**a**) What is the limit $\ell \to 0$?

(**b**) For spherically symmetric, static metrics in Schwarzschild-like coordinates, the event horizon is determined by the equation $g^{rr} = 0$. Find the location of the event horizon. If there is a qualitative (not just quantitative) difference to the Schwarzschild case, discuss it.

(**c**) Is the vector $\xi = \partial_t$ a Killing vector? If yes, is there a Killing horizon?

(**d**) Determine the surface gravity for a metric of the form Eq. (13.1) for an unspecified function $f(r)$; then insert $f(r)$ for the Hayward metric and discuss the difference to a Schwarzschild black hole.

(**e**) What are the implications of your previous results for the Hawking temperature of the Hayward black hole?

(**a**) In the limit $\ell \to 0$, we obtain $f = 1 - 2GM/r$, yielding the Schwarzschild metric.
(**b**) The condition $g^{rr} = f(r_\mathrm{H}) = 0$ amounts to the cubic equation

$$r_\mathrm{H}^3 + 2G\,M\,\ell^2 - 2G\,M\,r_\mathrm{H}^2 = 0 \tag{13.2}$$

This equation has two real, positive solutions if $\ell < 4GM/3\sqrt{3}$, namely

$$r_+ = \frac{2}{3}GM\left(1 + 2\cos\Xi\right), \qquad r_- = \frac{2}{3}GM\left(1 - \cos\Xi + \sqrt{3}\sin\Xi\right), \tag{13.3}$$

where $\Xi = \arccos(1 - 27\ell^2/8G^2M^2)/3 \leq \pi/3$ as long as $\ell \leq 4GM/3\sqrt{3}$. The two radii become equal, *i. e.* both horizons merge in the limit $\ell \to 4GM/3\sqrt{3}$. In this limit, the black hole is extremal. While we recover the single Schwarzschild horizon in the limit $\ell \to 0$, , with $\lim_{\ell\to 0} r_+ = 2GM$ and $\lim_{\ell\to 0} r_- = 0$, the Hayward black hole, thus, generally differs qualitatively from the Schwarzschild solution.

(**c**) We proved on the last exercise sheet that if the metric is independent of a coordinate, the corresponding vector is a Killing vector. The metric is independent of t, so $\xi = \partial_t$ is a Killing vector. A Killing horizon is a surface where the Killing vector becomes null. Computing the norm of ξ and setting it equal to zero, we obtain

$$\xi^2 \equiv \xi^\mu g_{\mu\nu}\xi^\nu = g_{tt} = f(r) = 0. \tag{13.4}$$

This is the same equation as the defining equation of the horizon. Thus, the horizon is a Killing horizon.

(**d**) This is analogous to exercise 10 (**d**). The surface gravity is defined such that

$$\xi^\nu \nabla_\nu \xi_\mu|_{r_\pm} = \kappa \xi_\mu|_{r=r_\pm}. \tag{13.5}$$

As Killing vector ξ satisfies the Killing equation

$$\nabla_{(\mu}\xi_{\nu)} = 0. \tag{13.6}$$

Therefore, we can express Eq. (13.5) as

$$\xi^\nu \nabla^\mu \xi_\nu|_{r=r_\pm} = \frac{1}{2}\nabla_\mu(\xi^2)|_{r=r_\pm} = -\kappa\xi_\mu|_{r=r_\pm}, \tag{13.7}$$

Projecting on some vector V which is not collinear with ξ on the horizon, we obtain

$$\kappa = -\left.\frac{V^\mu \nabla_\mu(\xi^2)}{2V^\mu \xi_\mu}\right|_{r=r_\pm}. \tag{13.8}$$

In order to be able to evaluate this equation, we need coordinates, which are well defined on the horizon, akin to the Eddington-Finkelstein coordinates in exercise 10 (**c**). Therefore, we choose

$$\mathrm{d}u = \mathrm{d}t - \mathrm{d}r_*, \tag{13.9}$$

with the tortoise coordinate satisfying $\mathrm{d}r_* = \mathrm{d}r/f(r)$. Replacing the time coordinate with the light-like coordinate u, the metric reads

$$\mathrm{d}s^2 = f\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2\mathrm{d}\Omega^2. \tag{13.10}$$

Now we can contract with $V = \partial_r$ to obtain

$$\kappa = -\left.\frac{V^\mu\nabla_\mu(\xi^2)}{2V^\mu\xi_\mu}\right|_{r=r_\pm} = \frac{f'(r_\pm)}{2}. \tag{13.11}$$

For the Hayward black hole, we obtain

$$\kappa = GMr_\pm\frac{r_\pm^3 - 4GM\ell^2}{(2GM\ell^2 + r_\pm^3)^2} = \frac{3}{4GM} - \frac{1}{r_\pm}, \tag{13.12}$$

where in the last equality we used Eq. (13.2). At the outer horizon ($r = r_+$), the surface gravity reads explicitly

$$\kappa = \frac{3}{4GM}\left(1 - \frac{2}{1 + 2\cos\Xi}\right) \leq \frac{1}{4GM}, \tag{13.13}$$

where the inequality holds for all $\ell \leq 4GM/3\sqrt{3}$, and κ_S denotes the surface gravity of the horizon of a Schwarzschild black hole, which is, of course, recovered for $\ell \to 0$. Thus, the surface gravity of the Hayward black hole is lower than that of the Schwarzschild black hole. In total, it interpolates between the Schwarzschild value $\kappa|_{\ell=0} = \kappa_\mathrm{S}$ and $\kappa|_{\ell=4GM/3\sqrt{3}} = 0$. Thus, in the extremal limit the temperature vanishes, just as it did for the Kerr black hole last week.

(**e**) The Hawking temperature of the outer horizon of the Hayward black hole reads

$$T = \frac{\kappa}{2\pi} = \frac{3}{8\pi GM} - \frac{1}{2\pi r_\pm} = \frac{3}{8\pi GM}\left(1 - \frac{2}{1 + 2\cos\Xi}\right) \leq \frac{1}{8\pi GM} = T_\mathrm{S}, \tag{13.14}$$

where T_S denotes the temperature of the Schwarzschild black hole. Thus, the temperature is lower than that of the Schwarzschild black hole, and vanishes in the extremal limit, just like for the Kerr black hole. This has major implications for black hole evaporation: Black hole evaporation is dominated by the Stefan-Boltzmann law

$$\frac{\mathrm{d}E}{\mathrm{d}t} \propto T^4, \tag{13.15}$$

where E is the radiated energy. The Schwarzschild black hole increases its temperature while evaporating, which further increases the energy loss due to evaporation. Hence, a Schwarzschild black hole enters a vicious circle which only ends when the black hole has fully evaporated which is in finite time. Instead, the Hayward metric gets closer and closer to extremality, effectively lowering its temperature once it is small enough. Thus, when the horizon radius of the Hayward black hole becomes comparable to the regulator scale ℓ, its evaporation will slow down drastically. As it is impossible to lower the temperature of any object to zero by a finite number of steps, the Hayward black hole will never evaporate fully, but slowly enter an adiabatic stage, becoming a remnant. Note, though, that any physics at the final stages of Hawking evaporation hinges on concepts derived from QFT in curved spacetime, which is not a good approximation any more – at the final stages, quantum gravity cannot be neglected.

Quantum field theory in curved spacetime

Assignment 6 – June 4

Exercise 14: Superradiance

Motivation: Two weeks ago, we learned about particles in the ergoregion. These effects get supercharged once we talk about fields. This will be an explosive adventure!

Superradiance amounts to extraction of energy from a black hole by perturbing the Kerr geometry. First, let's find a thermodynamic argument that makes superradiance plausible in GR. In the last weeks, we have often encountered the laws of black hole mechanics/thermodynamics. Here, we need the first law

$$\delta M = \frac{\kappa}{8\pi}\delta A_{\mathrm{H}} + \Omega_{\mathrm{H}}\delta J, \tag{14.1}$$

with the change in the area δA_{H}, the surface gravity κ, the angular velocity of the horizon Ω_{H} and the change in the black hole's angular momentum δJ. Besides, we need the second law $\delta A_{\mathrm{H}} \geq 0$. The perturbation of angular momentum and energy by an incident wave of frequency ω and azimuthal number m reads

$$\frac{\delta J}{\delta M} = \frac{m}{\omega}. \tag{14.2}$$

(a) Compute the frequency range, within which we can extract mass from the black hole.

Next, we compute this explicitly for a scalar test field. The Klein-Gordon equation for a minimally coupled massless scalar ϕ on the Kerr geometry is a complicated beast. But there is a surprising extra piece of information, which allows to simplify it enormously.

(b) Consider the tensor

$$K_{\mu\nu} = r^2 g_{\mu\nu} - 2\rho^2 l_{(\mu} n_{\nu)}, \tag{14.3}$$

where l and n are null vectors satisfying $l^\mu n_\mu = 1$. In Boyer-Lindquist coordinates

$$l = \frac{r^2+a^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\phi, \qquad n = \frac{r^2+a^2}{2\rho^2}\partial_t - \frac{\Delta}{2\rho^2}\partial_r + \frac{a}{2\rho^2}\partial_\phi. \tag{14.4}$$

Using your favourite symbolic equation manipulation software, verify that K satisfies the generalized Killing equation

$$\nabla_{(\rho} K_{\mu\nu)} = 0 \tag{14.5}$$

on the Kerr geometry. Such a tensor is called a Killing tensor.

Hint: Do not do this computation by hand unless you really like to grind.

(c) Consider a general Killing tensor $k_{\mu_1\mu_2\dots\mu_n}$, satisfying the generalization of Eq. (14.6)

$$\nabla_{(\rho} k_{\mu_1\mu_2\dots\mu_n)} = 0. \tag{14.6}$$

Show that given the momentum $p^\mu = m u^\mu$, with mass m and four-velocity u^μ, one can construct a scalar $s = k_{\mu_1\mu_2\dots\mu_n} p^{\mu_1} p^{\mu_2} \dots p^{\mu_n}$, which is conserved on geodesics.

Thus, the Kerr geometry has a hidden symmetry. This symmetry is instrumental in solving the Klein-Gordon equation because it allows to construct a hermitian derivative operator which commutes with the d'Alembertian, namely $\nabla_\mu K^{\mu\nu}\nabla_\nu$.[a] As a result, one can simultaneously diagonalize the operators $\Box$, $K^\mu\nabla_\mu$, $R^\mu\nabla_\mu$ (here we use the notation of exercise 11) and $\nabla_\mu K^{\mu\nu}\nabla_\nu$.
Therefore, it does not come as a surprise that the Klein-Gordon equation on Kerr can be brought into the form

$$\begin{aligned}&\left[\frac{(r^2+a^2)^2}{\Delta}-a^2\sin^2\theta\right]\partial_t^2\Psi+\frac{4GMar}{\Delta}\partial_t\partial_\phi\Psi\\&+\left[\frac{a^2}{\Delta}-\frac{1}{\sin^2\theta}\right]\partial_\phi^2\Psi-\partial_r\left(\Delta\partial_r\Psi\right)-\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\Psi)=0.\end{aligned}\tag{14.7}$$

(**d**) Show that Eq. (14.7) is separable: Making the ansatz

$$\Psi=\frac{1}{2\pi}\int \mathrm{d}\omega e^{-i\omega t}e^{im\phi}S(\theta)R(r)\tag{14.8}$$

show that the radial part satisfies the scalar Teukolsky equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\bar{V}R=0,\tag{14.9}$$

with

$$\bar{V}=\frac{\left[\omega(r^2+a^2)-am\right]^2}{\Delta}-a^2\omega^2+2am\omega-A_{\ell m},\tag{14.10}$$

and where $A_{\ell m}$ are the eigenvalues to the equation

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}S}{\mathrm{d}\theta}\right)+\left(a^2\omega^2\cos^2\theta-\frac{m^2}{\sin^2\theta}+A_{\ell m}\right)S=0.\tag{14.11}$$

For example, for small $a\omega$ the eigenvalue becomes $A_{\ell m}=\ell(\ell+1)$.

Hint: In order to get the conventional shape of $A_{\ell m}$ we aim at here, you may transform θ-dependent parts in Eq. (14.7) into combinations of constant and θ-dependent terms using trigonometric identities, and shift the constant contributions into the radial differential equation.

(**e**) Introduce the tortoise coordinate we already know from earlier assignments

$$\mathrm{d}r_\star=\frac{r^2+a^2}{\Delta}\mathrm{d}r.\tag{14.12}$$

Recall for later that $r_\star\to-\infty$ as $r\to r_+$, while $r_\star\to\infty$ as $r\to\infty$. Find a redefinition $R(r_\star)\to\psi(r_\star)$ such that Eq. (14.9) becomes a Schrödinger-like differential equation

$$\psi''(r_\star)+V_{\text{eff}}\psi(r_\star)=0,\tag{14.13}$$

with the effective potential

$$V_{\text{eff}}=\frac{\Delta}{(a^2+r^2)^4}\left[a^4-2a^2\Delta+\Delta r^2-r^4+(a^2+r^2)^2\bar{V}\right].\tag{14.14}$$

Consider a scattering experiment with a monochromatic wave. The boundary conditions for such an experiment have to be set at the outer horizon $r \to r_+$, and at infinity, *i. e.* $r \to \infty$. At infinity, there is an incoming wave with amplitude $\mathcal{I}$ and a reflected wave with amplitude $\mathcal{R}$. At the horizon, there is only a transmitted wave, *i. e.* one that enters the horizon with amplitude $\mathcal{T}$ because the horizon is a one-way surface.

(f) Show that Eq. (14.13) at the boundaries ($r \to r_+$, *i. e.* $r_\star \to -\infty$ and $r \to \infty$, *i. e.* $r_\star \to \infty$) allows for the solutions

$$\psi \sim \begin{cases} \mathcal{T} e^{-ik_{\rm H} r_\star} & r_\star \to -\infty, \\ \mathcal{I} e^{-i\omega r_\star} + \mathcal{R} e^{i\omega r_\star} & r_\star \to \infty, \end{cases} \tag{14.15}$$

where $k_{\rm H} = \omega - m\Omega_{\rm H}$ with, again, the angular velocity of the horizon $\Omega_{\rm H} = a/2GMr_+$.

(g) As the potential is real, the complex conjugate of Eq. (14.30) is a linearly independent solution of the equation of motion. Then, the Wronskian of ψ is independent of $r_\star$. Use this fact to compute that

$$|\mathcal{R}|^2 = |\mathcal{I}|^2 - \frac{k_{\rm H}}{\omega}|\mathcal{T}|^2. \tag{14.16}$$

(h) The energy flux at infinity reads

$$\frac{\mathrm{d}E_{\rm out}}{\mathrm{d}t} = \frac{\omega^2}{2}|\mathcal{R}|^2, \qquad \frac{\mathrm{d}E_{\rm in}}{\mathrm{d}t} = \frac{\omega^2}{2}|\mathcal{I}|^2, \tag{14.17}$$

where $E_{\rm out}$ and $E_{\rm in}$ stand for ingoing and outgoing energy. What happens if $\omega < m\Omega_{\rm H}$?

(i) **Bonus Sci-fi-question:** What happens if we surround the black hole by a perfectly reflecting (and extremely durable) mirror and send in an initial wave of frequency $\omega < \Omega_{\rm H}$? Not without reason, this is called a black-hole bomb.

[a] Note that K being a Killing tensor is a necessary but not a sufficient condition for $\nabla_\mu K^{\mu\nu}\nabla_\nu$ to commute with the d'Alembertian. It also has to be compatible with the curvature on the geometry, which K on Kerr actually is.

(a) Plugging the angular momentum into the first law of black hole mechanics

$$\delta M = \frac{\kappa}{8\pi}\frac{\delta A_{\rm H}}{1 - m\frac{\Omega_{\rm H}}{\omega}}. \tag{14.18}$$

Mass is extracted if $\delta M < 0$. As $\delta A_{\rm H} \geq 0$, mass extraction therefore requires $\omega < m\Omega_{\rm H}$.

(b) See ancillary Mathematica notebook.

(c) The time derivative along a geodesic equals $\mathrm{d}/\mathrm{d}\tau = u^\mu \nabla_\mu$ (here τ is the affine parameter along the curve). Thus, the time derivative of the scalar reads

$$\frac{\mathrm{d}k_{\mu_1\mu_2\ldots\mu_n}p^{\mu_1}p^{\mu_2}\ldots p^{\mu_n}}{\mathrm{d}\tau} = m^{-1}\nabla_{(\nu}k_{\mu_1\mu_2\ldots\mu_n)}p^\nu p^{\mu_1}p^{\mu_2}\ldots p^{\mu_n} + nk_{(\mu_1\mu_2\ldots\mu_n)}\frac{\mathrm{d}p^{\mu_1}}{\mathrm{d}\tau}p^{\mu_2}\ldots p^{\mu_n} = 0, \tag{14.19}$$

where in the last equality we used Eq. (14.6) and that test particles on geodesics follow unaccelerated motion, *i. e.* $\mathrm{d}p^\mu/\mathrm{d}\tau = 0$.

(**d**) Plugging in the ansatz, Eq. (14.8), we obtain

$$
\begin{aligned}
&-\omega^2\left[\frac{(r^2+a^2)^2}{\Delta}-a^2\sin^2\theta\right]+\frac{4GMar}{\Delta}m\omega \\
&-\left[\frac{a^2}{\Delta}-\frac{1}{\sin^2\theta}\right]m^2-R^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta\frac{\mathrm{d}R}{\mathrm{d}r}\right)-\frac{1}{S\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}S}{\mathrm{d}\theta}\right)=0.
\end{aligned}
\tag{14.20}
$$

Observe that terms depending on θ and r appear separately, never mixed. Hence, we can separate these variables: The contributions to the Klein-Gordon equation dependent on r have to be constant and so have to be the contributions dependent on θ to cancel each other for all values of the variables. Thus, we set the contributions dependent on r equal to the constant $A_{\ell m}$. Then, those contributions dependent on θ have to be equal to $-A_{\ell m}$ for the Klein-Gordon equation to be satisfied.

As suggested in the hint, we write $\sin^2\theta = 1-\cos^2\theta$ in the first line and shift the constant bit to the differential equation for R. Thus, for R, we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\left[\omega^2\frac{(r^2+a^2)^2}{\Delta}+\frac{a^2m^2}{\Delta}-a^2\omega^2-\frac{4GMar}{\Delta}m\omega-A_{\ell m}\right]R=0. \tag{14.21}
$$

We can rewrite this equation as

$$
\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\left[\frac{\left[(r^2+a^2)\omega-am\right]^2}{\Delta}+2am\omega\frac{r^2-2GMr+a^2}{\Delta}-a^2\omega^2-A_{\ell m}\right]R=0. \tag{14.22}
$$

Plugging in the definition of Δ, we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\left[\frac{\left[(r^2+a^2)\omega-am\right]^2}{\Delta}+2am\omega-a^2\omega^2-A_{\ell m}\right]R=0. \tag{14.23}
$$

Apart from that, for the differential equation for S we obtain

$$
\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}S}{\mathrm{d}\theta}\right)+\left(a^2\omega^2\cos^2\theta-\frac{m^2}{\sin^2\theta}+A_{\ell m}\right)S=0. \tag{14.24}
$$

(**e**) In terms of the tortoise coordinate, the derivative part of Eq. (14.9) reads

$$
\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta\frac{\mathrm{d}R}{\mathrm{d}r}\right)=\frac{r^2+a^2}{\Delta}\frac{\mathrm{d}}{\mathrm{d}r_\star}\left((r^2+a^2)\frac{\mathrm{d}R}{\mathrm{d}r_\star}\right), \tag{14.25}
$$

$$
=\frac{(r^2+a^2)^2}{\Delta}\frac{\mathrm{d}^2R}{\mathrm{d}r_\star^2}+2r\frac{\mathrm{d}R}{\mathrm{d}r_\star}. \tag{14.26}
$$

To obtain a Schrödinger-like equation, we have to eliminate the first-derivative term. Indeed, if we redefine $\psi\equiv R/\sqrt{r^2+a^2}$, we obtain

$$
\frac{(r^2+a^2)^2}{\Delta}\frac{\mathrm{d}^2R}{\mathrm{d}r_\star^2}+2r\frac{\mathrm{d}R}{\mathrm{d}r_\star}=\frac{(a^2+r^2)^{3/2}}{\Delta}\psi''+\frac{a^4-2a^2\Delta+\Delta r^2-r^4}{(a^2+r^2)^{5/2}}\psi. \tag{14.27}
$$

The Schrödinger-like equation Eq. (14.13) follows immediately.

(**f**) At the horizon $\Delta=0$, so terms proportional to Δ can be neglected. The surviving terms in the effective potential given in Eq. (14.14) read

$$
V_{\text{eff}}|_{r=r_+}=\left(\omega-\frac{am}{r_+^2+a^2}\right)^2=(\omega-m\Omega_{\mathrm{H}})^2=k_{\mathrm{H}}^2. \tag{14.28}
$$

Thus, as $r \to r_+$ the field ψ behaves like a plane wave with frequency $k_{\rm H}$.

At large r, we can neglect terms at higher than zeroth order in $1/r$, and we have $\Delta \simeq r^2$. Thus, the effective potential reads

$$V_{\rm eff} = \omega^2, \tag{14.29}$$

implying plane wave behaviour at infinity with frequency ω.

Thus assuming that the horizon is a one-way surface, we obtain

$$\psi \sim \begin{cases} \mathcal{T} e^{-ik_{\rm H} r_\star} & r_\star \to -\infty, \\ \mathcal{I} e^{-i\omega r_\star} + \mathcal{R} e^{i\omega r_\star} & r_\star \to \infty. \end{cases} \tag{14.30}$$

(**g**) The Wronskian of ψ and ψ^* reads

$$W(\psi, \psi^*) = \psi \frac{{\rm d}\psi^*}{{\rm d}r_\star} - \psi^* \frac{{\rm d}\psi}{{\rm d}r_\star} \tag{14.31}$$

On the horizon, the Wronskian becomes

$$W(\psi, \psi^*) = 2|\mathcal{T}|^2 i k_{\rm H}, \qquad r_\star \to -\infty, \tag{14.32}$$

while at large $r_\star$, we obtain

$$W(\psi, \psi^*) = -2i|\mathcal{R}|^2 i\omega + 2i\omega|\mathcal{I}|^2, \qquad r_\star \to \infty. \tag{14.33}$$

Thus, constancy of the Wronskian imposes the condition

$$|\mathcal{R}|^2 = |\mathcal{I}|^2 - \frac{k_{\rm H}}{\omega}|\mathcal{T}|^2. \tag{14.34}$$

(**h**) Using Eq. (14.16), the outgoing energy flux satisfies

$$\frac{{\rm d}E_{\rm out}}{{\rm d}t} = \frac{\omega^2}{2}\left(|\mathcal{I}|^2 - \frac{k_{\rm H}}{\omega}|\mathcal{T}|^2\right) = \frac{{\rm d}E_{\rm in}}{{\rm d}t} - \frac{k_{\rm H}\omega}{2}|\mathcal{T}|^2. \tag{14.35}$$

If $\omega < m\Omega_{\rm H}$, we have $k_{\rm H} < 0$. This immediately implies that

$$\frac{{\rm d}E_{\rm out}}{{\rm d}t} > \frac{{\rm d}E_{\rm in}}{{\rm d}t}, \tag{14.36}$$

i. e. net energy is extracted from the black hole.

(**i**) As the outgoing wave from the black hole has the same frequency as the incident wave, when reflected back by the mirror it will produce another, further amplified outgoing wave and so on. Like the Penrose process, this process decreases the angular momentum of the black hole, *i. e.* it decreases the angular velocity of the horizon $\Omega_{\rm H}$. Thus, it will do so until $\omega = m\Omega_{\rm H}$, where the energy extraction stops, and the whole system equilibrates. As we can, in principle, choose ω to be arbitrarily small (which requires making the mirror arbitrarily large), similarly to the Penrose process, one can basically reduce the angular momentum of the black hole to zero this way, releasing the same amount of energy (29% of the initial black-hole mass). Now imagine, the mirror breaks or is made to disappear at the very end of the process. This would release all of this energy at once, *i. e.* for Sagittarius A* if it was extremal $\sim 0.25\%$ of all energy that all stars in the galaxy will emit during their whole lifetime. Sounds like a galaxy-sized nuke, doesn't it?

Exercise 15: Superradiance in analogue systems

Motivation: Last time I checked, we couldn't send waves to black holes and pick up the amplified reflected waves. But we can see the effect already in a draining bath tub. Here's how.

The idea of analogue gravity is to study the effects of fields on curved spacetime in more accessible systems realizable in a lab. An example is water going down a drain. Similar to a black hole, the speed required to avoid falling down the drain exceeds the sound speed of the liquid such that perturbations experience an effective horizon. Here we try to derive the effective curved-spacetime Klein-Gordon equation these perturbations satisfy.

We describe the fluid by a velocity potential Φ in terms of which one defines the velocity field $\vec{v} = -\vec{\nabla}\Phi$. Besides, the fluid has vanishing viscosity and a barotropic equation of state $p(\rho)$ relating pressure p and density ρ, and the evolution is adiabatic, *i. e.* slow enough for the system to remain in local equilibrium. Then, the fluid satisfies the continuity equation

$$\partial_t \rho + \vec{\nabla}(\rho \vec{v}) = 0 \tag{15.1}$$

and, absent external driving forces, the Euler equation

$$-\partial_t \Phi + \frac{v^2}{2} + \int \frac{\mathrm{d}p}{\rho} = 0. \tag{15.2}$$

(a) Assume $\rho = \rho_0 + \epsilon\rho_1$, $\vec{v} = \vec{v}_0 - \epsilon\vec{\nabla}\phi$, $\Phi = \Phi_0 + \epsilon\phi$, where $\epsilon \ll 1$. Obtain the linearized continuity equation

$$\partial_t \rho_1 + \vec{\nabla}(\rho_1 \vec{v}_0 - \rho_0 \vec{\nabla}\phi) = 0, \tag{15.3}$$

and the linearized Euler equation

$$-\partial_t \phi - \vec{v}_0 \cdot \vec{\nabla}\phi + c_\mathrm{s}^2 \frac{\rho_1}{\rho_0} = 0, \tag{15.4}$$

with the background speed of sound $c_\mathrm{s}^2 = \mathrm{d}p/\mathrm{d}\rho|_{\rho=\rho_0}$.

(b) Use Eqs. (15.3) and (15.4) to obtain the differential equation for the perturbation

$$\partial_t \left(c_\mathrm{s}^{-2} \rho_0 (\partial_t \phi + \vec{v}_0 \cdot \vec{\nabla}\phi) \right) - \vec{\nabla} \cdot \left[\rho_0 \vec{\nabla}\phi - c_\mathrm{s}^{-2} \rho_0 \vec{v}_0 (\partial_t \phi + \vec{v}_0 \cdot \vec{\nabla}\phi) \right] = 0. \tag{15.5}$$

(c) Show that this equation can be equivalently obtained as the curved-space Klein-Gordon equation derived from the metric

$$\mathrm{d}s^2 = \frac{\rho_0}{c_\mathrm{s}} \left[c_\mathrm{s}^2 \mathrm{d}t^2 - (\mathrm{d}\vec{x} - \vec{v}_0 \mathrm{d}t)^2 \right]. \tag{15.6}$$

(d) As an example, consider a fluid in $2+1$ dimensions with constant c_s and a background-fluid velocity (in terms of polar coordinates r and ϕ)

$$\vec{v}_0 = -\frac{A}{r}\vec{e}_r + \frac{B}{r}\vec{e}_\phi, \tag{15.7}$$

where A, B are constants characterizing the direction of the flow. Here, we assume that $A, B > 0$ such that the fluid is moving inward with an additional clockwise rotation, thus mimicking a rotating black hole. For example, water in a draining bath tub flows this way.

Show that the resulting metric reads

$$\mathrm{d}s^2 = c_\mathrm{s}^2 \mathrm{d}t^2 - \left(\mathrm{d}r + \frac{A}{r}\mathrm{d}t \right)^2 - \left(r\mathrm{d}\phi - \frac{B}{r}\mathrm{d}t \right)^2. \tag{15.8}$$

(e) Show that the resulting geometry has an ergosphere at radii below

$$r_{\partial \mathrm{E}} = \frac{\sqrt{A^2 + B^2}}{c_\mathrm{s}}, \tag{15.9}$$

and a sound horizon at

$$r_\mathrm{H} = \frac{A}{c_\mathrm{s}}. \tag{15.10}$$

(f) There is an ergoregion outside the sound horizon. What happens if you introduce a perturbation of sufficiently low frequency at the outer boundary of the experiment? What could the "sufficiently low frequency" be precisely? Think about your answer, then check out this link.

(a) At first order in ϵ, the linearized continuity equation, Eq. (15.3), follows immediately from Eq. (15.1). To obtain Eq. (15.4), we need to take a closer look at the specific enthalpy

$$h(p) = h(p_0 + \epsilon p_1) \simeq h(p_0) + \epsilon p_1 h'(p_0) = h(p_0) + \epsilon \frac{p_1}{\rho_0}, \tag{15.11}$$

where $\rho_0 = \rho(p_0)$. At the same time, Taylor expanding $\rho(p)$ fixes $p_1 = c_\mathrm{s}^2 \rho_1$, where $c_\mathrm{s}^2 = \mathrm{d}p/\mathrm{d}\rho(\rho_0)$. Therefore, we obtain

$$h \simeq h(p_0) + \epsilon \frac{c_\mathrm{s}^2 \rho_1}{\rho_0}. \tag{15.12}$$

This immediately implies Eq. (15.4).

(b) We first solve Eq. (15.4) such that

$$\rho_1 = \frac{\rho_0}{c_\mathrm{s}^2} \left(\partial_t \phi + \vec{v}_0 \cdot \vec{\nabla} \phi \right). \tag{15.13}$$

Plugging into Eq. (15.3), we obtain Eq. (15.5).

(c) The metric and its inverse have the components

$$g^{tt} = \frac{1}{c_\mathrm{s} \rho_0}, \qquad g^{it} = \frac{v_0^i}{c_\mathrm{s} \rho_0}, \qquad g^{ij} = \frac{v_0^i v_0^j - c_\mathrm{s}^2 \delta^{ij}}{c_\mathrm{s} \rho_0}. \tag{15.14}$$

The determinant of the metric equals $g = -\rho_0^4 / c_\mathrm{s}^2$. Thus, the d'Alembertian reads

$$\Box \phi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right), \tag{15.15}$$

$$= \frac{c_\mathrm{s}}{\rho_0^2} \partial_t \left[\frac{\rho_0}{c_\mathrm{s}^2} \left(\partial_t \phi + \vec{v}_0 \cdot \vec{\nabla} \phi \right) \right] + \frac{c_\mathrm{s}}{\rho_0^2} \vec{\nabla} \cdot \left[\frac{\rho_0}{c_\mathrm{s}^2} \left(\vec{v}_0 (\partial_t \phi + \vec{v}_0 \cdot \vec{\nabla} \phi) - c_\mathrm{s}^2 \vec{\nabla} \phi \right) \right]. \tag{15.16}$$

Finally, the Klein-Gordon equation $\Box \phi = 0$ can be expressed as

$$\partial_t \left[c_\mathrm{s}^{-2} \rho_0 \left(\partial_t \phi + \vec{v}_0 \cdot \vec{\nabla} \phi \right) \right] - \vec{\nabla} \cdot \left[\rho_0 \vec{\nabla} \phi - c_\mathrm{s}^{-2} \rho_0 \vec{v}_0 \left(\partial_t \phi + \vec{v}_0 \cdot \vec{\nabla} \phi \right) \right], \tag{15.17}$$

which is exactly Eq. (15.5).

(d) To get to polar coordinates, we transform $x^1 = r \cos \phi$, $x^2 = r \sin \phi$. Writing $\vec{v}_0 = v_0^r \vec{e}_r + v_0^\phi \vec{e}_\phi$, we can express the metric as

$$\mathrm{d}s^2 = \frac{\rho_0}{c_\mathrm{s}} \left[c_\mathrm{s}^2 \mathrm{d}t^2 - \left(\mathrm{d}r^2 - v_0^r \mathrm{d}t \right)^2 - (r \mathrm{d}\phi - v_0^\phi \mathrm{d}t)^2 \right]. \tag{15.18}$$

Plugging in for v_0^r and v_0^ϕ, we obtain Eq. (15.8).

(**e**) An ergosphere is a region where the timelike Killing vector becomes spacelike. The boundary of the ergosphere is defined where the norm of the timelike Killing vector becomes zero. Thus, at $r = r_{\partial\mathrm{E}}$, the vector $K = \partial_t$ is null. In other words

$$g_{tt}|_{r=r_{\partial\mathrm{E}}} = \frac{\rho_0}{c_\mathrm{s}}\left[c_\mathrm{s}^2 - \left(\frac{A}{r_{\partial\mathrm{E}}}\right)^2 - \left(\frac{B}{r_{\partial\mathrm{E}}}\right)^2\right] = 0. \tag{15.19}$$

This equation has one positive solution namely Eq. (15.9).

At the horizon, the absolute value of the radial velocity is as large as the speed of sound, similarly to the escape velocity on the horizon of a black hole being the speed of light. Thus, we set

$$|v_0^r|_{r=r_\mathrm{H}} = \frac{A}{r_\mathrm{H}} = c_\mathrm{s}, \tag{15.20}$$

which immediately implies Eq. (15.10).

(**f**) Provided the frequency is small enough, the wave is going to get amplified when it is reflected. Thus, there is going to be superradiance exactly as we derived it in the preceding exercise. The paper linked on the sheet is the first observation of this process in a lab, indeed done with a draining bath tub.

To find out what the “sufficiently low frequency" is, we have to compute the angular velocity at the horizon Ω_H. Since the angular motion of the fluid defines the frame-dragging at the horizon, Ω_H is simply the angular velocity of the fluid at the horizon, *i. e.*

$$\Omega_\mathrm{H} = \frac{v_\phi}{r}\bigg|_{r=r_\mathrm{H}} = \frac{B}{A^2}c_\mathrm{s}^2. \tag{15.21}$$

By analogy with the result in exercise 14, low frequencies are now $\omega < mBc_\mathrm{s}^2/A^2$, where m denotes the azimuthal wave number.