Quantum Field Theory I

Assignment Week 9

Classroom Exercise 1: Rotating spinors

Motivation: A nice way to think about fermions and their difference to bosons, is the behaviour of the first with respect to rotations. Usually, rotating an object by 2π you recover the same object. We will see that fermions, which are described by spinors, behave differently. One needs to do a 4π rotation of the fermion to recover it back.

You saw in the lecture that spinors transform in the spinor representation of the Lorentz algebra which can be written with the help of the γ matrices of the Clifford algebra

$$M^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] \tag{1.1}$$

Consider now a finite spatial rotations of a spinor with respect to the z-axis. These are generated by the $J^3 = M^{12}$ generator of the Lorentz algebra (see Sheet 2).

a) Calculate the generator of rotations J^3 in the spinor representation using the γ matrices in the Weyl representation. You should find that $J^3 = \frac{1}{2}\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}$, whereσ^i are the usual 2-dimensional Pauli matrices.

b) Write the finite rotation of a spinor (by an angle θ)generated by J^3. What happens if you rotate the spinor by $\theta = 2\pi$? What about $\theta = 4\pi$?

Exercise 1: Spin representation of the Lorentz group

Motivation: You mentioned in the lecture that the γ matrices satisfying the Clifford algebra, furnish a representation of the Lorentz algebra and that the γ matrices transform as a Lorentz 4-vector. In this exercise, you are going to prove these statements, which are crucial if we want to understand how to build relativistic theories quantum theories of fermions through spinors.

In the lecture, you introduced the matrices γ^μ, with $\mu = 0, 1, 2, 3$ that satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}_{4\times 4}\ , \tag{1.2}$$

for $(1+3)$ spacetime dimensions. The relevance of the Clifford algebra and its representations comes from the fact that one can build representations of the proper orthochronous Lorentz group $SO^+(1,3)$ from representations of the Clifford algebra, i.e. choice of γ matrices. The induced representation we are interested it is called the **spinor representation** and is defined as:

$$M^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] \tag{1.3}$$

a) First we show that the vector $\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$, in which each component of the vector is a matrix (in Dirac indices), transforms as a four-vector, i.e. that it transforms in the fundamental representation of the Lorentz group (recall Sheet 2, Ex.1a-1b). To that end, show that:

$$[M^{\mu\nu}, \gamma^\rho] = \left(M^{\mu\nu}_{\text{(fund)}}\right)^\rho{}_\sigma \gamma^\sigma \,, \tag{1.4}$$

where $\left(M^{\mu\nu}_{\text{(fund)}}\right)^{\rho\sigma} \equiv -i\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\nu\rho}\eta^{\mu\sigma}\right)$. Note that Eq. (1.4) is essentially the infinitesimal version of Eq. (2.9) in Sheet 1, or directly Eq. (2.10) with $P^\mu \to \gamma^\mu$.

b) Show that Eq. (1.3) defines a representation of the Lorentz algebra by showing that satisfies[a]

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho} + \eta^{\mu\sigma}M^{\nu\rho}\right) \tag{1.5}$$

Note that this is a bit of a lengthy calculation in which you need to repeatedly apply the anticommutation rule of the γ matrices however, you can simplify it significantly by using question a).

c) A relevant example for condensed matter physics is the restriction of the above in the 3-dimensional Euclidean space where $\eta^{\mu\nu} \to \delta^{ij}$ with $i, j = 1, 2, 3$. Show that the Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;,\; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;,\; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;, \tag{1.6}$$

which satisfy the known $su(2)$ algebra $[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$, satisfy also the 3-dimensional Clifford algebra (denoted as $Cl(3,0)$). This expresses the mathematical fact,

$$Cl(3,0) \cong SU(2) \cong SO(3) \tag{1.7}$$

[a] This Lorentz algebra is different up to a minus sign to the one we showed in Sheet 1. This is equivalent since we have the freedom to reparameterize the generators with an extra minus sign.This choice is the consistent one with the sign conventions for $M_{\text{(fund)}}$ and M_{spinor} above.

Exercise 2: All the γ-matrices

Motivation: You have seen in the lecture that one can construct a matrix γ^5 that anticommutes with all other γ matrices satisfying the Clifford algebra. In this exercise, we are going to showcase the utility of γ^5 that allows us to understand Dirac fermions through their Weyl componponents. Also γ^5 is instrumental when it comes to describing the weak interactions which only involve left-handed fermions.

You also saw in the lecture that one can construct another matrix, the γ^5, from the four γ^μ matrices, as

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \,. \tag{1.8}$$

a) Show that $\{\gamma^5, \gamma^\mu\} = 0$ and that $\left(\gamma^5\right)^2 = \mathbb{I}_{4\times4}$.

b) Show that in the (Weyl) representation for the γ-matrices, the matrix γ^5 takes the form:

$$\gamma^5 = \begin{pmatrix} -\mathbb{I}_{2\times2} & 0 \\ 0 & \mathbb{I}_{2\times2} \end{pmatrix}. \tag{1.9}$$

c) Show that the left and right projectors,

$$\mathbb{P}_L = \frac{1}{2}\left(\mathbb{I} - \gamma^5\right) \quad , \quad \mathbb{P}_R = \frac{1}{2}\left(\mathbb{I} + \gamma^5\right) \ , \tag{1.10}$$

are orthogonal, i.e., show that $\mathbb{P}_L \cdot \mathbb{P}_L = \mathbb{P}_L$, $\mathbb{P}_R \cdot \mathbb{P}_R = \mathbb{P}_R$ and $\mathbb{P}_L \cdot \mathbb{P}_R = \mathbb{P}_R \cdot \mathbb{P}_L = 0$. Show also that their action on a Dirac spinor $\Psi = (\Psi_L, \Psi_R)^{\mathrm{T}}$, where Ψ_L is a left-handed Weyl spinor and Ψ_R is a right-handed Weyl spinor,

$$\mathbb{P}_L \Psi = \Psi_L \quad , \quad \mathbb{P}_R \Psi = \Psi_R \ . \tag{1.11}$$

Now we can understand the Dirac Lagrangian $\mathcal{L} = \bar{\Psi}\left(i\gamma^\mu \partial_\mu - m\right)\Psi$, with $\bar{\Psi} = \Psi^\dagger \gamma^0$, in terms of left-handed and right-handed spinors.

d) First, show the following identities:

$$\gamma^\mu \mathbb{P}_L = \mathbb{P}_R \gamma^\mu \quad , \quad \gamma^\mu \mathbb{P}_R = \mathbb{P}_L \gamma^\mu \ , \tag{1.12}$$

and then that $\bar{\Psi}_L \equiv \Psi_L^\dagger \gamma^0 = \bar{\Psi}\mathbb{P}_R$ and $\bar{\Psi}_R \equiv \Psi_R^\dagger \gamma^0 = \bar{\Psi}\mathbb{P}_L$.

e) Show that in terms of left-handed and right-handed spinors, the kinetic term and the mass term can be written as

$$i\bar{\Psi}\gamma^\mu \partial_\mu \Psi = i\bar{\Psi}_L \gamma^\mu \partial_\mu \Psi_L + i\bar{\Psi}_R \gamma^\mu \partial_\mu \Psi_R \ , \tag{1.13}$$

$$m\bar{\Psi}\Psi = m\left(\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L\right) \ . \tag{1.14}$$

This last calculation shows that the kinetic term **decouples** the two Weyl spinors that build up the Dirac spinor, thus massless Dirac spinors are equivalently a system of two decoupled Weyl spinors with different handedness. On the contrary, massive Dirac spinors are not composed of independent Weyl spinors because the mass term necessarily mixes them up!