Quantum Field Theory I

Assignment Week 6

Classroom Exercise 1: Spontaneous symmetry breaking

Motivation: Here we present a toy example of spontaneous symmetry breaking. This mechanism has very important applications in the theory of phase transitions. In particle physics, it provides the mechanism under which the fundamental particles acquire mass, known as the Higgs mechanism. In QCD, symmetry breaking is behind hadronization. As a result, it is important to see how this mechanism plays out in field theory.

Consider a theory of a 3-component real scalar field $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ and a complex scalar χ, with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})^2 + \frac{1}{2}\mu^2 \vec{\phi}^2 - \frac{1}{4}\lambda \left(\vec{\phi}^2\right)^2 + (\partial_\mu \chi)^\dagger (\partial^\mu \chi) - g\, \vec{\phi}^2\, \chi^\dagger \chi, \tag{1.1}$$

where $\lambda > 0, g > 0$.

a) Set $\chi = 0$ for now and identify the potential of the real scalar field. Show that this potential is minimized for

$$\vec{\phi}^2 = v^2 \equiv \frac{\mu^2}{\lambda} \tag{1.2}$$

The above relation is satisfied by the vacuum of the theory, and the $O(3)$ global symmetry allows us to choose the vacuum state to be $\vec{\phi}_0 = (0, 0, v)$ without loss of generality. You see already that the vacuum is not $O(3)$-symmetric anymore, the symmetry was broken because the field acquired a non-zero vacuum expectation value. Do we have any remaining symmetry though?

b) Now consider fluctuations of the ϕ field around the vacuum, i.e.,

$$\vec{\phi} = (0, 0, v) + \delta\vec{\phi}\,. \tag{1.3}$$

Expand the χ-part of the Lagrangian using Eq. (1.3) and show that the χ-field obtains a mass-term and specifically the mass parameter

$$m_\chi^2 = g v^2. \tag{1.4}$$

c) Now write explicitly the fluctuation $\delta\vec{\phi}$ in terms of components

$$\delta\vec{\phi}(x) = (\pi_1(x), \pi_2(x), h(x))\,, \tag{1.5}$$

Expand the purely $\vec{\phi}$-part of the Lagrangian to quadratic order in the fluctuations, according to the above expansion and show that the dynamical mode $h(x)$ which is longitudinal to the vacuum acquires a mass

$$m_h^2 = 2\lambda v^2\,, \tag{1.6}$$

while the fluctuations (π_1, π_2), which are orthogonal directions to the vacuum, are massless. These massless fluctuations are known as Goldstone modes and they appear whenever a global symmetry is spontaneously broken (this is also known as Goldstone's theorem).

Exercise 1: The quantum effective action of the $O(N)$-model

Motivation: In this exercise we review and generalize the calculation of the effective potential for the $O(N)$ - model. This is a useful calculation to understand how the computation goes through with multiple degrees of freedom. It is also important for your intuition regarding the concept of renormalization, which is of paramount importance in Quantum field theory.

In the lecture, you introduced the **quantum effective action** $\Gamma[\langle\phi\rangle]$ as the functional that allows us to extract the equations that define the dynamics for the expectation value of the field operators. These expectation values behave classically in the sense that they satisfy deterministic equations of motion that however have taken into account quantum fluctuations. In this exercise, we see how quantum effects affect the shape of the effective potential of the $O(N)$-model or in other words the multi-scalar field theory.

You also calculated the effective potential $V_{\text{eff}} \equiv -\frac{\Gamma[\langle\phi\rangle]}{\mathcal{V}}$ whose minimum defines the vacuum (or vacua in the case of many minima) around which our quantum fields fluctuate. Here we extend this calculation to the slightly more complicated case of the $O(N)$ model, whose Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})^2 - \frac{1}{2}\kappa\vec{\phi}^2 - \frac{1}{4}\lambda\left(\vec{\phi}^2\right)^2 \ , \tag{1.7}$$

where $\lambda > 0$ and κ has an undefined sign for now[a]. We start by writing the the classical action as an expansion $S[\vec{\phi}]$ as an expansion around $S[\vec{\phi}_{\text{cl}}]$ where $\vec{\phi}_{cl}$ is a solution of the classical equations of motion. To that end,

a) Calculate the equations of motion for the components of the classical solution, ϕ^i_{cl}.

Now one can attempt to calculate the path-integral by expanding around a classical solution. However, one cannot obtain an exact expression since our action is not quadratic and so we cannot use Gaussian integrals. To understand this, consider fluctations of the fields around a classical solution by writing $\phi^i = \phi^i_{\text{cl}} + \varphi^i$. Then the action can be written as follows:

$$S[\phi_{\text{cl}} + \varphi] = S[\phi_{\text{cl}}] + \int d^4x d^4y \varphi^i(x) \frac{\delta^2 S}{\delta\varphi^i(x)\varphi^j(y)}\varphi^j(y)\Big|_{\phi=\phi_{\text{cl}}} + \dots \ . \tag{1.8}$$

For quadratic actions the above expansion is exact, but in our case, it is only approximate. Nevertheless, we are going to ignore higher orders, in what is called the "1-loop approximation" for reasons that will become clear when we introduce Feynman diagrams. For now,

b) Show that the second functional derivative of the action is given by

$$\begin{aligned}\frac{\delta^2 S}{\delta\varphi^i(x)\delta\varphi^j(y)}\Big|_{\phi=\phi_{\text{cl}}} \quad &= \quad \Box \cdot \delta^{(4)}(x-y)\delta^{ij} + \kappa \cdot \delta^{(4)}(x-y)\delta^{ij} \\ &\quad +\lambda\left(|\vec{\phi}_{\text{cl}}(x)|^2 + 2\phi^i_{\text{cl}}(x)\phi^j_{\text{cl}}(x)\right)\delta^{(4)}(x-y) \ ,\end{aligned} \tag{1.9}$$

c) Use the Gaussian integrals we introduced last week to show that the generating functional at vanishing sources $\mathcal{Z}[0]$ is given by

$$\mathcal{Z}[\,0\,] = \exp\left[\frac{i}{\hbar}S[\vec{\phi}_{\text{cl}}] - \frac{1}{2}\,\text{Tr}\left[\log\Big(\hat{D}\Big)\right]\right] \ , \tag{1.10}$$

where the functional trace is understood as a sum over all continuous or discrete indices[b] in some basis, i.e.,

$$\mathrm{Tr}\left[\hat{A}\right] = \sum\!\!\!\!\!\!\int_{\lambda,i} \langle\lambda, i|\hat{A}|\lambda, i\rangle \ , \tag{1.11}$$

and the operator $\hat{D}$ in position space is:

$$\langle y, i|\hat{D}|x, j\rangle = \left[\delta^{ij}\Box + \kappa\delta^{ij} + \lambda(|\vec{\phi}_{\mathrm{cl}}(x)|^2 + \phi^i_{\mathrm{cl}}(x)\phi^j_{\mathrm{cl}}(x))\right]\delta^{(4)}(x-y) \tag{1.12}$$

d) The global $O(N)$ symmetry allows us to parametrize the classical solution as follows:

$$\vec{\phi}_{\mathrm{cl}} = (0, \dots, \phi_{\mathrm{cl}})^{\mathrm{T}} \tag{1.13}$$

Write the functional trace explicitly in momentum space and show that

$$\begin{aligned}\frac{1}{\mathcal{V}}\mathrm{Tr}\left[\log\left(\hat{D}\right)\right] = \ & (N-1)\int \tfrac{d^4p}{(2\pi)^4}\ln\left(-p^2+\kappa+\lambda\phi^2_{\mathrm{cl}}\right) \\ & + \int \tfrac{d^4p}{(2\pi)^4}\ln\left(-p^2+\kappa+3\lambda\phi^2_{\mathrm{cl}}\right) \ ,\end{aligned} \tag{1.14}$$

where $\mathcal{V} = \int d^4x$ the spacetime volume of our theory.

Now we would like to calculate the above momentum integrals. However, these integrals are diverging for high momenta hence we would like to impose an energy cut-off Λ^2. There is a problem though; the square of the 4-momentum p can be zero or even negative while at the same time corresponding to high energies. This happens precisely because spacetime is Minkowskian rather than Euclidean. To solve this problem we perform an analytic continuation[c] called Wick rotation, where

$$t \to -i\tau \implies p^0 \to -ip^0_{\mathrm{E}} \ . \tag{1.15}$$

We then perform the calculation and at the end we Wick rotate back to obtain the QFT result. To that end,

e) Show that after Wick rotating and changing to spherical Euclidean coordinates the 1-loop contribution to the quantum effective action becomes

$$\begin{aligned}\frac{1}{\mathcal{V}}\mathrm{Tr}\left[\log\left(\hat{D}\right)\right] = \ & i\frac{(N-1)}{32\pi^2}\int_0^{\Lambda^2} dp^2_{\mathrm{E}}\left[p^2_{\mathrm{E}}\ln\left(\frac{p^2_{\mathrm{E}}+\kappa+\lambda\phi^2_{\mathrm{cl}}}{\Lambda^2}\right)\right] \\ & + i\frac{1}{32\pi^2}\int_0^{\Lambda^2} dp^2_{\mathrm{E}}\left[p^2_{\mathrm{E}}\ln\left(\frac{p^2_{\mathrm{E}}+\kappa+3\lambda\phi^2_{\mathrm{cl}}}{\Lambda^2}\right)\right] \ ,\end{aligned} \tag{1.16}$$

Where did the Λ^2 factor come from inside the logarithm?

Note that for $N = 1$ we obtain the expression you computed in the lecture (up to conventions), while for $N > 1$ we have extra contributions. Which kind of fluctuations are responsible for these extra contributions? These are usually called Goldstone modes and we are going to discuss them in the classroom exercise as well.

f) Calculate the above momentum integrals which are now finite and depend on the cutoff scale Λ^2. You should find that:

$$\begin{aligned}\frac{1}{\mathcal{V}}\,\mathrm{Tr}\left[\log\left(\hat{D}\right)\right] \;=\;& i\frac{(N-1)}{4\cdot 32\pi^2}\Bigg[\Lambda^2(2\kappa+2\lambda\phi_{\rm cl}^2-\Lambda^2)+2\Lambda^4\ln\left(\frac{\Lambda^2+\kappa+\lambda\phi_{\rm cl}^2}{\Lambda^2}\right)\\ &\qquad -2\left(\kappa+\lambda\phi_{\rm cl}^2\right)^2\ln\left(\frac{\Lambda^2+\kappa+\lambda\phi_{\rm cl}^2}{+\kappa+\lambda\phi_{\rm cl}^2}\right)\Bigg]\\ &+i\frac{1}{4\cdot 32\pi^2}\Bigg[\Lambda^2(2\kappa+6\lambda\phi_{\rm cl}^2-\Lambda^2)+2\Lambda^4\ln\left(\frac{\Lambda^2+\kappa+3\lambda\phi_{\rm cl}^2}{\Lambda^2}\right)\\ &\qquad -2\left(\kappa+3\lambda\phi_{\rm cl}^2\right)^2\ln\left(\frac{\Lambda^2+\kappa+3\lambda\phi_{\rm cl}^2}{+\kappa+3\lambda\phi_{\rm cl}^2}\right)\Bigg] \end{aligned} \tag{1.17}$$

Hint: You can calculate first the general integral

$$\int_0^{\Lambda^2} dp_{\rm E}^2\left[p_{\rm E}^2\;\ln\left(\frac{p_{\rm E}^2}{\Lambda^2}+A\right)\right]=\frac{\Lambda^4}{4}\left(-2A^2\ln\left(\frac{A+1}{A}\right)+2\ln\left(A+1\right)-1+2A\right)\;, \tag{1.18}$$

where A is a dimensionless quantity that takes the values

$$A=\begin{cases}\frac{\kappa}{\Lambda^2}+\lambda\frac{\phi_{\rm cl}^2}{\Lambda^2}\;, & \text{for Goldstone modes}\\ \\ \frac{\kappa}{\Lambda^2}+3\lambda\frac{\phi_{\rm cl}^2}{\Lambda^2}\;, & \text{for the massive scalar mode}\end{cases} \tag{1.19}$$

Assuming now that $\phi_{\rm cl}$ is a constant expectation value that corresponds to our vacuum we have that the effective potential is of the following form:

$$V_{eff} \;=\; -\frac{\Gamma[\phi_{\rm cl}]}{\mathcal{V}}=\frac{S[\phi_{\rm cl}]}{\mathcal{V}}+\frac{i}{2}\frac{1}{\mathcal{V}}\,\mathrm{Tr}\left[\log\left(\hat{D}\right)\right] \tag{1.20}$$

$$\begin{aligned} =\;& \frac{1}{2}\kappa\phi_{\rm cl}^2+\frac{\lambda}{4}\phi_{\rm cl}^4+\frac{(N-1)}{8\cdot 32\pi^2}\Bigg[\Lambda^2(2\kappa+2\lambda\phi_{\rm cl}^2-\Lambda^2)+2\Lambda^4\ln\left(\frac{\Lambda^2+\kappa+\lambda\phi_{\rm cl}^2}{\Lambda^2}\right)\\ &\qquad -2\left(\kappa+\lambda\phi_{\rm cl}^2\right)^2\ln\left(\frac{\Lambda^2+\kappa+\lambda\phi_{\rm cl}^2}{+\kappa+\lambda\phi_{\rm cl}^2}\right)\Bigg]\\ &+\frac{1}{8\cdot 32\pi^2}\Bigg[\Lambda^2(2\kappa+6\lambda\phi_{\rm cl}^2-\Lambda^2)+2\Lambda^4\ln\left(\frac{\Lambda^2+\kappa+3\lambda\phi_{\rm cl}^2}{\Lambda^2}\right)\\ &\qquad -2\left(\kappa+3\lambda\phi_{\rm cl}^2\right)^2\ln\left(\frac{\Lambda^2+\kappa+3\lambda\phi_{\rm cl}^2}{+\kappa+3\lambda\phi_{\rm cl}^2}\right)\Bigg] \end{aligned} \tag{1.21}$$

This potential is divergent for $\Lambda\to\infty$ and that limit is the one we want to take in the end if we want to take into account all quantum effects. Similarly to the lecture, if you expand the potential in powers of ϕ^2 you will see that only the coefficients of the quadratic and quartic terms diverge while the higher orders are finite. This allows us to **renormalize** the theory by redefining the parameters of the theory, that is the mass parameter κ and the coupling λ.

g) **[OPTIONAL TASK]** Expand the potential to first and second order in $\phi_{\rm cl}^2$ (you might want to use mathematica for that), and find the expressions of the renormalized mass and coupling in terms of κ,λ and Λ, using the conditions[d]:

$$\kappa_{\rm phys}=\left.\frac{\partial V_{\rm eff}}{\partial\phi_{\rm cl}^2}\right|_{\phi_{\rm cl}=0}\;,\quad \lambda_{\rm phys}=\left.\frac{\partial^2 V_{\rm eff}}{\partial\phi_{\rm cl}^2\partial\phi_{\rm cl}^2}\right|_{\phi_{\rm cl}=0} \tag{1.22}$$

The above calculation reveals something crucial. If you want your new parameters to be finite in the limit $\Lambda \to \infty$ then the original parameters κ and λ are really functions of the cutoff Λ that would cancel the divergent contributions of the quantum fluctuations. In other words, we can add particular counter-terms in the original Lagrnagian of the form

$$\delta\kappa\,\phi^2 \;\;,\;\; \delta\lambda\,\phi^4 \tag{1.23}$$

where the coefficients $\delta\kappa$ and $\delta\lambda$ are diverging in the limit $\Lambda \to \infty$. You have actually figured out already how these counter terms should look like, haven't you?

If we proceed as above and absorb the divergences properly into the new parameters κ_{phys} and λ_{phys} ending up with a potential of the form

$$V_{\text{eff}} = \frac{\kappa_{\text{phys}}}{2}\phi_{\text{cl}}^2 + \frac{\lambda_{\text{phys}}}{4}\phi_{\text{cl}}^4 + V_{\text{finite}}^{1-\text{loop}} \;, \tag{1.24}$$

or more explicitly:

$$\begin{aligned} V_{\text{eff}} \;=\; & \frac{1}{2}\kappa_{\text{phys}}\phi_{\text{cl}}^2 + \frac{\lambda_{\text{phys}}}{4}\phi_{\text{cl}}^4 + \frac{1}{4\cdot 32\pi^2}\Bigg[(\kappa_{\text{phys}} + 3\lambda_{\text{phys}}\phi_{\text{cl}}^2)^2 \ln\left(\frac{\kappa_{\text{phys}} + 3\lambda_{\text{phys}}\phi_{\text{cl}}^2}{\mu^2}\right) \\ & + (N-1)(\kappa_{\text{phys}} + \lambda_{\text{phys}}\phi_{\text{cl}}^2)^2 \ln\left(\frac{\kappa_{\text{phys}} + \lambda_{\text{phys}}\phi_{\text{cl}}^2}{\mu^2}\right)\Bigg] \;, \end{aligned} \tag{1.25}$$

where μ is some physical scale of the system (e.g. the mass or vacuum expectation value of the field). It comes out of the renormalization process which we will discuss in detail later in the course.
Note that if we set $N = 1$ we obtain the result of the lecture up to our convention ($3\lambda_{\text{here}} = \lambda_{\text{lecture}}$). The extra fields give extra contributions to the effective potential similar to the case of $N = 1$.

h Calculate the minimum of the effective potential as a function of $\kappa_{\text{phys}}, \lambda_{\text{phys}}$ and μ. Use a plotting software and see what happens as we increase the number of scalar fields. How does the potential changes upon introducing quantum corrections in the case where $\kappa_{\text{phys}} < 0$ and $\kappa_{\text{phys}} > 0$?

i) For the case of $\kappa = 0$ write the form of the potential and find the new minimum. Observe that the minimum is induced by a cancellation between a classical leading contribution and the quantum correction. Is this a problem? What does this mean for our theory?

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[a] For $\kappa > 0$ we can write as a square of a number that would correspond to the mass parameter. But if $\kappa < 0$ then this naively looks like an imaginary mass.

[b] The states here carry an extra index associated to the internal $O(N)$-symmetry, or in other words to the different/distinguishable scalar fields.

[c] This extends the domain of a function to complex numbers such that the complex function is analytic. Such a condition is so restrictive that such continuation is unique. Here we will assume that we can use this technique to assign a meaningful value to an otherwise divergent integral. Another example of such assignment is the Cauchy principal-value for divergent real integrals with infinite boundaries.

[d] These are called **renormalization conditions** and essentially prescribe how the finite parameters should be defined. The choice of conditions is actually not unique.