Quantum Field Theory I

Assignment Week # 1

Classroom Exercise 1: Group structure of Poincaré transformations

Motivation: In this exercise, you will show that the Poincaré transformations form a group, called the Poincaré group, which is a fundamental symmetry group of nature. As you can probably imagine, it is very important to understand its properties, if we want to build a relativistic quantum theory. A nice exposition is again A. Hebecker's QFT 1 notes Ch. 1.2-1.3

We saw this week that the Poincaré transformations are combinations of Lorentz transformations and spacetime translations. A Poincaré transformation is denoted as (Λ, a) where Λ is a Lorentz transformation and a^μ a translation vector. On spacetime coordinates, this transformation acts as follows

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu. \tag{1.1}$$

Additionally Λ satisfies the defining property

$$\eta_{\mu\nu} = \eta_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu. \tag{1.2}$$

a) Show that the composition of two Poincare transformations is another Poincare transformation. In other words, calculate $(\Lambda_1, a_1) \circ (\Lambda_2, a_2)$ and show that the resulting transformation satisfies equation (1.2). Is the unit matrix an element of the Poincaré group? This question is relevant, because a group needs a unit element.

b) Show that every Poincare transformation has an inverse and calculate $(\Lambda, a)^{-1}$. Hint: Compute the determinant of (1.2) and think about when a matrix is invertible.

The above three properties (combination of two elements to form a third element, existence of a unit element, existence of an inverse) together with **associativity** define a group.

Exercise 1: The Lorentz group and its Lie algebra

Motivation: In our quest for a relativistic quantum theory, it is essential to study the structure of the symmetries of special relativity. In this exercise, we show that Poincare transformations (i.e., Lorentz transformations + spacetime translations) form a group and study some of its properties. A clear and complete reference is S. Weinberg's "Quantum Field Theory of Field" Vol. 1 Chapter 2.3-2.4, although keep in mind that Weinberg uses the convention with $\eta = \mathrm{diag}(-1, +1, +1, +1)$

Focusing on the Lorentz transformations ($\alpha = 0$), these form a group of spatial rotations and spacetime boosts in 4D Minkowski space denoted as $SO(1,3)$. This group is continuous and it belongs to the special class of matrix Lie groups.
In quantum mechanics we are interested in transformations of state vectors in a Hilbert space (roughly a vector space with an inner product) and especially the unitary transformations which

preserve inner products, since for $|\Psi'\rangle = U|\Psi\rangle$

$$\langle\Psi'|\Psi'\rangle = \langle\Psi|U^\dagger U|\Psi\rangle = \langle\Psi|\Psi\rangle\,, \quad \text{if U is unitary} \tag{2.3}$$

In relativistic quantum mechanics, we are naturally interested in the Hilbert space ($\mathcal{H}$) transformations $U(\Lambda)$ induced by Lorentz transformations. These map, for example, momentum eigenstates $|p\rangle$ to eigenstates with $|\Lambda p\rangle$. Then $U(\Lambda)$ should be a **unitary representation of the Lorentz group**, i.e.

$$U(\Lambda)\circ U(\Lambda') = U(\Lambda\circ\Lambda') \;\; \& \;\; U(\mathbb{I}_{\text{Lorentz}}) = \mathbb{I}_{\mathcal{H}}. \tag{2.4}$$

You also learned that every Lie group has an associated Lie algebra. In our case this is the **the Lorentz algebra**, which generates those (finite) Lorentz transformations that can be (continuously) connected to the identity, through the exponential map[a]. In this exercise we are going to determine this algebra.

a) Use the defining relation (1.2) to show that an infinitesimal Lorentz transformation

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \;\;, |\omega| \ll 1, \tag{2.5}$$

has only 6 independent components. The infinitesimal version of the representation $U(\Lambda)$ near the identity element can then be written as [b]

$$U(1+\omega) = \mathbb{I}_{\mathcal{H}} + \frac{i}{2}\omega_{\mu\nu}\Pi^{\mu\nu} + \mathcal{O}(\omega^2), \tag{2.6}$$

where the overall $\frac{i}{2}$ factor is conventional. The operators $\Pi^{\mu\nu}$ (you can think of them as 4×4 matrices) are called the **generators of the Lorentz algebra**. What is the symmetry of $\Pi^{\mu\nu}$? How many independent generators are there? What is the dimension of the Lorentz algebra $\mathbf{so(1,3)}$?

b) Use the representation relation (2.4) to evaluate

$$U^{-1}(\Lambda)\circ U(1+\omega)\circ U(\Lambda) \tag{2.7}$$

and show the relation

$$U^{-1}(\Lambda)\Pi^{\rho\sigma}U(\Lambda) = \Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu\Pi^{\mu\nu} \tag{2.8}$$

This relation essentially shows that the $\Pi^{\mu\nu}$ operator transforms as a tensor.

c) Now expand the remaining Lorentz transformations in (2.8) to prove the following defining relation of the $\mathfrak{so}(1,3)$ Lie algebra:

$$i\left[\Pi^{\mu\nu},\Pi^{\rho\sigma}\right] = \eta^{\nu\rho}\Pi^{\mu\sigma} - \eta^{\mu\rho}\Pi^{\nu\sigma} - \eta^{\sigma\mu}\Pi^{\rho\nu} + \eta^{\nu\sigma}\Pi^{\rho\mu}. \tag{2.9}$$

d) Show that the components of the tensor Π

$$J^i = \frac{1}{2}\epsilon^i{}_{jk}\Pi^{jk} \;\; \& \;\; K^i = \Pi^{i0}, \tag{2.10}$$

with $i,j,k = 1,2,3$ and ϵ_{ijk} the Levi-Civita tensor, generate rotations and boosts respectively.

e) A representation of the Lorentz transformation that you might be familiar with, is the one acting on functions of spacetime through the differential operators

$$\Pi^{\mu\nu} = -i\left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}\right), \tag{2.11}$$

with $\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}}$, which satisfy the commutation relations (2.9). Hint: This is similar to quantum mechanics in position space (or Schrödinger representation) where the momentum operator is represented as a spatial derivative. Consequently, to verify the commutation relations, you need to use a test function that the commutator acts on.

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[a] The Lorentz transformations that can be continuously connected to the identity form the subgroup of the Lorentz group that has $\det\Lambda = 1$ and $\Lambda^{0}{}_{0} \geq +1$, called the **proper orthochronous Lorentz group** and this is the one we are interested in at the moment. The transformations that do not belong to this subgroup, are the discrete transformations of **space inversion (parity)** and **time-reversal**.

[b] Note we use the symbol $\Pi^{\mu\nu}$ for the algebra generators instead of $M^{\mu\nu}$ used in the lecture.

Exercise 2: Boundedness and Stability – A curious example

Motivation: As you discussed in the lecture, higher derivative Lagrangians correspond to Hamiltonians which are unbounded from below. This is Ostrogradsky's theorem and it has been a fundamental guiding principle for the construction of fundamental theories. However, it might be too restrictive and we might miss interesting and still meaningful physics coming from higher derivative theories. In particular, the intuition that a Hamiltonian which is unbounded from below, results in an unstable system, is not always correct. Below we see such an example. (As you will see, the example avoids instability in a "trivial" way. There are, however (more complicated) examples that exhibit the same property.)

Consider a higher derivative oscillator system with a Lagrangian that depends on the position, its first derivative and its second derivative, i.e., $\mathcal{L}(\ddot{x}, \dot{x}, x)$ with

$$\mathcal{L}\left(\ddot{x}, \dot{x}, x\right) = -\frac{\epsilon m}{2\omega^2}\ddot{x}^2 + \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2 \tag{3.12}$$

where we have introduced a dimensionless parameter ϵ that quantifies the deviation of this system from the normal harmonic oscillator ($\epsilon = 0$).
We are now going to compute the Hamiltonian of this system and show that it is unbounded from below. For this we are going to use the generalized Legendre transform

$$\mathcal{H} = \sum_i \dot{X}_i P_i - \mathcal{L} \tag{3.13}$$

where X_i, P_i are the canonical phase space variables of the theory, i.e., generalized positions and their conjugate momenta. In the more general case of higher-derivative theories, the phase space variables (as introduced by Ostrogradsky[a]) are the following:

$$X_1 = x \quad , \quad P_1 = \frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\ddot{x}}\right) \tag{3.14}$$

$$X_2 = \dot{x} \quad , \quad P_2 = \frac{\partial\mathcal{L}}{\partial\ddot{x}} \tag{3.15}$$

a) Use the definitions (3.14) and calculate the corresponding Hamiltonian of the Lagrangian (3.12). Show that it takes the form

$$\mathcal{H} = P_1X_2 - \frac{\omega^2}{2\epsilon m}P_2^2 - \frac{m}{2}X_2^2 + \frac{m\omega^2}{2}X_1^2 \tag{3.16}$$

In this last calculation you can easily see, where the unboundedness comes from. It comes from the first term in (3.16) which is linear in P_1 and hence unbounded; it can become arbitrarily positive or negative. Such a term exists for any higher derivative theory in classical mechanics, leading to **Ostrogradsky's theorem**.
Up to this point, we have convinced ourselves that introducing higher-order derivatives in the Lagrangian is generally a bad idea, because we would prefer Hamiltonians to be bounded from below. The reason is that if the Hamiltonian is unbounded from below, then the time-evolution of the system might not stable. We expect that when the mode associated to P_1 can contribute to $\mathcal{H}$ arbitrarily negatively, then the other mode can grow exponentially, even if energy (i.e., the total Hamiltonian) is conserved. This intuition is the reason that people sometimes refer to "Ostrogradsky instabilities".
However, we want to be careful and avoid wrong conclusions. Here, we look at one example (further examples exist in classical mechanics and also in classical field theories) where an unbounded Hamiltonian does not necessarily imply an unstable time evolution. In fact, the system we are considering here is a counterexample. To see this, we now solve the equations of motion.

b) Calculate the Euler-Lagrange equations to show that the equation of motion of this classical system is the following:

$$\frac{\epsilon}{\omega^2}\ddddot{x} + \ddot{x} + \omega^2 x = 0 \tag{3.17}$$

Hint: Calculate the variation of the action

$$\delta S[x, \dot{x}, \ddot{x}] = S[x + \delta x, \dot{x} + \delta\dot{x}, \ddot{x} + \delta\ddot{x}] - S[x, \dot{x}, \ddot{x}] \tag{3.18}$$

and apply Hamilton's principle, i.e., set the variation to zero. The usual formula of the Euler-Lagrange equations does not apply for higher-derivative theories. Instead, a generalization applies where you have to derive the Lagrangian with respect to $\ddot{x}$ as well.

c) Assuming a general solution of the form

$$A\cos(kt) + B\sin(kt)\ , \tag{3.19}$$

show that a general solution admits 2 different frequencies

$$k_\pm = \omega\sqrt{\frac{1 \mp \sqrt{1 - 4\epsilon}}{2\epsilon}} \tag{3.20}$$

This would mean that the general solution takes the form

$$x(t) = A\cos(k_+ t) + B\sin(k_+ t) + C\cos(k_- t) + D\sin(k_- t) \tag{3.21}$$

Is this solution stable?

d) How many initial conditions do we need to specify for a unique solution? Find the constants A, B, C, and D in terms of the initial value data.

e) Show that the Hamiltonian of the system takes the form

$$\mathcal{H} = \frac{m}{2}\sqrt{1 - 4\epsilon}\,k_+^2\left(A_+^2 + B_+^2\right) - \frac{m}{2}\sqrt{1 - 4\epsilon}\,k_-^2\left(A_-^2 + B_-^2\right) \tag{3.22}$$

This calculation shows that the oscillating k_+-modes are "physical" since they contribute positevely to the total energy. The k_- oscillating modes contribute negatively and hence they are considered unphysical (we call them "ghost modes").

Our example is, in fact, a bit trivial. The theory is well behaved because the "physical" k_+ modes are decoupled from the "ghost modes", i.e., our system is actually two decoupled/independent oscillators in disguise.

Generically (but not always) when a coupling between such modes is introduced, the system becomes unstable. This is the reason why we stick with second-order time derivatives throughout this lecture. It is, however, a subject of ongoing research, whether this condition can be relaxed; and examples that it can exist in classical mechanics and in field theory, but (not yet) in quantum field theory. So you see that even in the most basic assumptions that we are making, there is actually ongoing research to understand whether or not these assumptions can be relaxed.

[a] You might wonder where the definitions (3.14) come from. The answer lies in the boundary terms introduced by the variational principle. The conjugate momenta are the coefficients of the endpoints of the variations $\delta x(t)$ and $\delta \dot{x}(t)$. The reason for identifying these coefficients with the canonical momenta is deep and lies in the symplectic nature of the phase-space geometry.