Quantum Field Theory I

Assignment Week 2

Classroom Exercise 1: Active vs. Passive transformations

Motivation: The goal of this exercise is to clarify the geometric notion of passive and active transformations and generalize them in the context of a Hilbert spaces under the action of a symmetry group. Our prototypical example will be again the Poincaré group.

Imagine you are in the national gallery looking at your favourite painting. As you stand in front of it you realize that the painting is tilted and not perfectly aligned with the floor. You decide to break the rules and attempt to fix it yourself by rotating the painting, the object of observation, accordingly. You just performed an **active transformation on the system** you wanted to observe.

Alternatively, you could follow the rules of the museum and not touch any of the exhibits. You decide then to change your point of view and instead of rotating the painting, you tilt your head accordingly so that your eyes (the observational apparatus) are aligned with the painting. You have now changed the way you observe or in other words you performed a passive transformation. you can think of this passive transformation as a change of the coordinate system that the observer uses.

Most importantly, for you the observer both solutions are equivalent since you end up perceiving the painting in the correct orientation, aligned with your eyes!

A more mathematical version of these passive and active transformations involves 3 dimensional vectors, which in the analogy above could be the position vector of one of the corners of the painting. Let a vector be

$$\vec{v} = v^1 \hat{e}_1 + v^2 \hat{e}_2 + v^3 \hat{e}_3 \, . \tag{1.1}$$

This vector can be transformed in two different ways, one can change the basis vectors (i.e. the way we measure its components) or one can change the vector components themselves while keeping the basis unchanged. The former is the passive transformation while the latter is active transformation.

a) Perform an infinitesimal rotation of the basis vectors around the z-axis and write the transformed vector. Now transform instead the vector itself while keeping the basis fixed.

b) What happens to the vector if we do both types of transformations, active and passive?

Another example is the Hilbert space of quantum states together with a symmetry group action. Suppose we have a state $|\Psi\rangle$ in a Hilbert space that transforms under a symmetry transformation (e.g. Poincaré) via an operator U,

$$|\Psi'\rangle = U|\Psi\rangle \quad \& \quad \langle\Psi'| = \langle\Psi| U^\dagger \tag{1.2}$$

Let also $\mathcal{O}$ be an operator representing a measurement (i.e. an observable). The observable quantity associated to $\mathcal{O}$ is its expectation value

$$\langle\mathcal{O}\rangle = \langle\Psi|\mathcal{O}|\Psi\rangle \tag{1.3}$$

c) Assuming that U belongs to a unitary representation of a symmetry group, we demand that $\langle\Psi|\mathcal{O}|\Psi\rangle$ is invariant under U, meaning that

$$\langle\Psi|\mathcal{O}|\Psi\rangle = \langle\Psi'|\mathcal{O}'|\Psi'\rangle \ . \tag{1.4}$$

Given (1.2) and (1.4), what is the transformation rule of $\mathcal{O}$ under U? Is this transformation (1.2) passive or active? What about the transformation of the state vectors $|\Psi\rangle$?

Exercise 1: The Poincaré algebra and its Casimir operators

Motivation: In the lecture, we learned that the reason that fields and the corresponding particles are categorized by their mass and spin lies in the structure of the Poincare group. More specifically, this is determined by the two Casimir operators, which commute with all elements of the Lie algebra. Their values provide quantities that remain unchanged under the action of all generators, i.e. under all possible symmetry transformations of Minkowski spacetime. The Casimir operators are therefore central objects that we will investigate in more depth in this exercise. We also saw in sheet 1 that physical states on a Hilbert space must transform unitarily under the action of the Poincaré group. As a result, the possible physical states (i.e. particle states) can be classified via the unitary irreducible representations of the Poincaré group which can be understood using the Casimir operators of the corresponding Lie algebra.

In order to form the Poincaré algebra, we need to extend the Lorentz algebra that you derived in the last exercise sheet, by adding the generators of spacetime translations. You already know these generators, they are the Hamiltonian and the 3-momentum. We gather these operators and form the 4-momentum operator

$$P^\mu = \left(P^0, P^1, P^2, P^3\right)^T \tag{2.5}$$

We will denote a unitary representation of a Poincaré transformation $x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu$ acting on a Hilbert space $\mathcal{H}$ as $U(\Lambda, a)$. We write the infinitesimal Poincaré transformation on states as follows

$$U(1+\omega,\epsilon) = \mathbb{I}_{\mathcal{H}} + \frac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma} + i\epsilon_\mu P^\mu + \dots \quad \text{, for} \quad |\omega|, |\epsilon| \ll 1 \tag{2.6}$$

a) Similar to what we did in Sheet 1, use the defining property of a group representation, which says that the multiplication law in the group is respected by its representation,

$$U(\Lambda, a) \circ U(\Lambda', a') = U\left((\Lambda, a) \circ (\Lambda', a')\right) \tag{2.7}$$

to show the following relations:

$$U^{-1}(\Lambda, \Lambda a) M^{\rho\sigma} U(\Lambda, \Lambda a) = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \left(M^{\mu\nu} - a^\mu P^\nu + a^\nu P^\mu\right) \ , \tag{2.8}$$

$$U^{-1}(\Lambda, \Lambda a) P^\rho U(\Lambda, \Lambda a) = \Lambda^\rho{}_\mu P^\mu \ . \tag{2.9}$$

b) Show that the commutation relations involving the 4-momentum operator are the following:

$$i\left[P^\mu, M^{\rho\sigma}\right] = \eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho \ , \tag{2.10}$$

$$i\left[P^\mu, P^\nu\right] = 0 \ . \tag{2.11}$$

In the lecture you learned about the Casimir operators, i.e., those operators that commute with every Lie algebra generator. These operators are very important since their eigenspaces are invariant subspaces of our Hilbert space. In less fancy words, it means that states in the same eigenspace of a Casimir are not mixed with states of a different eigenspace under the action of the Poincaré group. This means that the eigenvalues of the Casimir operators cannot be changed by translations, rotations or boosts. In other words, they constitute quantities that are *invariant* characterizations of a state. They furnish an **irreducible representation of the Poincaré symmetry group**. Hence, knowing the Casimir operators and their possible eigenvalues is crucial for classifying all irreducible representations of our symmetry group.

In the lecture, you already found one Casimir, the squared momentum $P_\mu P^\mu$, whose (continuous) eigenvalue we denote as m^2 and call it "mass". Therefore, we classify the (unitary) irreducible representations of the Poincaré by the mass of the corresponding states.

Another operator that you introduced is the square of the Pauli-Lubanski-pseudovector $W^2 = W_\mu W^\mu$ where

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma \tag{2.12}$$

c) Find the commutation relation of the Pauli-Lubanski pseudovector with the 4-momentum operator and show that the operator $W^2 = W_\mu W^\mu$ commutates with the 4-momentum operator.

d) Compute W^2 in terms of the generators P^μ and $M^{\rho\sigma}$ only (i.e. no Levi-Civita symbols around) and then show that it commutes with the Lorentz generators $M^{\rho\sigma}$.

Exercise 2: Application of Noether's theorem

Motivation: This exercise serves as a reminder on the applications of one of the most (if not the most) important theorem of physics, Noether's theorem. Here, we will focus on Lorrentz boosts, since you already covered spacetime translations in the lecture. In later weeks we will also cover other examples of symmetries.

In class, you formulated the celebrated Noether's theorem in classical field theory, which states that, given a global symmetry of the action, there is always. For example, you saw that symmetry under time-translation leads to the conservation of energy (the Hamiltonian) and symmetry under translation leads to the conservation of 3-momentum. Overall this translates as the conservation of the energy-momentum tensor which is the associated conserved current of spacetime translations. However, as we have already mentioned, Lorentz boosts and spatial rotations are also symmetries of any relativistic theory. Therefore they should also have associated conserved currents. We are going to calculate these currents for the free theory of real scalar fields described by the action:

$$S = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right] \tag{3.13}$$

a) Using the fact that scalar fields transform in the trivial representation of the Lorentz group, i.e.

$$\phi'(x) = \phi(\Lambda^{-1}x)\ , \tag{3.14}$$

calculate the infinitesimal variation $\delta_\omega \phi$ induced by an infinitesimal Lorentz transformation with parameter $|\omega| \ll 1$.

b) Show that the current associated to Lorentz symmetry, i.e. associated to the infinitesimal transformation of the field in the previous question, is

$$J^{\mu,\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} \ , \tag{3.15}$$

where $T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}$ is the energy momentum tensor of theory, and show that it is conserved, i.e. $\partial_\mu J^{\mu,\rho\sigma} = 0$. Here the comma in the indices of $J^{\mu,\rho\sigma}$ does not mean a partial derivative but merely distinguishes the μ-index as the one along which the derivative should be contracted with.