Quantum Field Theory I

Assignment Week 7

Classroom Exercise 1: Gauge ~~symmetry~~ redundancy

Motivation: Gauge theories have proven of outmost importnace for our understanding of nature. However, gauge symmetry is not really a symmetry of nature, but rather a redundancy of our description of nature. We would like to clarify this further.

Gauge transformations in electromagnetism take the familiar form

$$A_\mu(x) \to A_\mu(x) - \partial_\mu \lambda(x) \; . \tag{1.1}$$

Think of the following questions from your classical electrodynamics course:

i) How do the electric and magnetic fields transform under (1.1)?

ii) Which objects are measurable, the fields or the potentials?

iii) Are two configurations A and A' physically equivalent or not?

Now let us do some counting. Think of how many degrees of freedom are encoded in a general vector field A_μ and impose the Lorentz gauge $\partial_\mu A^\mu = 0$, how many degrees of freedom are left? Finally, there is an additional gauge redundancy even after imposing the Lorentz gauge. Show that the Lorentz condition is invariant under transformations with,

$$\Box \lambda(x) = 0 \; . \tag{1.2}$$

How many degrees of freedom do we have after we make a choice of $\lambda(x)$ satisfying Eq.(1.2)? How many polarizations do the photons have?

Exercise 1: Gauge invariant Quantities

Motivation: We have seen already in classical electromagnetism the concept of gauge invariance. In short, gauge invariance means that our degrees of freedom (the fields) are more than the ones needed to describe our physical reality. If we want to then quantize electromagnetism using the fields A_μ we need start from a gauge invariant theory, or in other words, a gauge invariant action.

In electromagnetism, we can use the 4-vector potential $A^\mu \equiv (\Phi, \vec{A})$ to write down the Maxwell equations. However, the choice of A^μ is not unique; the 4-vector potential

$$A'_\mu = A_\mu + \partial_\mu \lambda \; , \tag{1.3}$$

is actually physically equivalent to A_μ for any scalar function $\lambda \equiv \lambda(x)$. Show that,

a) The quantity $(\partial_\mu A^\mu)(\partial_\nu A^\nu)$ is not gauge invariant.

b) Show that the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is gauge invariant. This implies that the Lagrangian of electromagnetism, $\mathcal{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ is gauge invariant.

c) Show that the field strength tensor can be written in terms of covariant derivatives $D_\mu = \partial_\mu + iA_\mu$ as

$$F_{\mu\nu} = -i\left[D_\mu, D_\nu\right] \ , \tag{1.4}$$

where the right-hand side is understood on acting on some test field.

d) Convince yourself that the only quantities that is quadratic in A, Lorentz invariant, gauge invariant and that contains only two derivatives are

$$F_{\mu\nu}F^{\mu\nu} \ \ \& \ \ \tilde{F}_{\mu\nu}F^{\mu\nu} \ . \tag{1.5}$$

So here it seems that we have two candidates for the kinetic term of electromagnetism. What is going on here?

Exercise 2: The Hamiltonian in Gupta-Bleuer quantization

Motivation: In the lecture, we discussed that the longitudinal polarization, which is related to zero-norm states, does not contribute to physical states. As an example, we wrote equations 324-326 about the Hamiltonian. To practise working with the creation and annihilation operators α, which are associated to the polarization vectors, we will fill in the gaps in this derivation. In addition, the main point of this exercise is to think about the physical meaning of the result and why it makes sense the way it is.

---

a) **This part of the exercise is *optional*. If you would like to practise calculations with the field and its mode expansions, you can do this exercise. However, as an alternative to the calculation, you can also *argue* why, in analogy to the Hamiltonian for the scalar field, eq. 325 must come out for the Hamiltonian of the gauge field in Lorenz gauge. (Note that we are free to drop infinite constants, because we can only measure energy differences.**
Start from Eq. 324 in the lecture notes, i.e., write the Hamiltonian

$$H = \int d^3x \left(\Pi^\mu A_\mu - \mathcal{L}\right), \tag{2.6}$$

and use the mode expansion for the field

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{\omega_{\vec{k}}}} \sum_\lambda \left[\epsilon_\mu^{(\lambda)}(\vec{k}) a_{\vec{k},\lambda} e^{-ikx} + \left(\epsilon_\mu^{(\lambda)}(\vec{k})\right)^* a^\dagger_{\vec{k},\lambda} e^{-ikx}\right], \tag{2.7}$$

and the corresponding mode expansion for

$$\Pi_\mu(x) = F^{\mu 0} - \eta^{\mu 0}\partial_\kappa A^\kappa, \tag{2.8}$$

and show that (upon dropping infinite constants) you arrive at Eq. 325 in the lecture notes, i.e.,

$$H = \int \frac{d^3k}{(2\pi)^3} k_0 \left(-a^\dagger_{\vec{k},\mu} a^\mu_{\vec{k}}\right). \tag{2.9}$$

Think about the meaning of this result. Also remember that we had both unphysical states (with negative norm) and zero-norm states, in the Fock space generated by the a_μ's. From the expression of the Hamiltonian above, is it clear that these drop out of the expression?

b) Use the creation and annihilation operators defined by:

$$\alpha^{\dagger}_{\vec{k},(u,L,1,2)} = \epsilon^{\mu\,(u,L,1,2)}(k) a^{\dagger}_{\vec{k},\mu}, \tag{2.10}$$

with the polarization vectors as they are defined in the lecture notes and show that when rewriting the mode expansion for the field in terms of α's, the Hamiltonian will take the form as in Eq. 326, namely

$$H = \int \frac{d^3k}{(2\pi)^3} k_0 \left(\sum_{i=1}^{2} \alpha^{\dagger}_{\vec{k},i} \alpha_{\vec{k},i} - \left[\alpha^{\dagger}_{\vec{k},u} \alpha_{\vec{k},L} + \alpha^{\dagger}_{\vec{k},L} \alpha_{\vec{k},u} \right] \right). \tag{2.11}$$

Now comes the crucial part of the whole exercise: Think about what will happen to the part in angular brackets, when the expectation value of H is evaluated in a *physical* state. What does this mean and does it make sense for this to happen?