The Standard Model of Particle Physics: Theory and Experiment

Examples

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updated 8.11.2002

Relativistic quantum mechanics

1 Show that in the presence of an electrostatic potential V the conserved Klein–Gordon density and current are

$$\begin{aligned} \rho &= i\hbar\left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right) - 2eV\phi^*\phi \, , \\ \mathbf{J} &= -i\hbar c^2 \left(\phi^* \boldsymbol{\nabla}\phi - \phi \boldsymbol{\nabla}\phi^*\right) \, . \end{aligned}$$

Spin-zero particles of charge e, mass m, are incident on a one-dimensional rectangular potential barrier of height V such that $eV > 2mc^2$. Show that when the particles have total energy $E = eV/2$ the barrier is perfectly transparent, independent of its thickness. Find ρ and J_x inside the barrier in this case, and interpret your results.

2 Show that in terms of the Mandelstam variables s, t and u the c.m. differential cross section for the scattering of two distinguishable spin-zero particles, a and b, via one photon exchange is

$$\frac{d\sigma}{d\Omega^*} = \left[\frac{e_a e_b (s-u)}{8\pi t\sqrt{s}}\right]^2 \, .$$

3 The Klein–Gordon equation can also be written in a two-component form, to emphasise the positive- and negative-energy degrees of freedom.

(a) Given that Φ satisfies the Klein–Gordon equation and also the Schrödinger-like equation

$$i\hbar\frac{\partial \Phi}{\partial t} = \eta m c^2 \Phi - \frac{\hbar^2}{2m}\zeta\, \nabla^2 \Phi \, ,$$

deduce that

$$\eta^2 = 1 \, , \quad \zeta^2 = 0 \, , \quad \eta\zeta + \zeta\eta = 2 \, .$$

Hence η and ζ must be at least 2×2 matrices, for example

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \, , \quad \zeta = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \, .$$

(b) Show that in this notation suitable definitions of the density and current are

$$\begin{aligned}\rho &= 2mc^2\Phi^\dagger\eta\,\Phi\ , \\ \mathbf{J} &= -i\hbar c^2\left[\Phi^\dagger\eta\,\zeta\boldsymbol{\nabla}\Phi - (\boldsymbol{\nabla}\Phi)^\dagger\,\eta\,\zeta\Phi\right]\ .\end{aligned}$$

(c) The density and current should change sign under the charge conjugation transformation

$$\Phi \to \Phi^c = C\Phi^*\ .$$

Find a suitable real matrix C.

(d) In terms of the usual Klein–Gordon wave function ϕ, we have

$$\Phi = \begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix} \quad \text{where } \chi_\pm = \frac{1}{2}\left(\phi \pm \frac{i\hbar}{mc^2}\frac{\partial\phi}{\partial t}\right)\ .$$

Show that $\chi_+ \approx 1$ and $\chi_- = O(v^2/c^2) \ll 1$ for a slowly-moving particle, and vice-versa for a slowly-moving antiparticle.

4 The Dirac wave function for the ground state of the hydrogen atom has the following form (in the standard Dirac matrix representation):

$$\psi(r,\theta,\phi) = R(r)\begin{pmatrix}1\\0\\ ia\cos\theta\\ iae^{i\phi}\sin\theta\end{pmatrix}$$

where $a \approx \alpha/2$.

(a) Investigate whether ψ is an eigenstate of L_z.

(b) Find the expectation value of L_z and comment on the result.

(c) Show that ψ is an eigenstate of J_z and find its eigenvalue.

[Don't forget to normalize to one electron.]

5 The Fourier decomposition of a Dirac wave function $\psi(\mathbf{r},t)$ is

$$\psi(\mathbf{r},t) = \int \frac{d^3\mathbf{k}}{2(2\pi)^3\omega}\sum_s\left[c_s(\mathbf{k})u_s(\mathbf{k})e^{-ik\cdot x} + d_s^*(\mathbf{k})v_s(\mathbf{k})e^{+ik\cdot x}\right]$$

where $\omega = \sqrt{\mathbf{k}^2+m^2}$ and $k^\mu = (\omega,\mathbf{k})$.

(a) Show that the positive- and negative-energy plane-wave amplitudes are given in terms of the wave function at time $t = 0$ by

$$\begin{aligned} c_s(\mathbf{k}) &= \int d^3\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}} u_s{}^\dagger(\mathbf{k})\psi(\mathbf{r},0) \\ d_s^*(\mathbf{k}) &= \int d^3\mathbf{r}\, e^{+i\mathbf{k}\cdot\mathbf{r}} v_s{}^\dagger(\mathbf{k})\psi(\mathbf{r},0)\ . \end{aligned}$$

(b) Consider a wave packet with the initial form

$$\psi(\mathbf{r},0) = \left(\pi d^2\right)^{-\frac{3}{4}} e^{-\mathbf{r}^2/2d^2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\ .$$

Show that

$$\begin{aligned} \sum_s |c_s(\mathbf{k})|^2 &= (\omega + m)\left(4\pi d^2\right)^{\frac{3}{2}} e^{-d^2\mathbf{k}^2} \\ \sum_s |d_s(\mathbf{k})|^2 &= (\omega - m)\left(4\pi d^2\right)^{\frac{3}{2}} e^{-d^2\mathbf{k}^2}\ . \end{aligned}$$

(c) Hence show that the wave packet has a substantial negative-energy component when $d < 1/m$, that is, when its width is less than the Compton wavelength of the particle. Discuss the interpretation of this result.

6 Derive the conservation equation

$$\partial_\mu J_V^\mu = 0$$

for the 4-vector current density $J_V^\mu = \bar{\psi}\gamma^\mu\psi$, using the covariant form of the Dirac equation and the relation $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.

Show that the axial 4-vector current density $J_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ is not conserved but instead satisfies the covariant equation

$$\partial_\mu J_A^\mu = 2im\bar{\psi}\gamma^5\psi\ .$$

7 Derive the *Gordon decomposition* of the Dirac transition current:

$$\bar{\psi}_f\gamma^\mu\psi_i = \frac{1}{2m}\bar{\psi}_f\left[(p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu\right]\psi_i\ ,$$

where $\sigma^{\mu\nu} = \frac{1}{2}i(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$.

The first term in the Gordon decomposition (the *convection current*) is similar to the current for a spin-zero particle. Show that in a magnetic field $\mathbf{B}$ the second term (the *spin current*) gives rise to the magnetic interaction matrix element

$$-\int \bar{\psi}_f \frac{e}{2m}\mathbf{\Sigma}\cdot\mathbf{B}\,\psi_i\, d^4x\ .$$

[Hint: Use the Dirac equations $\bar{\psi}_f(\not{p}_f - m) = (\not{p}_i - m)\psi_i = 0$, and note that $\sigma^{jk} = \varepsilon_{jkl}\Sigma_l$.]

8 Defining $\not{a} = \gamma^\mu a_\mu$ and using $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$, prove the following results:

(a)The trace of an odd number of γ-matrices is zero

(b) $\mathrm{Tr}(\not{a}\not{b}) = 4\, a \cdot b$

(c) $\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4[(a \cdot b)(c \cdot d) + (a \cdot d)(b \cdot c) - (a \cdot c)(b \cdot d)]$

(d)$\gamma_\mu \not{a} \gamma^\mu = -2\not{a}$

(e)$\gamma_\mu \not{a}\not{b} \gamma^\mu = 4\, a \cdot b$

(f)$\gamma_\mu \not{a}\not{b}\not{c} \gamma^\mu = -2\not{c}\not{b}\not{a}$.

[Hint: Useful tricks are to use $\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA)$, and to insert $(\gamma^5)^2 = 1$ into a trace and then use $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$.]

9 Consider the scattering of a non-relativistic particle a with 3-momentum p_a by a very massive particle b, initially at rest, via one photon exchange. Show that the differential cross section in the lab frame is given by the Rutherford formula

$$\frac{d\sigma}{d\Omega} = \left(\frac{e_a e_b m_a}{8\pi p_a^2}\right)^2 \frac{1}{\sin^4(\theta/2)} ,$$

independent of whether a and b have spin zero or one-half.

10 Show that contributions of the two diagrams for Compton scattering are not separately gauge invariant, but their sum is.

Show that the invariant differential cross section in the extreme relativistic limit is

$$\frac{d\sigma}{dt} = -\frac{2\pi\alpha^2}{s^2}\left(\frac{u}{s} + \frac{s}{u}\right) .$$

[Hint: For the first part, use the Dirac equations $\bar{u}'(\not{p}' - m) = (\not{p} - m)u = 0$.]

Relativistic fields

11 Show that if Ψ and Ψ^* are taken as independent classical fields, the Lagrangian density

$$\mathcal{L} = \frac{\hbar}{2i}\left(\Psi\frac{\partial\Psi^*}{\partial t} - \Psi^*\frac{\partial\Psi}{\partial t}\right) - \frac{\hbar^2}{2m}\boldsymbol{\nabla}\Psi \cdot \boldsymbol{\nabla}\Psi^* - V(\mathbf{r})\Psi^*\Psi$$

leads to the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\Psi + V(\mathbf{r})\Psi$$

and its complex conjugate. What are the momentum densities conjugate to Ψ and Ψ^*? Deduce the Hamiltonian density, and verify that integrating it over all space gives the usual expression for the energy.

12 The Klein-Gordon Lagrangian density for a Hermitian spin-zero quantum field $\hat{\phi}$ is

$$\hat{\mathcal{L}} = \tfrac{1}{2}\left[\left(\frac{\partial\hat{\phi}}{\partial t}\right)^2 - \left(\boldsymbol{\nabla}\hat{\phi}\right)^2 - m^2\hat{\phi}^2\right] .$$

(a) Show that the corresponding Hamiltonian density is

$$\hat{\mathcal{H}} = \tfrac{1}{2}\left[\hat{\pi}^2 + \left(\boldsymbol{\nabla}\hat{\phi}\right)^2 + m^2\hat{\phi}^2\right] .$$

where $\hat{\pi} = \partial\hat{\phi}/\partial t$ is the momentum density conjugate to $\hat{\phi}$.

(b) Introducing the Fourier representation

$$\hat{\phi}(\mathbf{r},t) = \int \frac{d^3\mathbf{k}}{2(2\pi)^3\omega}\left[\hat{a}(\mathbf{k})e^{-ik\cdot x} + \hat{a}(\mathbf{k})^\dagger e^{+ik\cdot x}\right]$$

where $\omega = \sqrt{\mathbf{k}^2 + m^2}$, show that the Hamiltonian can be written as

$$\hat{H} = \int d^3\mathbf{r}\,\hat{\mathcal{H}} = \int \frac{d^3\mathbf{k}}{4(2\pi)^3}\left[\hat{a}(\mathbf{k})\hat{a}(\mathbf{k})^\dagger + \hat{a}(\mathbf{k})^\dagger\hat{a}(\mathbf{k})\right] .$$

Interpret this result.

(c) Show that the annihilation operator $\hat{a}(\mathbf{k})$ can be expressed in terms of the field and momentum density as

$$\hat{a}(\mathbf{k}) = \int d^3\mathbf{r}\, e^{ik\cdot x}\left[\omega\hat{\phi}(\mathbf{r},t) + i\hat{\pi}(\mathbf{r},t)\right]$$

and hence deduce the commutation relation

$$\left[\hat{a}(\mathbf{k}), \hat{a}(\mathbf{k}')^\dagger\right] = (2\pi)^3\, 2\omega\, \delta^3(\mathbf{k} - \mathbf{k}')$$

from the canonical commutator

$$\left[\hat{\phi}(\mathbf{r},t), \hat{\pi}(\mathbf{r}',t)\right] = i\delta^3(\mathbf{r} - \mathbf{r}') .$$

13 *Number-phase uncertainty relation.* Suppose that $\hat{a}$ and $\hat{a}^\dagger$ are annihilation and creation operators for a normal mode of a bosonic field, so that

$$[\hat{a}, \hat{a}^\dagger] = 1 .$$

Then the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$ counts the number of quanta of excitation of the mode, which is a measure of its intensity. Define the *amplitude* and *phase* operators, $\hat{A}$ and $\hat{\Phi}$, by

$$\hat{a} = e^{i\hat{\Phi}}\hat{A} , \qquad \hat{a}^\dagger = \hat{A}e^{-i\hat{\Phi}} .$$

(a) Show that $\hat{A}\hat{A} = \hat{n}$. The amplitude operator may thus be defined as the (unique, Hermitian) operator with the same eigenstates as $\hat{n}$ but with eigenvalues that are the square roots of the corresponding eigenvalues of $\hat{n}$.

(b) The operator $\hat{\Phi}$ measures the phase of oscillation of the mode. The exponential function $e^{i\hat{\Phi}}$ may be defined by its Taylor expansion. Show that

$$[\hat{n}, \hat{\Phi}] = i \; .$$

(c) Hence show using the general form of the uncertainty relation that the r.m.s. uncertainties in n and Φ in any state satisfy

$$\Delta n \cdot \Delta\Phi \geq \frac{1}{2} \; .$$

Thus a bosonic (e.g. electromagnetic) wave with a definite phase, suitable for doing interference with other waves, must contain a very uncertain number of particles (photons).

[There are some subtleties about the rigorous definition of the phase operator. If you are interested, see, for example, Chapter 4 of *Lectures on Quantum Optics*, W. Vogel & D.-G. Welsch (Akademie Verlag, Berlin, 1994).]

14 In the lectures the (negative) Casimir pressure between plane conductors with separation a was calculated to be

$$P = \frac{\pi^2}{240}\frac{\hbar c}{a^4} \; .$$

In order to perform the calculation it was necessary to introduce a cutoff function $f(k)$ to remove the contribution of large wave numbers:

$$\begin{aligned} f(k) &= 1 \quad \text{for } k \ll \Lambda \\ &= 0 \quad \text{for } k \gg \Lambda \; . \end{aligned}$$

(a) Show that a more accurate expression for the Casimir pressure is

$$P = -\frac{\hbar c}{2\pi^2}\sum_{l=2}^{\infty}\frac{B_{2l}}{2l}\frac{f^{(2l-4)}(0)}{(2l-4)!}\left(\frac{\pi}{a}\right)^{2l}$$

where B_{2l} are the Bernoulli numbers, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, ..., and $f^{(n)}(0)$ represents the n-th derivative of the cutoff function $f(k)$ at $k = 0$.

(b) Consider the specific cutoff function

$$f(k) = \left[1 + e^{b(k-\Lambda)}\right]^{-1}$$

where $b\Lambda \gg 1$. Show that the first correction to P is δP where

$$\frac{\delta P}{P} = -\left(1 - \frac{5\pi^2}{21}\frac{b^2}{a^2} + \cdots\right)e^{-b\Lambda} \; .$$

Thus the dependence on the cutoff wave number Λ vanishes exponentially as $\Lambda \to \infty$.

15 Consider a string with mass per unit length σ, tension T, stretched along the z-axis and free to execute small transverse oscillations, with displacements ϕ_x and ϕ_y in the x- and y-directions respectively. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\sigma\left[\left(\frac{\partial\phi_x}{\partial t}\right)^2 + \left(\frac{\partial\phi_y}{\partial t}\right)^2\right] - \frac{1}{2}T\left[\left(\frac{\partial\phi_x}{\partial z}\right)^2 + \left(\frac{\partial\phi_y}{\partial z}\right)^2\right] .$$

Show that this Lagrangian is invariant under the transformation

$$\begin{aligned}\phi_x &\to \phi_x\cos\theta - \phi_y\sin\theta \\ \phi_y &\to \phi_x\sin\theta + \phi_y\cos\theta\end{aligned}$$

where θ is a constant. Show that the corresponding Noether current and density are

$$\begin{aligned}\rho &= \sigma\left(\phi_x\frac{\partial\phi_y}{\partial t} - \phi_y\frac{\partial\phi_x}{\partial t}\right) \\ J_z &= T\left(\phi_y\frac{\partial\phi_x}{\partial z} - \phi_x\frac{\partial\phi_y}{\partial z}\right) .\end{aligned}$$

Write down the corresponding conserved charge and explain its physical significance.

16 The Dirac Lagrangian density is

$$\mathcal{L}_D = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$$

and the Fourier representation of the Dirac field operator is

$$\hat{\psi}(\mathbf{r},t) = \int\frac{d^3\mathbf{k}}{2(2\pi)^3\omega}\sum_s\left[\hat{c}_s(\mathbf{k})u_s(\mathbf{k})e^{-ik\cdot x} + \hat{d}_s(\mathbf{k})^\dagger v_s(\mathbf{k})e^{+ik\cdot x}\right]$$

where $\omega = \sqrt{\mathbf{k}^2 + m^2}$ and $k^\mu = (\omega, \mathbf{k})$. The free-particle spinors have the properties

$$\sum_s u_s(\mathbf{k})\bar{u}_s(\mathbf{k}) = \not{k} + m\ , \quad \sum_s v_s(\mathbf{k})\bar{v}_s(\mathbf{k}) = \not{k} - m\ ,$$

and the creation and annihilation operators satisfy the anticommutation relations

$$\{\hat{c}_s(\mathbf{k}), \hat{c}_{s'}(\mathbf{k}')^\dagger\} = \{\hat{d}_s(\mathbf{k}), \hat{d}_{s'}(\mathbf{k}')^\dagger\} = (2\pi)^3\, 2\omega\, \delta_{ss'}\delta^3(\mathbf{k}-\mathbf{k}')\ ,$$

all other anticommutators being zero. Show that this implies that the field and its conjugate momentum density satisfy the anticommutation relation

$$\{\hat{\psi}_\alpha(\mathbf{r},t), \hat{\pi}_\beta(\mathbf{r}',t)\} = i\,\delta_{\alpha\beta}\delta^3(\mathbf{r}-\mathbf{r}')$$

where α and β are spinor component labels.

Gauge fields

17 The Lagrangian density for a free Dirac field ψ is

$$\mathcal{L}_D = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \,.$$

Show that $\mathcal{L}_D$ is not invariant under a local phase transformation

$$\psi \to \psi' = \exp(-ie\chi)\psi$$

where $-e$ is the charge associated with the field ψ and where χ varies with respect to time and space. Introduce the electromagnetic interaction of ψ with a vector field A_μ in the Lagrangian density. Explain how the invariance under the above phase transformation can be restored by simultaneously performing a gauge transformation of the vector field A_μ. What is the full Lagrangian density? Why has the vector field A_μ to be massless?

18 The Lagrangian density for the Higgs field $\hat{\phi}$ contains a self-interaction term of the form $\lambda\hat{\phi}^3$ where λ is a constant. Derive the Feynman rule for the three-Higgs-boson vertex.

[Ans: Vertex factor $= 6i\lambda$.]

19 Starting from the Standard Model Lagrangian, show that the Z^0 boson couples to the following current of a fermion f:

$$J_f^\mu = \frac{e}{\sin(2\theta_W)} \bar{\psi}_f \gamma^\mu (g_V - g_A\gamma^5)\psi_f$$

where

$$g_V = I_f - 2Q_f \sin^2\theta_W \;, \qquad g_A = I_f \;,$$

I_f being the third component of the weak isospin of the left-handed state of the fermion and Q_f its charge in units of e. Evaluate the vector and axial couplings g_V and g_A for $f = u, d, \nu_e, e$, assuming $\sin^2\theta_W = 0.232$.

[Ans: $g_V = +0.191, -0.345, +0.5, -0.036$, $g_A = +0.5, -0.5, +0.5, -0.5$]

20 Show that the spontaneous symmetry breaking in the Standard Model releases an energy per unit volume

$$U \approx 10^6 M_H^2 \;\; \mathrm{GeV/fm}^3 \;,$$

where M_H is the Higgs boson mass in GeV/c^2.

21 Show that the sum over polarization states P of a massive spin-one boson with mass M and four-momentum p^μ can be written as

$$\sum_P \varepsilon_P^\mu \varepsilon_P^{*\nu} = -g^{\mu\nu} + \frac{p^\mu p^\nu}{M^2} \,.$$

[Hint: look at it in the rest frame of the boson.]

Use this result to calculate the rate for the decay of the Higgs boson into W^+W^-:

$$\Gamma(H \to WW) = \frac{G_F M_H^3}{8\pi\sqrt{2}} \left(1 - 4\frac{M_W^2}{M_H^2} + 12\frac{M_W^4}{M_H^4}\right) \sqrt{1 - 4\frac{M_W^2}{M_H^2}} \,.$$

22 Calculate the rate for the decay of the Higgs boson into a fermion-antifermion pair:

$$\Gamma(H \to f\bar{f}) = \frac{C G_F m_f^2 M_H}{4\pi\sqrt{2}} \left(1 - 4\frac{m_f^2}{M_H^2}\right)^{\frac{3}{2}}$$

where C is a colour factor ($C = 1$ for leptons, 3 for quarks). Evaluate the corresponding partial width for decay into $t\bar{t}$ when $M_H = 4m_t = 700$ GeV. [Ans: 27 GeV]

23 The electron self-energy mass shift δm is given to order e^2 by

$$\bar{u}(\mathbf{p})u(\mathbf{p})\,\delta m = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(\mathbf{p})\gamma^\mu(\not{p} - \not{k} + m)\gamma_\mu u(\mathbf{p})}{(k^2 + i\epsilon)[(p-k)^2 - m^2 + i\epsilon]} \,.$$

(a) Show that, after averaging over electron spin states, the numerator of the integrand is equal to $4(k \cdot p + m^2)$. [Hint: Use the spinor property given in qu. 16.]

(b) Perform the k^0-integration by the method of residues, to obtain

$$\delta m = -2\frac{e^2}{m}\left(\int \frac{d^3\mathbf{k}}{2(2\pi)^3\omega} \frac{\omega p^0 - \mathbf{k}\cdot\mathbf{p} + m^2}{(p^0 - \omega)^2 - E^2} + \int \frac{d^3\mathbf{k}}{2(2\pi)^3 E} \frac{(p^0 + E)p^0 - \mathbf{k}\cdot\mathbf{p} + m^2}{(p^0 + E)^2 - \omega^2}\right)$$

where $\omega = |\mathbf{k}|$ and $E = \sqrt{(\mathbf{p} - \mathbf{k})^2 + m^2}$. Show that the two integrands tend to cancel when $\omega \gg |\mathbf{p}|, m$, in such a way that δm diverges logarithmically.

(c) Discuss the interpretation of this cancellation from the viewpoint of old-fashioned (time-ordered) perturbation theory. [Hint: Use partial fractions and compare with the OFPT denominators $p^0 - E_n$ for intermediate states n.]