Quantum Field Theory I

Assignment Week 8

Classroom Exercise 1: Faddeev-Popov in finite dimensions

Motivation: In the lecture, you discussed the Faddeev-Popov gauge-fixing procedure of the path-integral as a way to avoid overcounting equivalent field configurations. In this exercise we will see a finite-dimensional example where we consider two-dimensional rotational symmetry as gauge symmetry.

Consider a two-dimensional integral

$$I = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \; f(x_1^2 + x_2^2). \tag{1.1}$$

If you think of the function $f(x_1^2 + x_2^2)$ as the gauge invariant $e^{\frac{i}{\hbar}S}$, then configurations described by the vectors $\vec{x} = (x_1, x_2)$ that have the same magnitude would be considered equivalent. Thus, we view the rotational symmetry which preserves $|\vec{x}|$ as a gauge symmetry, i.e., vectors that are simply rotated by some angle ϕ correspond to equivalent physical configurations.
Use the Faddeev-Popov trick to remove this “gauge degree of freedom”. To that end, insert in the integral a factor one in the form:

$$1 = \Delta_{\mathrm{FP}} \int_0^{2\pi} d\phi \; \delta(g(\phi)) \;, \tag{1.2}$$

with an appropriate gauge-fixing function $g(\phi)$. Choose the condition $g(x_1, x_2)$ such that we restrict the integration along configurations/paths (x_1, x_2) with fixed polar angle ϕ, i.e., along lines where $y/x = \text{const.}$. You can now calculate Δ_{FP} explicitly and plug it in Eq. (1.1).
Lastly, use the rotational gauge symmetry to argue why $\int_0^{2\pi} d\phi \; \delta(g(\phi)) = \int_0^{2\pi} d\phi \; \delta(x_2)$ inside Eq. (1.1) and integrate out one direction by using the δ-function.

Exercise 1: Quantization in the radiation gauge

Motivation: In the lecture you discussed Gupta-Bleuler quantization of the electromagnetic field. This procedure involves quantising the theory first and then implementing the constraints implied from the $U(1)$ gauge symmetry to determine the physical field configurations. In this exercise, we will implement the constraints first to obtain the physical fields and then proceed with the quantization of the physical degrees of freedom of the theory, i.e. we do the reverse process.

The exercise has a calculational part (a-e) and a conceptual part (f). The conceptual part is very important. Even if you do not manage to get through all calculational parts, work on (f).
We proceed in analogy with the quantization of the real scalar field. The difference comes from gauge redundancy which requires us to fix some gauge conditions (constraints). In this exercise we quantize the electromagnetic field in the radiation (or Coulomb) gauge,

$$\Phi(t, \vec{x}) = A^0 = 0 \;\; , \;\; \vec{\nabla} \cdot \vec{A} = 0 \;, \tag{2.3}$$

where Φ is the classical electromagnetic scalar potential and $\vec{A}$ the vector potential.
The free electromagnetic Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

a) Argue why imposing the commutation relations

$$\left[A_i(t,\vec{x}), \Pi^j(t,\vec{y})\right] = i\delta_i^{\ j}\delta^{(3)}(x-y) \ , \tag{2.4}$$

is **inconsistent** with the Coulomb gauge.

In order to find the consistent relations we use the following general ansatz for the relations:

$$\left[A^i(t,\vec{x}), \Pi^j(t,\vec{y})\right] = -i\Delta^{ij}\delta^{(3)}(x-y) \ , \tag{2.5}$$

where Δ^{ij} would be a tensor of rank 2 built from the Kornecker delta δ^{ij} and derivative operators ∂^i.

b) Impose the Coulomb gauge condition on Eq. (2.5) and show that the Fourier transform of the rank 2 operator Δ^{ij} satisfy the following condition

$$\tilde{\Delta}^{ij}(k) = \delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2} \ . \tag{2.6}$$

As a result the operator in position space takes the following form:

$$\Delta^{ij} = \delta^{ij} - \partial_i \frac{1}{\nabla^2}\partial_j \ , \tag{2.7}$$

where $\frac{1}{\nabla^2} \equiv (\nabla^2)^{-1}$ or in other words the Green's function of the Laplacian. The rest of the comm. relations are of course:

$$\left[A^i(t,\vec{x}), A^j(t,\vec{y})\right] = 0 = \left[E^i(t,\vec{x}), E^j(t,\vec{y})\right] \ . \tag{2.8}$$

Since we have set $A^0 = 0$, then the only dynamical field is the 3-vector $\vec{A}$ satisfying the wave equation $\Box\vec{A} = 0$. Thus, the mode expansion of the components of $\vec{A}$ in the Heisenberg picture is the known,

$$A_i(\mathbf{x},t) = \sum_{\lambda=1}^{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left[\varepsilon_i^{(\lambda)}(\vec{k})\, a_{\vec{k},\lambda}\, e^{i\mathbf{k}\cdot\mathbf{x}-i\omega_{\mathbf{k}}t} + \left(\varepsilon_i^{(\lambda)}(\vec{k})\right)^* a^\dagger_{\vec{k},\lambda}\, e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega_{\mathbf{k}}t}\right] . \tag{2.9}$$

d) Observe that the mode expansion (2.9) is different then the one in Eq. (304), because it involves already only two polarizations. Explain why here the index λ takes only two discrete values, by imposing the gauge condition on Eq. (2.9).

e) (**OPTIONAL**) Using Eq. (2.5), Eq. (2.6), and Eq. (2.9), show the commutation relations

$$\begin{aligned}\left[a_\lambda(\vec{k}), a^\dagger_{\lambda'}(\vec{k}')\right] &= (2\pi)^3\delta^3(k-k')\delta_{\lambda\lambda'}, \\ \left[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k}')\right] &= 0 = \left[a^\dagger_\lambda(\vec{k}), a^\dagger_{\lambda'}(\vec{k}')\right] \ ,\end{aligned} \tag{2.10}$$

using also the completeness relation for the polarization vectors,

$$\sum_{\lambda} \varepsilon^i_\lambda(\vec{k})\varepsilon^j_\lambda(\vec{k})^* = \delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2} \,. \tag{2.11}$$

The important bit here is that the commutation relations that we obtained are identical to the ones in the Gupta-Bleuler quantization but the internal indices λ and λ' take only two possible values corresponding to the physical (transverse) polarization of the photons.

Finally the Hamiltonian takes the form:

$$H = \sum_{\lambda} \int \frac{d^3k}{(2\pi)^3} k^0 \sum_{\lambda=1}^{2} \left(a^\dagger_{\lambda,\,\mu} a^\mu_\lambda \right) \tag{2.12}$$

f) **Non-optional, think about this even if you did not go through all calculations.** What is the advantage of this method of quantization over Gupta-Bleuler quantization? What is its disadvantage?