HAMILTONIAN ANALYSIS OF RG-IMPROVED E-H ACTION AND THE JORDAN/EINSTEIN EQUIVALENCE DEBATE

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QUANTUM AND GRAVITY AND MATTER
HEIDELBERG, GERMANY, 9-13/06/2019

work in partial collaboration with Alfio Bonanno (INAF and INFN Catania), based on gr-qc/1710.06317 and gr-qc/1805.02318, gr-qc/1902.03014 and more to come...
Outline of the talk

- Quantum Gravity and Asymptotic Safety in Q.G.

- RG improved modified Einstein-Hilbert Action (Reuter-Weyer) and its Hamiltonian Analysis.


- Constraint Analysis of the Brans-Dicke degenerate case $\omega = -\frac{3}{2}$.

- Jordan Frame vs. Einstein Frame debate.

- Conclusions.
QUANTUM GRAVITY

- General Relativity is considered an effective theory. It is not pertubatively renormalizable (the Newton constant $G$ has a $(\text{length})^{-2}$ dimension).
  There exist fundamental (=infinite cut off limit) theories which are not “perturbatively renormalizable (along the line of Wilson theory of renormalization).

- They are constructed by taking the infinite-cut off limit (continuum limit) at a Non-Gaussian Fixed Point (NGFP) ($u_* \neq 0$, pert. theories have trivial Gaussian point $u_* = 0$).

- The “Asymptotic safety approach to Quantum Gravity” is based on “Weinberg conjecture” (1979). He suggested to run the coupling constants of the theory, find a non (NGFP)-Gaussian fixed point in this space of parameters, define the Quantum theory at this point.

- $d=2+\epsilon$: F. P. exists (Weinberg); $d=4$ NGFP in the Einstein-Hilbert truncation exists (Reuter and Sauressing 2002).
The main idea of Functional Renormalization Group (FRG) in Asymptotic Safety is to make sense of the Path Integral in Riemannian Quantum Gravity. The guideline is to use in the continuum the methods of discrete quantum gravity (EDT or CDT for example).

We define a (non-perturbative) Wilson-type (coarse grained) effective average action

$$\Gamma_k[g_{\mu\nu}]$$

$$\Gamma_k[g_{\mu\nu}]$$ has an IR cutoff at k; $$\Gamma_k$$ contains all the quantum fluctuation for p>k, and not yet those for p<k.

Modes with p<k are suppressed in the path integral by a (mass)²=\(R_k(p^2)\),
ASYMPTOTIC SAFETY

- $\Gamma_k$ interpolates between: $\Gamma_{k\to\infty} = S$, classical (bare) action, and $\Gamma_{k\to0} = \Gamma$, standard effective action.

- $\Gamma_k$ satisfies the RG Functional equation, symbolically

$$k \partial_k \Gamma_k = \frac{1}{2} Tr \left[ (\delta^2 \Gamma_k + R_k)^{-1} k \partial_k R_k \right]$$

$\delta^2 \Gamma_k$ being a Hessian matrix.

- Powerful nonperturbative approximation scheme: the space of the action functionals is “Truncated”

$$\Gamma_k[\cdot] = \sum_{i=0}^{N} \tilde{a}_i(k) k^{d_i} I_i[\cdot]$$

$I_i[\cdot]$ are given “local or non-local functionals” of the fields, $\tilde{a}_i(k)$ are the adimensional coefficients that carry the scale dependence.
In the case of gravity the following truncation ansatz has been made

\[ I_0[g] = \int d^4x \sqrt{g}, \quad I_1[g] = \int d^4x \sqrt{g} R, \quad I_2[g] = \int d^4x \sqrt{g} R^2, \quad \text{etc.} \]

The Renormalization Group (RG) equation is

\[ k \partial_k \tilde{a}_i(k) = \beta_i(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \ldots) \]

\( \tilde{a}_{i \neq 0} \) is a NGFP if \( \beta_j(\tilde{a}_{i \neq 0}) = 0, \forall j. \)

Around the NGFP there exists relevant directions \( S_{uv} : \{ \text{RG trajectories hitting the FP as } k \to \infty \} \) and irrelevant directions which leave the critical surface as \( k \to \infty. \) Quantum Gravity theory defined by relevant directions.
Modified Einstein Hilbert action has external, non-geometrical fields $G(x)$ and $\Lambda(x)$ (Reuter and Weyer, Phys. Rev.D 69 2004; Cai and Easson, Phys. Rev.D 84 2011)

$$S_{mEH}[g, G(x), \Lambda(x)] \equiv \frac{1}{16\pi} \int_M d^4x \sqrt{-g} \left( \frac{R}{G(x)} - 2 \frac{\Lambda(x)}{G(x)} \right) + \frac{1}{8\pi} \int_{\partial M} d^3x \sqrt{h} \frac{K}{G(x)}$$

$\phi(x) \equiv \frac{1}{G(x)}$ as a Brans-Dicke field.

The dynamic (Reuter-Weyer) could impose integrability condition on $k(x), G(x)$ and $\Lambda(x)$.

We start from an ADM decomposition

$$g = -(N^2 - N_i N^i) dt \otimes dt + N_i (dx^i \otimes dt + dt \otimes dx^i) + h_{ij} dx^i \otimes dx^j$$

variables $(N, N_i, h_{ij})$
Bonanno et al. (CGQ 21 2004) proposed a Hamiltonian formalism with G and \( \Lambda \) as dynamical variables. They modified the boundary and get a geometrodynamics that, finally, is like Einstein GR.

If we perform an ADM analysis of the previous (Reuter-Weyer) action with \( G(x) \) and \( \Lambda(x) \) non-geometrical and non-dynamical fields. The ADM metric enables to derive ADM Lagrangian density \( \mathcal{L}_{ADM} \) and its Hamiltonian

\[
\mathcal{H}_{ADM} = \pi^{ij} \dot{h}_{ij} - \mathcal{L}_{ADM}
\]

We immediately get primary constraints, \( \pi, \pi_i \) associated to \( N \) and \( N^i \)

\[
\pi = \frac{\partial \mathcal{L}_{ADM}}{\partial \dot{N}} \approx 0 \quad \pi_i = \frac{\partial \mathcal{L}_{ADM}}{\partial \dot{N}^i} \approx 0 .
\]

The total Hamiltonian density is then

\[
\mathcal{H}_T = \lambda \pi_i + \lambda^i \pi_i + \mathcal{H}_{ADM}
\]
ADM FORMALISM WITH G AND \( \Lambda \) VARIABLE

- Preserving the two primary constraints \( \pi, \pi_i \) we get the secondary constraints

\[
\{ \pi, \int dx^3 \mathcal{H}_T \} = -\mathcal{H} \approx 0 \quad \{ \pi_i, \int d^3x \mathcal{H}_T \} = -\mathcal{H}_i \approx 0 .
\]

- These are the Hamiltonian and momentum constraints, whose form is the following

\[
\mathcal{H} = \left( \frac{16\pi GN}{\sqrt{h}} \left( \pi^{ij} \pi_{ij} - \frac{1}{2} \pi_h^2 \right) - \frac{\sqrt{h}}{16\pi G} R \right) + \frac{\Lambda \sqrt{h}}{8\pi G} - \frac{1}{8\pi} \sqrt{h} h^{ij} \nabla_j \left( \frac{G_{,i}}{G^2} \right)
\]

- Non zero Poisson brackets with \( \pi, \pi_i \), so they look second class!
There is the possibility to simplify a bit the previous calculation with the following transformation

\[ \tilde{\pi}^{ij} = \pi^{ij} - \frac{\sqrt{\hbar}}{16\pi NG^2} (G,_{0} - G,_{k}N^{k}) = -\frac{\sqrt{\hbar}}{16\pi G} (K^{ij} - h^{ij}K) \]

The following transformation is canonical

\[ (N, N^{i}, h_{ij}, \pi, \pi_{i}, \pi^{ij}) \mapsto (N, N^{i}, h_{ij}, \pi, \pi_{i}, \tilde{\pi}^{ij}) \]

The Hamiltonian and momentum constraints are now

\[ H = (16\pi G)G_{abcd}\tilde{\pi}^{ab}\tilde{\pi}^{cd} - \frac{\sqrt{\hbar}(3R - 2\Lambda)}{16\pi G} - \frac{\sqrt{\hbar}(G,_{0} - G,_{k}N^{k})\nabla_{a}N^{a}}{8\pi G^{2}N^{2}} - \nabla_{i} \left( \frac{G,i\sqrt{\hbar}h^{ij}}{8\pi G^{2}} \right) \]

\[ H_{i} = -2\nabla^{a}\tilde{\pi}_{ai} + \frac{\sqrt{\hbar}(-G,i)\nabla_{a}N^{a}}{8\pi G^{2}N} - \sqrt{\hbar}\nabla_{i} \left( \frac{G,_{0} - G,_{k}N^{k}}{8\pi G^{2}N} \right) \]

They are still second class. Diffeomorphism are lost since a non geometrical (and non-dynamical) field G(x) distinguishes points in Space-Time.
Previous Hamiltonian analysis triggers the question why the constraint algebra looks so complicated and do not close. Some authors claim AS close to NGFP should be described by Brans-Dicke with $\omega = 0$.

To answer this question one can consider a Brans-Dicke theory in which $\phi(x)$ is a dynamical variable and a York boundary term.

The corresponding ADM Lagrangian is

\[
S = \frac{1}{4q^2} \left[ \int_M d^4x \sqrt{-g} \left( \phi^4 R - \frac{\omega}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - U(\phi) \right) + 2 \int_{\partial M} d^3x \sqrt{h} \phi K \right]
\]

\[
\mathcal{L}_{ADM} = \frac{\sqrt{h}}{4q^2} \left[ N \left\{ \phi^3 R + \frac{(4q^2)^2}{h\phi} \left( \Pi_{ab} \Pi_{ab} - \frac{\Pi^2_h}{2} \right) \right. \right.
\]

\[
\left. \left. - \frac{N\omega}{\phi} D_c \phi D^c \phi + 2D_c \phi D^c N - NU(\phi) + \frac{8q^4}{h\phi} \frac{N}{(3 + 2\omega)} (\Pi_h - \phi \pi_\phi)^2 \right] \right]
\]

\[
\Pi_h = h_{ab} \Pi^{ab}
\]
Here we have that the momenta are defined as

\[
\Pi^{ab} = \frac{\partial L_{ADM}}{\partial \dot{h}_{ab}} = -\frac{\sqrt{h}}{4q^2} \phi \left( K^{ab} - h^{ab} K \right) - \frac{\sqrt{h} h^{ab}}{4q^2 N} \left( \partial_0 \phi - \partial_a \phi N^a \right)
\]

\[
\pi_\phi = \frac{\partial L_{ADM}}{\partial \phi} = \frac{\omega \sqrt{h}}{2q^2 N \phi} \left( \partial_0 \phi - N^a \partial_a \phi + \frac{N \phi}{\omega} \phi K \right)
\]

Following Dirac’s constraint procedure, primary constraints are

\[
\pi = \frac{\partial L_{ADM}}{\partial \dot{N}} \approx 0 \quad \pi_a = \frac{\partial L_{ADM}}{\partial \dot{N}^a} \approx 0
\]

The total Hamiltonian \( H_T \) is (\( \lambda, \lambda^a \) are Lagrangian Multipliers)

\[
H_T = \int d^3 x \left( \lambda \pi + \lambda^a \pi_a + N \mathcal{H} + N^a \mathcal{H}_a \right)
\]
ADM ANALYSIS OF BRANS-DICKE THEORY

- As in Einstein General Relativity one has a momentum constraints $\mathcal{H}_a$ and Hamiltonian constraint $\mathcal{H}$

$$\mathcal{H}_a = -2h_{ab}D_c \Pi^{cb} + \pi_\phi \partial_a \phi$$

$$\mathcal{H} = \frac{\sqrt{h}}{4q^2} \left\{ -\phi^3 R + \frac{(4q^2)^2}{h\phi} \left( \Pi^{ab} \Pi_{ab} - \frac{\Pi_h^2}{2} \right) \right\}$$

$$+ \omega_\phi \frac{D_c \phi D^c \phi - \frac{2}{N} D_c \phi D^c N + U(\phi) + \frac{(4q^2)^2}{h\phi(3 + 2\omega)} (\Pi_h - \phi^2 \pi_\phi)^2}{N}$$

- One could also confront these results with Olmo and Sanchis Phys. Rev D83.10436, arXiv.org:1101.3403.

- Following Menotti (arXiv:1703.05155) as for canonical analysis of GR we notice the following: the momentum constraints are the generators of the space diffeomorphisms on the three-dimensional spatial surfaces.
From this relations, like in standard Hamiltonian General Relativity, follows

\[ \{ \mathcal{H}_a(x), \mathcal{H}_b(x') \} = \mathcal{H}_a(x') \partial_b \delta(x, x') - \mathcal{H}_b(x) \partial'_a \delta(x, x') \]

The Poisson brackets of the Hamiltonian constraint with the momentum constraints are

\[ \{ \mathcal{H}(x), \mathcal{H}_a(x') \} = -\mathcal{H}(x') \partial'_a \delta(x', x) \]

The last step is the calculation of the Poisson brackets (P.B.) of the Hamiltonian-Hamiltonian constraint (distributions in two different points)

\[ \left\{ \int d^3x N(x) \mathcal{H}(x), \int d^3x' N'(x') \mathcal{H}(x') \right\} \]
ADM ANALYSIS OF BRANS-DICKE THEORY

The calculation gives the following results (note: $\phi = 1$ we get GR constraint algebra)

$$\left\{ \int d^3x N(x)\mathcal{H}(x), \int d^3x' N'(x')\mathcal{H}(x') \right\} =$$

$$\int d^3y (-N' D^a N + ND^a N')(1 + 4\log\phi)\mathcal{H}_a + \int d^3y (-N' D^a N + ND^a N')\chi_a$$

$$\chi_a = 4\log\phi \frac{(2 + 2\omega)D_a \pi_h - (2 + 2\omega)\pi_\phi D_a \phi + D_a (\pi_\phi) \phi}{(3 + 2\omega)}$$

Finally we have to impose the momenta and Hamiltonian constraint be preserved on the constraint manifold

$$\{\mathcal{H}_a(x), H_T\} \approx 0 \quad \{\mathcal{H}(x), H_T\} \approx 0$$

The momentum constraints are preserved, while for the Hamiltonian constraint we get the following equation for the lapse function $N(x)$

$$2\chi^a(x) \partial_a (x) + N(x) \partial_a \chi^a (x) = 0$$

It fixes only the spatial part of the Lapse, there is still a diffeomorphism invariance in the time variable since $N = N(t,x)$. The constraints are Dirac’s first class constraints since they commute among themselves.
DEGENERATE CASE $\omega = -\frac{3}{2}$

- Equation of motion for the field $\phi$
  \[(3 + 2\omega) \Box \phi + 2U(\phi) - \phi \frac{dU(\phi)}{d\phi} = 0\]

- Case $\omega = -\frac{3}{2}$ extra primary constraint
  \[C_{\phi} = \Pi_{\phi} - \phi \pi_{\phi}\]

- Preserving this constraint along the dynamics, we get a secondary constraint
  \[C^{II}_{\phi} = \frac{3}{\phi} D_c \phi D^c \phi - 4D_c D^c \phi + \frac{3}{N} D^c N D_c \phi\]

- This constraints are second class among them.

- Preliminary results show that using a particular combination of previous constraints, following Dirac, finally we have 8 first class constraints and 2 second class constraints. The degrees of freedom relative to the scalar field $\phi$ are eliminated by defining Dirac’s brackets.
The different commutation relations of the Hamiltonian constraint respect to Einstein GR address the issue of the Equivalence Jordan-Einstein frame.

If we consider the following action with a Gibbons-Hawking-York boundary term (Dyer & Hinterbicher Phy.RevD.79.024028 arXiv:0809.4033)

\[ S = \int_M d^m x \sqrt{-g} \left( f(\phi) R - \frac{1}{2} \lambda(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - U(\phi) \right) \]
\[ S_{GHY} = 2 \int_{\partial M} d^{m-1} \sqrt{h} f(\phi) K \]

If we perform the following Weyl transformation

\[ \tilde{g}_{\mu\nu} = \left( 16\pi G f(\phi) \right)^{\frac{2}{n-2}} g_{\mu\nu} \]
The action transforms, including the boundary term, as

\[
S = \int_M d^n x \sqrt{-\tilde{g}} \left( \frac{1}{16\pi G} \tilde{R} - A(\phi) \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) + \frac{1}{8\pi G} \int_{\partial M} d^{n-1} \sqrt{h} \tilde{K}
\]

where \( A(\phi) = \frac{1}{16\pi G} \left( \frac{\lambda(\phi)}{2f(\phi)} + \frac{n - 1}{n - 2} \frac{(f'(\phi))^2}{f^2(\phi)} \right) \) and \( V(\phi) = \frac{U(\phi)}{[16\pi G f(\phi)]^{\frac{n}{n-2}}} \)

In the Einstein frame the previous action has the same constraint algebra of Einstein General Relativity.

It is easy, although lengthy, the Weyl transformation is not canonical, does not preserve Hamiltonian structure.
The way of proceeding is to express the canonical variables of the conformal transformation as function of the old variables.

The Poisson Brackets of the canonical variables in the Einstein frame do not preserve the symplectic structure

\[ \{Q^i(q,p), P_j(q,p)\}_{q,p} = \delta^i_j \]
\[ \{Q^i(q,p), Q^j(q,p)\}_{q,p} = \{P_i(q,p), P_j(q,p)\}_{q,p} = 0; \]

In the literature there is a big debate: Deruelle, Sendouda, Yousseff Phys. Rev. D. 80, 22-10-2019 claim there is a canonical transformation. (I found they missed a Poisson Bracket…)

Ezawa et al. Nuov. Cim. B125 2010 disagree and say the transformation from Jordan to Einstein frame is not canonical.
Our considerations state clearly the transformation from Jordan to Einstein is not canonical.

Kuchar (1974) found that matter fields $\phi^A$, with momenta $\pi_A$, coupled through “a non derivative gravitational coupling”

$$H_T = H(g_{ij}, \pi^i j) + H_{(M)}(g_{ij}, \phi^A, \pi_A)$$

($H_T$ is the total Hamiltonian, $H_{(M)}$ the matter part) has the same algebra of constraint algebra of Einstein Geometrodynamics.

Brans-Dicke does not fulfil the above condition (Kuchar, Hojman, Teitelboim (1976) p. 131) (We thank A.Kamenshchik for pointing us these references).
This is the constraint algebra of Einstein’s Geometrodynamics

\[
\{ \mathcal{H}_i(x), \mathcal{H}_j(x') \} = \mathcal{H}_i(x') \partial_j \delta(x, x') - \mathcal{H}_i(x) \partial'_j \delta(x, x')
\]

\[
\{ \mathcal{H}(x), \mathcal{H}_i(x') \} = -\mathcal{H}(x') \partial'_i \delta(x, x')
\]

\[
\{ \mathcal{H}(x), \mathcal{H}(x') \} = \mathcal{H}_i(x) \partial^i \delta(x, x') - \mathcal{H}_i(x') \partial'^i \delta(x, x')
\]

Branse-Dicke theory has the following constraint algebra which show a completely different ADM-diffeomorphism group

\[
\{ \mathcal{H}_i(x), \mathcal{H}_j(x') \} = \mathcal{H}_i(x') \partial_j \delta(x, x') - \mathcal{H}_i(x) \partial'_j \delta(x, x')
\]

\[
\{ \mathcal{H}(x), \mathcal{H}_i(x') \} = -\mathcal{H}(x') \partial'_i \delta(x, x')
\]

\[
\{ \mathcal{H}(x), \mathcal{H}(x') \} = (1 + 4 \log \phi) \left( \mathcal{H}_i(x) \partial^i \delta(x, x') - \mathcal{H}_i(x') \partial'^i \delta(x, x') \right) + \left( \chi_i(x) \partial^i \delta(x, x') - \chi_i(x') \partial'^i \delta(x, x') \right) \approx \left( \chi_i(x) \partial^i \delta(x, x') - \chi_i(x') \partial'^i \delta(x, x') \right)
\]

and remember that the lapse \( N(t,x) \) has the spatial part fixed, gauge only in time \( t \).
Brans-Dicke has a Hamiltonian theory non equivalent to Einstein General Relativity.

Falls and Herrero-Valea (EPJC July 2019, arXiv:1812.08187) has noted that the non-equivalence (Jordan/Einstein) is in the measure of the Path-Integral, but the problem is already at the classical level since there are two inequivalent Hamiltonian theories.

Benedetti and Guarnieri (New J. Phys. 16 2014) studies Brans-Dicke theory in the framework of asymptotic safety and in the limit $\omega = 0$ and found inequivalence with $F(R)$ theory at quantum level although there is an equivalence at the classical level.
Ohta (Progress of Theoretical and Experimental Physics, 3 March 2018) claims a full equivalence of Jordan and Einstein Frame at classical and quantum level.

From our point of view the quantum measure (for simplicity in the classical mechanical case)

\[
\int Dq(t)Dp(t)\exp\left(\frac{i}{\hbar} \int dt \left( pdq - H(q,p) dt \right)\right)
\]

is invariant only under canonical transformations. Therefore if Jordan and Einstein frames are non connected by canonical transformations the equivalence does not hold.
CONCLUSIONS

- RG improved Einstein-Hilbert action with non geometrical field $G(x) \Lambda(x)$ breaks diffeomorphism invariance at classical level (second class constraints).

- ADM-analysis of Brans-Dicke theory has a good geometrodynamical behavior (first class constraints) but is different from Einstein General Relativity constraint algebra.

- Some studies have already singled out the inequivalence between Jordan and Einstein Frames at Quantum Level (Benedetti & Gualtieri, Falls and Herrero Valea).

- We have shown the inequivalence (B-D/E-H and then Jordan/Einstein) exists already at classical level, because their Hamiltonian theories are inequivalent (cannot be related by canonical transformations), which hints to the Quantum Inequivalence as well.