Quantum Field Theory I

Assignment#0

This sheet is to review relativistic four-vector notation, the notion of tensors and related concepts, which we will use extensively throughout the course.

Exercise 1: Spacetime coordinates and Lorentz transformations in the index notation

Motivation: Quantum field theory aims to reconcile quantum mechanics with special relativity. In the latter, we are interested in covariant and contra-variant vectors that transform with Lorentz transformations, which we denote with lower and upper indices. This exercise aims to remind you of the index notation which will be used extensively throughout this course. If you have not covered it in previous courses, you can read A. Hebecker's notes on Electrodynamics Chapter 1.2 &1.3 (in German), D. Tong's notes on Electromagnetism Ch. 4, or S. Weinberg's "Quantum Theory of Fields" Vol. 1, Chapter 2.3

Recall that in classical mechanics one talks about vectors $\vec{x}, \vec{y} \in \mathbb{R}^3$. The length of the difference of these vectors $|\vec{x}-\vec{y}|^2 = (x^1-y^1)^2 + (x^2-y^2)^2 + (x^3-y^3)^2$ is invariant under rotations and translations. In special relativity, time and space are both coordinates of 4 dimensional spacetime $\mathbb{M} = \mathbb{R}^{1,3}$. Points in this spacetime are sometimes called "events" and their distance from the origin is a 4-vector (i.e., a vector in $\mathbb{M}$ with 4 components) denoted as $x = (t, \vec{x}) \equiv (x^0, \vec{x}) \equiv \{x^\mu\}$ with $\mu = 0, 1, 2, 3$. The length of the difference of two such 4-vectors is defined as

$$|(x-y)|^2 = +(x^0-y^0)^2 - |\vec{x}-\vec{y}|^2, \tag{1.1}$$

which is invariant under spatial rotations, translations and Lorentz boosts. In the index notation covariant 4-vectors always have upper indices. One can also introduce the (Minkowski) metric tensor

$$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.2}$$

with which we define co-vector components denoted with lower indices, $x_\mu \equiv \eta_{\mu\nu} x^\nu$, i.e., we can lower indices (only!) with contractions with the metric $\eta_{\mu\nu}$. [a]

a) Calculate the components x_μ and the quantity $x_\mu x^\mu$.

b) Similarly we can define another object, denoted as $\eta^{\mu\nu}$, with which we can raise indices, i.e.,

$$x^\mu = \eta^{\mu\nu} x_\nu \; . \tag{1.3}$$

Show that $\eta^\mu{}_\nu = \delta^\mu{}_\nu$ (i.e. that $\eta^{\mu\nu}$ is the inverse metric) using eq. (1.3) and calculate the components $\eta^{\mu\nu}$.

Equation (1.1) expresses that all "inertial" frames of reference are equivalent under rotations, translations or boosts since they do not alter the lengths of 4-vectors. The set of all these coordinate transformations $x^\mu \to x'^\mu$ are collectively called "Poincaré transformations" and take the form

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu. \tag{1.4}$$

c) Using eq. (1.1) and requiring that this expression is invariant under Lorentz transformations, show the defining relation of the Lorentz transformations is

$$\eta_{\mu\nu}\Lambda^{\mu}{}_{\kappa}\Lambda^{\nu}{}_{\lambda} = \eta_{\kappa\lambda}. \tag{1.5}$$

d) Show that $(\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu}$. I know what you are thinking! Isn't $\Lambda^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu}$ true? The answer is NO! They are indeed different mathematical objects: the first acts on vectors and the other on c-vectors.

[a] Here we have used the Einstein convention where repeated upper and lower indices are summed over, i.e. $x_\mu x^\mu = x_0 \cdot x^0 + x_1 \cdot x^1 + x_2 \cdot x^2 + x_3 \cdot x^3$. This operation is also called contraction.

Exercise 2: Tensors, their Lorentz transformations, Einstein convention and index notation

Motivation: This exercise continues from the main theme of exercise 1, namely relativistic four-vector notation. The spacetime coordinate x^μ of an event that we considered in exercise 1 is the simplest example of a four-vector. Now, we generalize and review other four-vectors as well as tensors. We also review some properties of three-vectors, i.e., the spatial parts of four-vectors.

In special relativity we introduced 4-vectors as 4-tuples of numbers organised as a column. We denote the components of a vector A^μ with upper indices $\mu = 0, 1, 2, 3$

$$A = (A^0, A^1, A^2, A^3). \tag{2.6}$$

An example is the 4-velocity of a relativistic particle $u^\mu \equiv \frac{dx^\mu}{d\tau} = (c\gamma, \gamma\vec{v})$, where $\vec{v}$ is the usual 3-dim. velocity and $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$. Vectors are not just any collection of four components, but their defining property is that they transform in specific ways under a Lorentz transformation: Vectors and co-vectors transform under a Lorentz transformations $x^\mu \to \Lambda^{\mu}{}_{\nu} x^\nu$ as

$$v^\mu \;\to\; \Lambda^{\mu}{}_{\nu} v^\nu \,, \tag{2.7}$$

$$u_\nu \;\to\; \Lambda_{\nu}{}^{\mu} u_\mu \,. \tag{2.8}$$

Objects with multiple upper and lower indices are called tensors and each index transforms accordingly, i.e.,

$$T^{\mu\nu}{}_{\rho} \to \Lambda^{\mu}{}_{\kappa}\Lambda^{\nu}{}_{\lambda}\Lambda_{\rho}{}^{\sigma} T^{\kappa\lambda}{}_{\sigma} \,. \tag{2.9}$$

Just like for vectors, a tensor is not just any collection of quantities, but its defining property is the above transformation under Lorentz transformations.

a) Show that the $T^{\mu\nu}\partial_\mu\partial_\nu\phi$ is Lorentz invariant and that it vanishes if $T^{\mu\nu}$ is an antisymmetric tensor and ϕ a function of x^μ. (Hint: In the Einstein convention repeated indices are summed and thus they can be renamed, hence their name "dummy" indices. Convince yourself of that by expanding the abstract quantities $A_\mu B^{\mu\nu}$ and $A_\alpha B^{\alpha\nu}$. Note that the free indices in an equation like $A_\mu B^{\mu\nu} = C^\nu$ must always match.)

Another useful tensor apart from the metric is the Levi-Civita tensor. It can be defined as a rank d tensor in d dimensions. Its rank-3-incarnation has three indices is defined as follows:

$$\epsilon_{ijk} = \begin{cases} +1 & , \text{ for even perm. of } (i,j,k) = (1,2,3) \\ -1 & , \text{ for odd perm. of } (i,j,k) = (1,2,3) \\ 0 & , \text{ otherwise} \end{cases} \tag{2.10}$$

b) Show that for three-vectors $\vec{C}$ and $\vec{D}$ in Euclidean space the components of the cross product vector $\vec{A} \times \vec{B}$ can be written with the Levi-Civita tensor as follows

$$(C \times D)^i = +\epsilon^i{}_{jk} C^j D^k \ , \tag{2.11}$$

where we use the metric $\eta = \text{diag}\,(+1, +1, +1)$ to raise or lower indices (i.e., in 3d Euclidean space the components of the contravariant and covariant vectors do not differ by a sign).

c) Show that $\epsilon^{ijk}\epsilon_{mnk} = (\delta^i{}_m \delta^j{}_n - \delta^i{}_n \delta^j{}_m)$, $\epsilon^{ijk}\epsilon_{mjk} = 2\delta^i{}_m$, and $\epsilon^{ijk}\epsilon_{ijk} = 6$.

Another example of a 4-vector is the electromagnetic 4-potential where $A^0 = \phi$ (scalar potential) and $\vec{A} = (A^1, A^2, A^3)$ (3-vector potential). Then the electric and magnetic field can be calculated from the following:

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \ , \ \ \vec{B} = \nabla \times \vec{A} \ , \tag{2.12}$$

d) Use the index notation and the Minkowski metric to identify the components of the field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ in terms of the fields $\vec{E}$ and $\vec{B}$. Then calculate the quantity $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and show that it is Lorentz invariant.

e) Show that $\partial_\mu F^{\mu\nu} = 0$ is equivalent to the inhomogeneous Maxwell's equations in vacuum (absence of sources). Is $\partial_\mu F^{\mu\nu}$ Lorentz invariant? Does $\partial_\mu F^{\mu\nu} = 0$ hold in every inertial frame?