Klein-Gordon Equation

- The existence of plane waves

$$\phi(\boldsymbol{r}, t) \propto \exp(i\boldsymbol{k} \cdot \boldsymbol{r} - i\omega t)$$

satisfying de Broglie and Einstein relations

$$\boldsymbol{p} = \hbar \boldsymbol{k} \ , \qquad E = \hbar\omega$$

implies the quantum operator interpretation

$$\boldsymbol{p} \to -i\hbar \boldsymbol{\nabla} \ , \qquad E \to i\hbar \frac{\partial}{\partial t} \ .$$

- Then the relativistic energy-momentum equation

$$E^2 = \boldsymbol{p}^2 c^2 + m^2 c^4$$

implies the Klein-Gordon equation

$$-\hbar^2 \frac{\partial^2 \phi}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \phi + m^2 c^4 \phi$$

i.e. in covariant notation (see handout)

$$\left[\partial_\mu\partial^\mu + \left(\frac{mc}{\hbar}\right)^2\right]\phi = 0$$

where

$$\partial_\mu\partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

- KG wave function ϕ is a Lorentz-invariant (scalar) function; Lorentz transformation $\boldsymbol{r}, t \rightarrow \boldsymbol{r}', t'$ implies $\phi \rightarrow \phi'$ where

$$\phi'(\boldsymbol{r}', t') = \phi(\boldsymbol{r}, t) \ .$$

  Hence it must represent a spin-zero particle (no orientation).

- Since $|\phi|^2$ is also invariant, this cannot represent a probability density. A density transforms as time-like (0-th) component of a 4-vector, due to Lorentz contraction of volume element.

- Correct definition of density follows from the conservation (continuity) equation

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\boldsymbol{J}$$

($\boldsymbol{J}=$ corresponding current vector), i.e.

$$\partial_\mu J^\mu = 0$$

where $J^\mu = (c\rho, \boldsymbol{J})$ is the 4-current.

- We can obtain an equation of this form from the KG equations for ϕ and ϕ^*,

$$\begin{aligned} i\hbar\left(\phi^*\frac{\partial^2\phi}{\partial t^2} - \phi\frac{\partial^2\phi^*}{\partial t^2}\right) &= i\hbar c^2\left(\phi^*\nabla^2\phi - \phi\nabla^2\phi^*\right) \ , \\ i\hbar\frac{\partial}{\partial t}\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right) &= i\hbar c^2\nabla\cdot\left(\phi^*\nabla\phi - \phi\nabla\phi^*\right) \ . \end{aligned}$$

Hence

$$\begin{aligned} \rho &= i\hbar\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right) \\ \boldsymbol{J} &= -i\hbar c^2\left(\phi^*\nabla\phi - \phi\nabla\phi^*\right) \end{aligned}$$

i.e. $J^\mu = i\hbar c^2(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)$.

- Normalization is such that the energy eigenstate $\phi = \Phi(\boldsymbol{r})e^{-iEt/\hbar}$ has $\rho = 2E|\Phi|^2$. Thus $|\Phi| = 1$ corresponds to $2E$ particles per unit volume (relativistic normalization).

- Compare with Schrödinger current

$$\boldsymbol{J}^{\,\mathsf{S}} = -\frac{i\hbar}{2m}\left(\phi^*\nabla\phi - \phi\nabla\phi^*\right) = \frac{1}{2mc^2}\boldsymbol{J}^{\,\mathsf{KG}}$$

which thus has in fact E/mc^2 particles per unit volume.

Problems with Klein-Gordon Equation

1. Density ρ is not necessarily positive (unlike $|\phi|^2$) $\Rightarrow$ equation was rejected initially.
2. Equation is second-order in t $\Rightarrow$ need to know both ϕ and $\frac{\partial\phi}{\partial t}$ at $t = 0$ in order to solve for ϕ at $t > 0$. Thus there is an extra degree of freedom, not present in the Schrödinger equation.
3. The equation on which it is based ($E^2 = \boldsymbol{p}^2 c^2 + m^2 c^4$) has both positive and negative solutions for E.

- Actually these problems are all related, since a solution $\phi = \Phi(\boldsymbol{r})e^{\mp iEt/\hbar}$ has $\rho = \pm 2E|\Phi|^2$, and so for the general solution

$$\phi = \Phi_+(\boldsymbol{r})e^{-iEt/\hbar} + \Phi_-(\boldsymbol{r})e^{+iEt/\hbar}$$

both $\phi(t = 0) = \phi_+ + \phi_-$ and $\frac{i\hbar}{E}\left.\frac{\partial\phi}{\partial t}\right|_{t=0} = \phi_+ - \phi$ are needed in order to specify ϕ_+ and ϕ_-.

Electromagnetic Waves

- In units where $\epsilon_0 = \mu_0 = c = 1$ ('Heaviside-Lorentz') Maxwell's equations are

$$\nabla \cdot \boldsymbol{E} = \rho_{\text{em}} \quad , \quad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \cdot \boldsymbol{B} = 0 \quad , \quad \nabla \times \boldsymbol{B} = \boldsymbol{J}_{\text{em}} + \frac{\partial \boldsymbol{E}}{\partial t}$$

where $(\rho_{\text{em}}, \boldsymbol{J}_{\text{em}}) = J^\mu_{\text{em}}$ is the electromagnetic 4-current.

- In terms of the scalar and vector potentials V and $\boldsymbol{A}$,

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla V \, , \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A} \, .$$

So we find

$$\begin{aligned} \nabla \times (\nabla \times \boldsymbol{A}) &\equiv \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} \\ &= \boldsymbol{J}_{\text{em}} - \frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla \frac{\partial V}{\partial t} \end{aligned}$$

- In terms of the 4-potential $A^\mu = (V, \boldsymbol{A})$

$$(\partial_\nu \partial^\nu) A^\mu - \partial^\mu (\partial_\nu A^\nu) = J^\mu_{\text{em}}$$

$$\Rightarrow \quad \partial_\nu F^{\nu\mu} = J^\mu_{\text{em}}$$

where the electromagnetic field-strength tensor is

$$F^{\nu\mu} = \partial^\nu A^\mu - \partial^\mu A^\nu = -F^{\mu\nu}\ .$$

- $\boldsymbol{E}$ and $\boldsymbol{B}$, and hence Maxwell's equations, are invariant under gauge transformations

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \chi$$

where $\chi(\boldsymbol{r}, t)$ is an arbitrary scalar function.

- Therefore we can always choose A^μ such that $\partial_\mu A^\mu = 0$ (Lorenz gauge). If $\partial_\mu A^\mu = f \neq 0$, we can change to $A'^\mu = A^\mu + \partial^\mu \chi$ where $\partial_\mu \partial^\mu \chi = -f$.

- Then in free space ($J^\mu = 0$) we have

$$\partial_\nu \partial^\nu A^\mu = 0$$

❖ Massless KG equation for each component of A^μ

❖ A^μ is 'wave function' of photon

❖ A^μ is a 4-vector $\Rightarrow$ photon has spin 1.

- Plane wave solutions

$$A^\mu = \varepsilon^\mu \exp(i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t) \equiv \varepsilon^\mu e^{-ik\cdot x}$$

where $\varepsilon^\mu =$ polarization 4-vector, $k \cdot x \equiv k^\mu x_\mu$, $k^\mu = (\omega, \boldsymbol{k})$ = wave 4-vector.

- From wave equation $k \cdot k = 0$ hence $\omega^2 = \boldsymbol{k}^2$, i.e. $E^2 = \boldsymbol{p}^2 c^2$ (massless photons).

- From Lorenz gauge condition $\varepsilon \cdot k = 0 \Rightarrow \varepsilon^0 = \boldsymbol{\varepsilon}\cdot\boldsymbol{k}/\omega$.

- Polarization 4-vector $\varepsilon'^\mu = \varepsilon^\mu + a k^\mu$ is equivalent to ε^μ for any constant a. Hence we can always choose $\varepsilon^0 = 0$. Then Lorenz condition becomes transversity condition: $\boldsymbol{\varepsilon}\cdot\boldsymbol{k} = 0$.

- E.g. for $\boldsymbol{k}$ along z-axis we can express ε^μ in terms of plane polarization states

$$\varepsilon_x^\mu = (0,1,0,0)\ , \qquad \varepsilon_y^\mu = (0,0,1,0)\ ,$$

or circular polarization states $\varepsilon^\mu_{R,L} = (0,1,\pm i,0)/\sqrt{2}$. N.B. only 2 polarization states for real photons.

Electromagnetic Interactions

- As in classical (and non-relativistic quantum) physics, we introduce e.m. interactions via the minimal substitution in the equations of motion:

$$E \to E - eV\,, \qquad \boldsymbol{p} \to \boldsymbol{p} - e\boldsymbol{A}$$

i.e.

$$\begin{aligned} p^\mu &\to p^\mu - eA^\mu \\ \partial^\mu &\to \partial^\mu + ieA^\mu \end{aligned}$$

- The Klein-Gordon equation becomes

$$(\partial_\mu + ieA_\mu)(\partial^\mu + ieA^\mu)\phi + m^2\phi = 0\,,$$

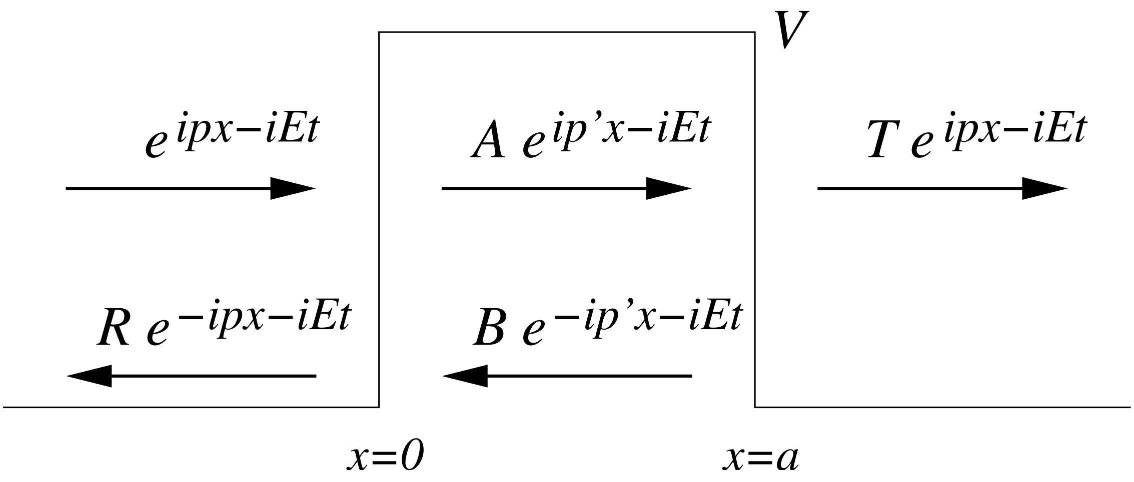
$$(\partial_\mu\partial^\mu + m^2)\phi = -ie[\partial_\mu(A^\mu\phi) + A_\mu(\partial^\mu\phi)] + e^2A_\mu A^\mu\phi$$

- The conserved current is now ($\hbar = c = 1$)

$$J^\mu = i(\phi^*\partial^\mu\phi - \phi\,\partial^\mu\phi^*) - 2eA^\mu\phi^*\phi$$

Klein Paradox

- Consider KG plane waves incident on electrostatic barrier, height V, width a



KG equation for $x < 0$, $x > a$ gives $E^2 = p^2 + m^2$

$$\Rightarrow \; p = +\sqrt{E^2 - m^2}$$

(sign from B.C.).

- In $0 < x < a$, $A^\mu = (V, \mathbf{0})$ and so $(E - eV)^2 = p'^2 + m^2$

$$\Rightarrow \; p' = +\sqrt{(E^{-}eV - m)(E - eV + m)}$$

(sign choice is arbitrary since we include $\pm p'$).

- Matching ϕ and $\partial\phi/\partial x$ at $x = 0$ and a gives (as for Schrödinger equation)

$$|T|^2 = \left|\cos p'a - \frac{i}{2}\left(\frac{p}{p'} + \frac{p'}{p}\right)\sin p'a\right|^{-2}$$

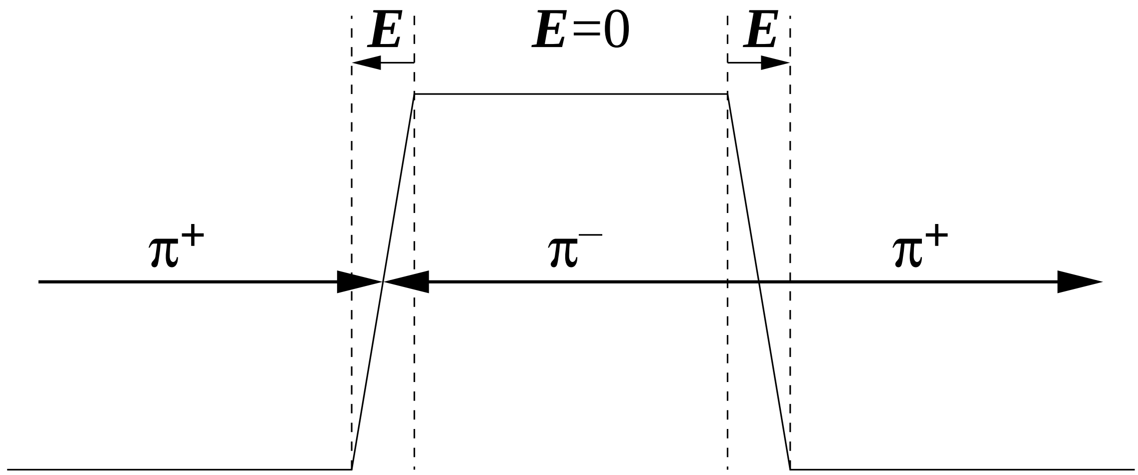
- Now consider behaviour as V is increased:
  - ❖ $eV < E - m$: p' is real, $|T| < 1$ ($|T| = 1$ when $p'a = n\pi$).
  - ❖ $E - m < eV < E + m$: p' is imaginary, $|T| < 1$, transmission by tunnelling.
  - ❖ $eV > E + m$: p' is real again! $|T| = 1$ when $p'a = n\pi$!?

- Note that when $eV > E + m$ density inside barrier is

$$\rho' = 2(E - eV)|\phi|^2 < -2m|\phi|^2$$

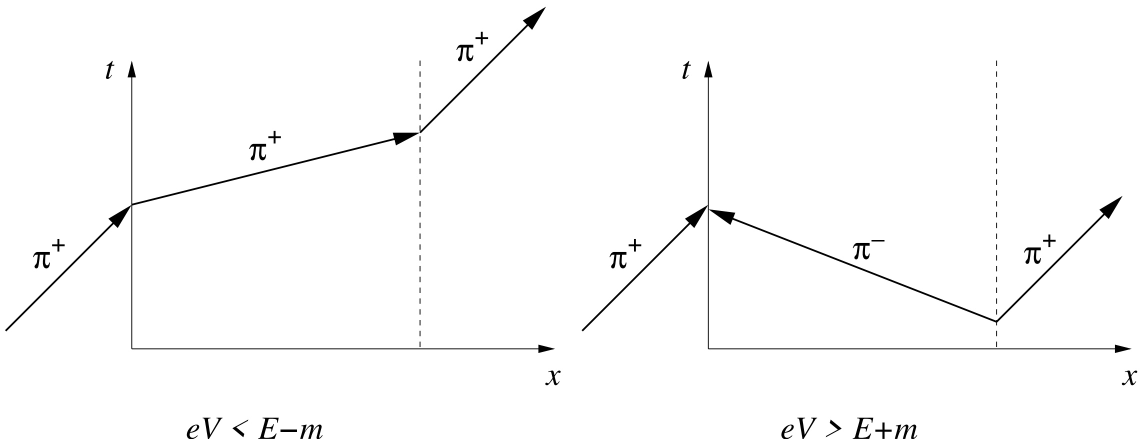
- Meanwhile, the current inside remains positive, $J'_x = 2p|T|^2$ (current conservation). Hence when $eV > E + m$ there is a negative density flowing from right to left, giving a positive current. We interpret this as a flow of antiparticles: $J^\mu_{\mathrm{em}} = eJ^\mu$ always.

- When $eV > E + m$ and $|T| = 1$, antiparticles created at the back of the barrier ($x = a$) travel to $x = 0$ and annihilate the incident particles. At the same time, particles created at $x = a$ travel to $x > a$, replacing the incident beam.



- Antiparticles are trapped inside the barrier, but field is zero there, so there can be perfect transmission for any thickness.

  Antiparticles are like particles propagating backwards in time



Charge Conjugation

- If ϕ is a negative-energy plane-wave solution of the KG equation, with momentum $\boldsymbol{p}$, $\phi = \exp(i\boldsymbol{p}\cdot\boldsymbol{r} + iEt)$ $(E > 0)$, then $\phi^* = \exp(-i\boldsymbol{p}\cdot\boldsymbol{r} - iEt)$ is a positive-energy wave with momentum $-\boldsymbol{p}$. Furthermore, in e.m. fields, ϕ^* behaves as a particle of charge $-e$:

$$
\begin{aligned}
& (\partial_\mu + ieA_\mu)(\partial^\mu + ieA^\mu)\phi + m^2\phi = 0 \\
\Rightarrow \quad & (\partial_\mu - ieA_\mu)(\partial^\mu - ieA^\mu)\phi^* + m^2\phi^* = 0
\end{aligned}
$$

- Thus if ϕ is a negative-energy solution, we take it to represent an antiparticle with wave function ϕ^* (and hence positive energy, opposite charge and momentum).

- Correspondingly, KG equation is invariant w.r.t. $\phi \to \phi^*$, $e \to -e$. This is called charge conjugation, C.

  N.B. Under C, $J^\mu \to -J^\mu$ as expected.

Electromagnetic Scattering

- We assume (for the moment) the same formula as in NRQM for the scattering amplitude in terms of the first-order perturbation due to e.m. field:

$$
\begin{aligned}
\mathcal{A}_{fi} &= -i \int \phi_f^* \{ie[\partial_\mu(A^\mu \phi_i) + A_\mu(\partial^\mu \phi_i)]\} d^4x \\
\text{by parts} \quad &= e \int A_\mu[\phi_f^*(\partial^\mu \phi_i) - (\partial^\mu \phi_f^*)\phi_i] d^4x \\
&= -ie \int A_\mu J_{fi}^\mu d^4x
\end{aligned}
$$

  where $J_{fi}^\mu = i[\phi_f^*(\partial^\mu \phi_i) - (\partial^\mu \phi_f^*)\phi_i]$ is generalization of J^μ to $\phi_f \neq \phi_i$ (transition curent). Note that to get $\mathcal{A}_{fi}$ to order e^1 we only need J_{fi}^μ to order e^0. Similarly, for A^μ we can use the free-field form

$$A^\mu = \varepsilon^\mu e^{-ik\cdot x}$$

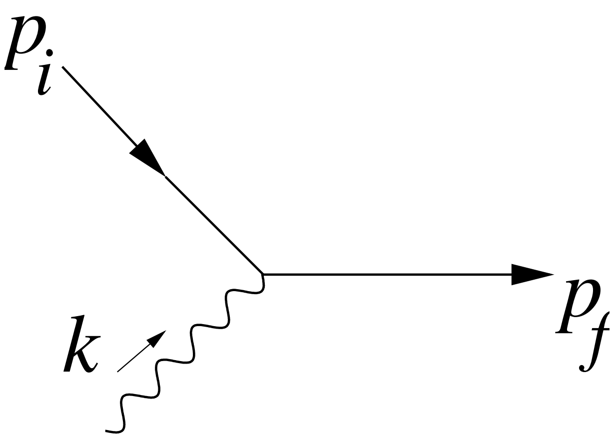
- For plane waves, $\phi_{f,i} = e^{-ip_{f,i}\cdot x}$,

$$J_{fi}^\mu = (p_i + p_f)^\mu e^{i(p_f - p_i)\cdot x}$$

Hence

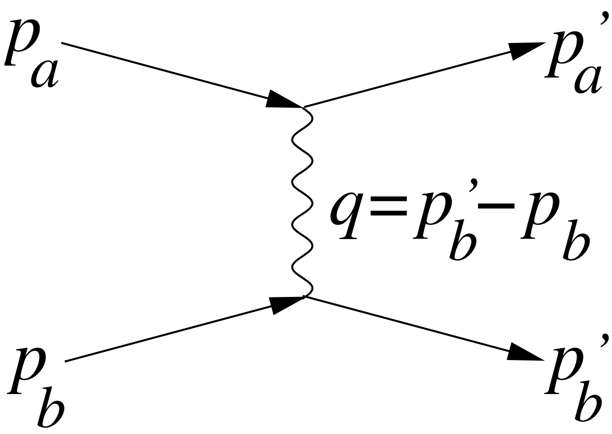
$$
\begin{aligned}
\mathcal{A}_{fi} &= -ie\varepsilon_\mu (p_i + p_f)^\mu \int e^{i(p_f - p_i - k)\cdot x} d^4x \\
&= -ie(2\pi)^4 \varepsilon \cdot (p_i + p_f)\, \delta^4(p_f - p_i - k)
\end{aligned}
$$

- This corresponds to the Feynman rules for the diagram



❖ An overall factor of $(2\pi)^4\, \delta^4(p_f - p_i - k)$ (momentum conservation)
❖ ε_μ for an external photon line
❖ $-ie(p_i + p_f)^\mu$ for a vertex involving a spin-0 boson of charge e.

N.B. 4-momentum cannot be conserved in this process for free particles! But we shall see that it can occur as part of a more complicated process, e.g. particle-particle scattering by photon exchange.



- We shall consider process $ab \to ab$ as scattering of a in e.m. field of b (both spin-0).

$$\mathcal{A}_{fi} = -ie_a \int A_\mu J^\mu_{a'a} d^4x$$

- Then (in Lorenz gauge) A^μ satisfies

$$\partial_\nu \partial^\nu A^\mu = e_b \, J^\mu_{b'b}$$

N.B. We assume correct source current is

$$J^\mu_{b'b} = (p_b + p_b')^\mu e^{i(p_b' - p_b)\cdot x}$$

- Solution for 4-vector potential is then

$$A^\mu = -\frac{1}{q^2} e_b (p_b + p_b')^\mu e^{iq\cdot x}$$

where $q = p_b' - p_b$ and $q^2 = q \cdot q$.

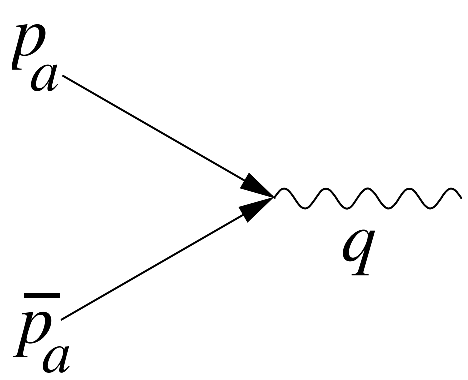
- Hence

$$\begin{aligned}\mathcal{A}_{fi} &= \frac{ie_a e_b}{q^2}(p_a + p_a') \cdot (p_b + p_b') \int e^{i(p_a'+p_b'-p_a-p_b)} d^4x \\ &= [-ie(p_a + p_a')^\mu] \left[\frac{-ig_{\mu\nu}}{q^2}\right] [-ie(p_b + p_b')^\nu] \\ &\quad \times (2\pi)^4 \delta^4(p_a' + p_b' - p_a - p_b)\end{aligned}$$

N.B. symmetry in a, b.

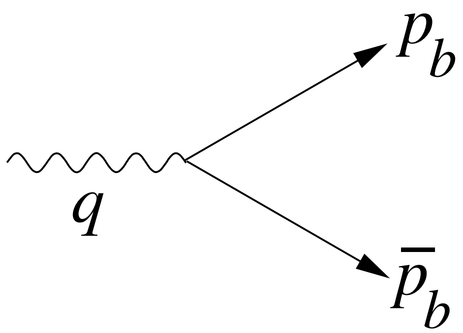
- Thus we have the additional Feynman rule:
  - ❖ $-ig_{\mu\nu}/q^2$ for an internal photon line.

- In processes involving antiparticles, remember we use particles with opposite energy and momentum; $p^\mu = -\bar{p}^\mu$.



$$\text{“}p_i\text{”}=p_a,\ \text{“}p_f\text{”}=-\bar{p}_a,\ \text{“}k\text{”}=-q,$$

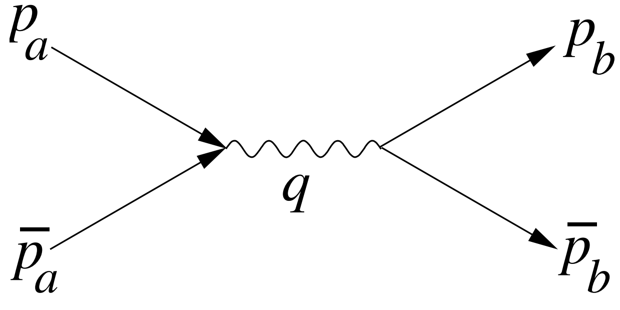
$$\mathcal{A}_{fi} = -ie_a(2\pi)^4\varepsilon\cdot(p_a-\bar{p}_a)\delta^4(q-p_a-\bar{p}_a)$$



$$\text{“}p_i\text{”}=-\bar{p}_b,\ \text{“}p_f\text{”}=p_b,\ \text{“}k\text{”}=q,$$

$$\mathcal{A}_{fi} = -ie_b(2\pi)^4\varepsilon\cdot(p_b-\bar{p}_b)\delta^4(p_b+\bar{p}_b-q)$$

- Annihilation process



$$\mathcal{A}_{fi} = i\frac{e_a e_b}{q^2}(p_a - \bar{p}_a) \cdot (p_b - \bar{p}_b)(2\pi)^4\delta^4(p_b + \bar{p}_b - q)$$

where $q = p_a + \bar{p}_a = p_b + \bar{p}_b$.

- Since we have already normalized to $2E$ particles per unit volume, we have

$$\mathcal{A}_{fi} = \mathcal{M}_{fi}\,(2\pi)^4\delta^4(\sum p_f - \sum p_i)$$

where $\mathcal{M}_{fi}$ is the invariant matrix element (see handout).

- Thus e.g. for annihilation process

$$\mathcal{M}_{fi} = i\frac{e_a e_b}{q^2}(p_a - \bar{p}_a) \cdot (p_b - \bar{p}_b)$$

- In terms of the Mandelstam variables

$$\begin{aligned}
s &= (p_a + \bar{p}_a)^2 = q^2 \\
t &= (p_b - p_a)^2 = (\bar{p}_a - \bar{p}_b)^2 \\
u &= (p_a - \bar{p}_b)^2 = (\bar{p}_a - p_b)^2
\end{aligned}$$

we get

$$\mathcal{M}_{fi} = i\frac{e_a e_b}{s}(u-t)$$

and hence the invariant differential cross section is

$$\frac{d\sigma}{dt} = \frac{e_a^2 e_b^2 (u-t)^2}{64\pi s^3 (p_a^*)^2}$$

where $p_a^* = \sqrt{s/4 - m_a^2}$ = c.m. momentum of a.

Dirac Equation

- Historically, Dirac (1928) was looking for a covariant wave equation that was first-order in time, to avoid the above 'problems' of the Klein-Gordon equation:

$$i\hbar\frac{\partial\psi}{\partial t} = \beta mc^2\psi - i\hbar c\,\boldsymbol{\alpha}\cdot\nabla\psi \equiv H_{\,\text{Dirac}}\psi$$

- We want ψ also to satisfy KG equation $\Rightarrow \beta, \alpha_x, \alpha_y, \alpha_z$ are matrices. Setting $\hbar = c = 1$:

$$\begin{aligned}
-\frac{\partial^2\psi}{\partial t^2} &= \beta mi\frac{\partial\psi}{\partial t} + \boldsymbol{\alpha}\cdot\nabla\frac{\partial\psi}{\partial t} \\
&= \beta^2 m^2\psi - im(\beta\boldsymbol{\alpha} + \boldsymbol{\alpha}\beta)\cdot\nabla\psi - (\boldsymbol{\alpha}\cdot\nabla)^2\psi \\
&= m^2\psi - \nabla^2\psi \qquad \text{(KG equation)}
\end{aligned}$$

Hence

$$\beta^2 = \alpha_x^2 = \alpha_y^2 = \alpha_z^2 = 1$$

and

$$\beta\alpha_j + \alpha_j\beta = \alpha_j\alpha_k + \alpha_k\alpha_j = 0$$

for all $j \neq k = x, y, z$. This means that $\beta, \alpha_x, \alpha_y, \alpha_z$ are (at least) 4×4 matrices.

- A suitable representation is

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array} \right)$$

$$\alpha_j = \left(\begin{array}{c|c} 0 & \sigma_j \\ \hline \sigma_j & 0 \end{array} \right)$$

where σ_j are the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Then ψ is represented by a 4-component object called a spinor (not a 4-vector!)

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

N.B. Each component $\psi_{1,2,3,4}$ satisfies the KG equation.

- For a particle at rest, $\psi = \phi \exp(-imc^2 t/\hbar)$, Dirac equation $\Rightarrow$ $\phi = \beta\phi$, and so

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ 0 \\ 0 \end{pmatrix}$$

where $\phi_{1,2}$ tell us the spin orientation.

- For antiparticle at rest, $\psi = \phi e^{+imc^2 t/\hbar} \Rightarrow \phi = -\beta\phi$, so

$$\phi = \begin{pmatrix} 0 \\ 0 \\ \phi_3 \\ \phi_4 \end{pmatrix}$$

where $\phi_{3,4}$ now give spin orientation.

Spin of Dirac Particles

- How do we prove that Dirac equation corresponds to spin one-half? We must show that there exists an operator $\boldsymbol{S}$ such that $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ is a constant of motion, and $(\hbar = 1)$ $\boldsymbol{S}^2 = S(S+1) = \frac{3}{4}I$.

- Note first that $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ is not a constant of motion:

$$\begin{aligned} H &= \beta m + \boldsymbol{\alpha} \cdot \boldsymbol{p} \\ [L_z, H] &= [x, H]p_y - [y, H]p_x \\ &= i\alpha_x p_y - i\alpha_y p_x \; . \end{aligned}$$

In general, $[\boldsymbol{L}, H] = i\boldsymbol{\alpha} \times \boldsymbol{p} \neq 0$.

- Thus we need $[\boldsymbol{S}, H] = -i\boldsymbol{\alpha} \times \boldsymbol{p}$.
This is true if $\boldsymbol{S} = \frac{1}{2}\boldsymbol{\Sigma}$ where

$$\Sigma_j = \left(\begin{array}{c|c} \sigma_j & 0 \\ \hline 0 & \sigma_j \end{array}\right) = -i\alpha_x\alpha_y\alpha_z\boldsymbol{\alpha} \; .$$

Then $\boldsymbol{S}^2 = \frac{1}{4}(\Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2) = \frac{3}{4}I$, proving that $S = \frac{1}{2}$.

Magnetic Moment

- In an electromagnetic field we make the usual minimal substitutions:

$$H \to H - eV \, , \qquad \boldsymbol{p} \to \boldsymbol{p} - e\boldsymbol{A}$$

in the Dirac equation, to obtain

$$H = \boldsymbol{\alpha} \cdot (\boldsymbol{p} - e\boldsymbol{A}) + \beta m + eV$$

- Note that we no longer get the KG equation when we “square”:

$$(H - eV)^2 = \sum_{j,k} \alpha_j \alpha_k (p_j - eA_j)(p_k - eA_k) + m^2$$

$$= (\boldsymbol{p} - e\boldsymbol{A})^2 + m^2 - e \sum_{j,k} (\alpha_j \alpha_k p_j A_k + \alpha_j \alpha_k A_j p_k)$$

Now for $j \neq k$,

$$\begin{aligned} \alpha_j \alpha_k &= i\varepsilon_{jkl}\Sigma_l \\ p_j A_k &= A_k p_j - i\nabla_j A_k \\ \varepsilon_{jkl}\Sigma_l \nabla_j A_k &= \boldsymbol{\Sigma} \cdot (\nabla \times \boldsymbol{A}) = \boldsymbol{\Sigma} \cdot \boldsymbol{B} \end{aligned}$$

Hence

$$
\begin{aligned}
(H - eV)^2 &= (\boldsymbol{p} - e\boldsymbol{A})^2 + m^2 - e\boldsymbol{\Sigma} \cdot \boldsymbol{B} \\
H - eV &\simeq m + \frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A})^2 - \frac{e}{2m}\boldsymbol{\Sigma} \cdot \boldsymbol{B}
\end{aligned}
$$

- This corresponds to a magnetic moment

$$
\boldsymbol{\mu} = \frac{e}{m}\boldsymbol{S} = g_e \left(\frac{e}{2m}\right) \boldsymbol{S}
$$

where $g_e = 2$ (experiment $\Rightarrow$ 2.0023193. . .).

Dirac Density and Current

- Write Dirac equation as

$$\frac{\partial \psi}{\partial t} = -im\beta\psi - \boldsymbol{\alpha} \cdot (\nabla\psi)$$

- Transpose and complex conjugate:

$$\frac{\partial \psi^\dagger}{\partial t} = +im\psi^\dagger\beta - (\nabla\psi^\dagger) \cdot \boldsymbol{\alpha}$$

N.B. $\beta, \boldsymbol{\alpha}$ are hermitian. Hence

$$\frac{\partial}{\partial t}(\psi^\dagger\psi) = -\nabla(\psi^\dagger\alpha\psi)$$

- Thus we can take

$$\begin{aligned} \rho &= \psi^\dagger\psi \equiv |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 \\ \boldsymbol{J} &= \psi^\dagger\alpha\psi \end{aligned}$$

N.B. Density ρ is positive definite! This is what Dirac wanted, but it is really a problem – what about antiparticles?!

- Answer will not come until we learn some quantum field theory.

Covariant Notation

- Nobody uses $\boldsymbol{\alpha}$ and β any more. Instead we define γ-matrices:

$$\gamma^0 = \beta \, , \qquad \gamma^j = \beta\alpha_j \qquad (j = 1, 2, 3)$$

$\Rightarrow \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \equiv \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. Also define

$$\bar{\psi} \equiv \psi^\dagger\beta \; = \; (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$$

in usual ('Bjorken and Drell') representation. Then

$$\begin{aligned} \rho &= \psi^\dagger\psi = \psi^\dagger\beta^2\psi = \bar{\psi}\gamma^0\psi \\ \boldsymbol{J} &= \psi^\dagger\boldsymbol{\alpha}\psi = \psi^\dagger\beta^2\alpha\psi = \bar{\psi}\boldsymbol{\gamma}\psi \end{aligned}$$

and J^μ is a 4-vector:

$$J^\mu = (\rho, \boldsymbol{J}) \; = \; \bar{\psi}\gamma^\mu\psi$$

- We can also show that

$$\bar{\psi}\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2$$

transforms like a scalar (invariant) under Lorentz transformations.

- Multiplying through by β, Dirac equation becomes

$$i\gamma^0 \frac{\partial \psi}{\partial t} = m\psi + i\gamma^j \nabla_j \psi$$

Hence

$$\begin{aligned}(\gamma^\mu \partial_\mu + im)\psi &= 0 \\ (\gamma^\mu p_\mu - m)\psi &= 0\end{aligned}$$

Free-Particle Spinors

- A positive-energy plane wave

$$\psi = u(E, \boldsymbol{p}) \exp(i\boldsymbol{p} \cdot \boldsymbol{r} - iEt)$$

satisfies $(\gamma^\mu p_\mu - m)u = 0$. Writing

$$u = \left(\frac{\phi}{\chi} \right) = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \chi_1 \\ \chi_2 \end{pmatrix}$$

this means that

$$\left(\begin{array}{c|c} E - m & -\boldsymbol{\sigma} \cdot \boldsymbol{p} \\ \hline +\boldsymbol{\sigma} \cdot \boldsymbol{p} & -E - m \end{array} \right) \left(\frac{\phi}{\chi} \right) = 0$$

Thus $\qquad \chi = \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E + m} \phi$

- Remember that

$$\boldsymbol{S} = \tfrac{1}{2} \boldsymbol{\Sigma} = \tfrac{1}{2} \left(\begin{array}{c|c} \boldsymbol{\sigma} & 0 \\ \hline 0 & \boldsymbol{\sigma} \end{array} \right)$$

Hence

$$
\begin{aligned}
\phi &= N\begin{pmatrix}1\\0\end{pmatrix} \qquad \text{for spin up (along } z\text{-axis)}\\
&= N\begin{pmatrix}0\\1\end{pmatrix} \qquad \text{for spin down}
\end{aligned}
$$

We have also

$$
\boldsymbol{\sigma}\cdot\boldsymbol{p}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}p_z\\p_x+ip_y\end{pmatrix}, \quad \boldsymbol{\sigma}\cdot\boldsymbol{p}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}p_x-ip_y\\-p_z\end{pmatrix}
$$

Thus

$$
u^{\uparrow} = N\begin{pmatrix}1\\0\\\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\end{pmatrix}, \qquad u^{\downarrow} = N\begin{pmatrix}0\\1\\\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\end{pmatrix}.
$$

- Normalization is as usual $\rho = \psi^{\dagger}\psi = u^{\dagger}u = 2E$ particles per unit volume. This gives

$$
N^2\left[1+\frac{p_x^2+p_y^2+p_z^2}{(E+m)^2}\right] = 2E
$$

Using $\boldsymbol{p}^2 = E^2 - m^2$ gives $N = \sqrt{E+m}$.

❖ Notice that the 'small' (3,4) components are $\mathcal{O}(v/c)$ relative to 'large' ones (1,2).

- For antiparticle of 4-momentum $(E, \boldsymbol{p})$ we need solution with $p^\mu \to (-E, -\boldsymbol{p})$:

$$\psi = v(E, \boldsymbol{p}) \exp(-i\boldsymbol{p} \cdot \boldsymbol{r} + iEt)$$

From the Dirac equation we now find

$$\left(\begin{array}{c|c} -E - m & \boldsymbol{\sigma} \cdot \boldsymbol{p} \\ \hline -\boldsymbol{\sigma} \cdot \boldsymbol{p} & E - m \end{array} \right) \left(\begin{array}{c} \phi \\ \hline \chi \end{array} \right) = 0$$

Thus $\qquad \phi = \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E + m} \chi$

- Like 4-momentum, spin must be reversed, so

$$v^\uparrow = N \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}, \qquad v^\downarrow = N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}.$$

Charge Conjugation

- Like the KG equation, the Dirac equation has charge conjugation symmetry. If ψ is a negative-energy solution, there is a transformation

$$\psi \to \psi^c = C\psi^*$$

such that ψ^c is a positive-energy solution for charge $-e$. To find C:

$$\begin{aligned} & \gamma^\mu(\partial_\mu + ieA_\mu)\psi + im\psi = 0 \\ \Rightarrow\quad & \gamma^{*\mu}(\partial_\mu - ieA_\mu)\psi^* - im\psi^* = 0 \\ \Rightarrow\quad & -C\gamma^{*\mu}C^{-1}(\partial_\mu - ieA_\mu)\psi^c + im\psi^c = 0\,. \end{aligned}$$

Hence we need $C\gamma^{*\mu}C^{-1} = -\gamma^\mu$, i.e. $\gamma^\mu C = -C\gamma^{*\mu}$.

- Since all γ^μ are real except γ^2 (which is pure imaginary) in our standard representation, we can take

$$C = i\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Explicitly, for free particles, $v^{\uparrow c} = u^\uparrow$, $v^{\downarrow c} = -u^\downarrow$

Parity Invariance

- Similarly if $\psi(\boldsymbol{r},t)$ is a solution of the Dirac equation, there exists a transformation

$$\psi(\boldsymbol{r},t) \to \psi^P(\boldsymbol{r},t) = P\psi(-\boldsymbol{r},t)$$

such that ψ^P is also a solution. Now

$$\left(\gamma^0 \frac{\partial}{\partial t} - \boldsymbol{\gamma} \cdot \nabla + im\right) \psi(\boldsymbol{r},t) = 0$$

$$\Rightarrow \quad \left(\gamma^0 \frac{\partial}{\partial t} + \boldsymbol{\gamma} \cdot \nabla + im\right) \psi(-\boldsymbol{r},t) = 0$$

$$\Rightarrow \quad \left(P\gamma^0 P^{-1} \frac{\partial}{\partial t} + P\boldsymbol{\gamma}P^{-1} \cdot \nabla + im\right) \psi^P(\boldsymbol{r},t) = 0\ .$$

Hence we need $P\gamma^0 P^{-1} = \gamma^0\ , \quad P\boldsymbol{\gamma}P^{-1} = -\boldsymbol{\gamma}\ ,$
i.e $P\gamma^0 = \gamma^0 P\ , \quad P\gamma^j = -\gamma^j P$ $(j = 1,2,3)$

- These relations are satisfied by

$$P = \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- For a particle at rest,

$$\psi = u(m, \mathbf{0})\, e^{-imt} \, , \qquad \psi^P = +\psi$$

but for an antiparticle at rest

$$\psi = v(m, \mathbf{0})\, e^{+imt} \, , \qquad \psi^P = -\psi$$

Thus particle and antiparticle have opposite intrinsic parity.

- Notice that for KG equation the parity transformation is simply

$$\phi(\boldsymbol{r}, t) \to \phi^P(\boldsymbol{r}, t) = \phi(-\boldsymbol{r}, t)$$

i.e. ϕ is a true scalar function, since

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \phi(\boldsymbol{r}, t) = 0$$
$$\Rightarrow \quad \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \phi(-\boldsymbol{r}, t) = 0$$

- For the Dirac equation, the scalar is not ψ but

$$\Phi = \bar{\psi}\psi = \psi^\dagger \gamma^0 \psi$$

Check:

$$\begin{aligned}
\Phi(\boldsymbol{r},t) &= \psi^\dagger(\boldsymbol{r},t)\gamma^0\psi(\boldsymbol{r},t) \\
\Phi^P(\boldsymbol{r},t) &= \psi^\dagger(-\boldsymbol{r},t)\gamma^{0\dagger}\gamma^0\gamma^0\psi(-\boldsymbol{r},t) \\
&= \psi^\dagger(-\boldsymbol{r},t)\gamma^0\psi(-\boldsymbol{r},t) \\
&= \Phi(-\boldsymbol{r},t)
\end{aligned}$$

- Similarly, J^μ is a true vector:

$$\begin{aligned}
J^\mu(\boldsymbol{r},t) &= \psi^\dagger(\boldsymbol{r},t)\gamma^0\gamma^\mu\psi(\boldsymbol{r},t) \\
J^{P\mu}(\boldsymbol{r},t) &= \psi^\dagger(-\boldsymbol{r},t)\gamma^{0\dagger}\gamma^0\gamma^\mu\gamma^0\psi(-\boldsymbol{r},t)
\end{aligned}$$

But $\gamma^{0\dagger}\gamma^0\gamma^\mu\gamma^0 = \gamma^\mu\gamma^0 = \gamma^0\gamma^\mu$ for $\mu = 0$, $= -\gamma^0\gamma^\mu$ for $\mu = 1, 2, 3$. Hence, as expected for a true vector,

$$J^{P0}(\boldsymbol{r},t) = J^0(-\boldsymbol{r},t)\ , \qquad \boldsymbol{J}^P(\boldsymbol{r},t) = -\boldsymbol{J}(-\boldsymbol{r},t)\ .$$

- Weak interactions involve the axial current

$$J_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$$

where

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \left(\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array}\right)$$

in our standard representation. Under parity transformations this is an axial vector:

$$J_A^{P\mu}(\boldsymbol{r},t) = \psi^\dagger(-\boldsymbol{r},t)\gamma^\mu\gamma^5\gamma^0\psi(-\boldsymbol{r},t)$$

Now $\gamma^5\gamma^0 = -\gamma^0\gamma^5$ (actually $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ for all $\mu = 0,1,2,3$), so

$$J_A^{P0}(\boldsymbol{r},t) = -J^0(-\boldsymbol{r},t)\ , \qquad \boldsymbol{J}_A^P(\boldsymbol{r},t) = \boldsymbol{J}(-\boldsymbol{r},t)$$

as expected for an axial vector.

- Similarly $\Phi_P = \bar{\psi}\gamma^5\psi$ is a pseudoscalar

$$\begin{aligned}
\Phi_P^P(\boldsymbol{r},t) &= \psi^\dagger(-\boldsymbol{r},t)\gamma^5\gamma^0\psi(-\boldsymbol{r},t) \\
&= -\bar{\psi}(-\boldsymbol{r},t)\gamma^5\psi(-\boldsymbol{r},t) \\
&= -\Phi_P(-\boldsymbol{r},t)
\end{aligned}$$

Massless Dirac Particles

- For $m = 0$ the positive-energy free particle solutions are

$$\psi = u(E, \boldsymbol{p}) \exp(i\boldsymbol{p} \cdot \boldsymbol{r} - iEt)$$

where $E = |\boldsymbol{p}|$ and so $u = \left(\frac{\phi}{\chi}\right)$ gives

$$\left(\begin{array}{c|c} |\boldsymbol{p}| & -\boldsymbol{\sigma} \cdot \boldsymbol{p} \\ \hline \boldsymbol{\sigma} \cdot \boldsymbol{p} & -|\boldsymbol{p}| \end{array}\right) \left(\frac{\phi}{\chi}\right) = 0$$

Hence $\chi = \Lambda\phi$ where $\Lambda = \boldsymbol{\sigma} \cdot \boldsymbol{p}/|\boldsymbol{p}|$ is the helicity operator: $\Lambda = \pm 1$ for spin aligned along/against direction of $\boldsymbol{p}$ ('right/left-handed')

- Note that if ψ represents a massless particle then

$$\gamma^5 \psi = \left(\frac{\Lambda\phi}{\phi}\right) = \Lambda\psi \qquad\qquad (\Lambda^2 = 1)$$

- Hence γ^5 is the helicity operator for massless particles (minus helicity for massless antiparticles).

- Weak interactions are 'V–A', i.e. they involve the current

$$(J^\mu - J^\mu_A)_{fi} = \bar{\psi}_f \gamma^\mu (1-\gamma^5)\psi_i$$

If i is a massless particle, then $(1-\gamma^5)\psi_i$ vanishes for helicity $+1$, i.e. only left-handed states interact. The same applies to particle f, since

$$\begin{aligned}\bar{\psi}_f \gamma^\mu (1-\gamma^5)\psi_i &= \psi_f^\dagger \gamma^0 (1+\gamma^5)\gamma^\mu \psi_i \\ &= \left[(1-\gamma^5)\psi_f\right]^\dagger \gamma^0\gamma^\mu)\psi_i\end{aligned}$$

- Thus if neutrinos are massless, only left-handed neutrinos (right-handed antineutrinos) interact.
- In the Standard Model, right-handed neutrinos do not exist.
- This is consistent with relativity, because helicity is frame-independent for massless particles.
- If neutrinos have mass, both helicities must exit, but the right-handed states interact more weakly (as for electrons).

Electromagnetic Interactions

- We already saw that in an e.m. field Dirac Hamiltonian is

$$\begin{aligned} H &= \boldsymbol{\alpha} \cdot (\boldsymbol{p} - e\boldsymbol{A}) + \beta m + eV \\ &= H_0 + e\gamma^0\gamma^\mu A_\mu \end{aligned}$$

where H_0 is the free-particle Hamiltonian.

- Hence first-order perturbation theory gives a transition amplitude

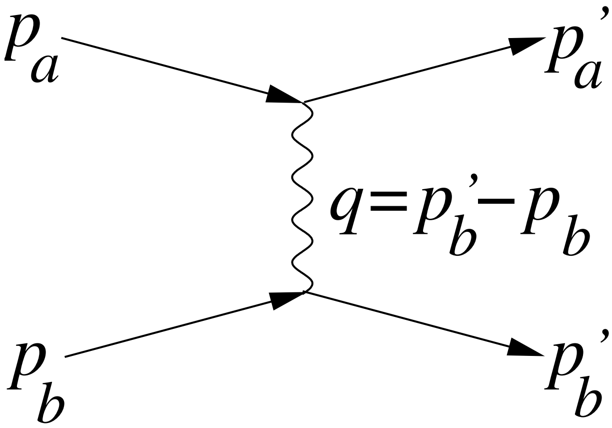
$$\begin{aligned} \mathcal{A}_{fi} &= -i \int \psi_f^\dagger (e\gamma^0\gamma^\mu A_\mu)\psi_i d^4x \\ &= -ie \int J_{fi}^\mu A_\mu d^4x \end{aligned}$$

where $J_{fi}^\mu = \bar{\psi}_f \gamma^\mu \psi_i$.

- For plane waves, $\psi_{f,i} = u_{f,i} e^{-ip\cdot x}$, and so the only difference from the KG (spin zero) case is that we need a vertex factor of

$$-ie\bar{u}_f \gamma^\mu u_i$$

for spin one-half instead of $-ie(p_f + p_i)^\mu$ for spin 0.



- Invariant matrix element is

$$\mathcal{M}_{fi} = \frac{ie_a e_b}{q^2}(\bar{u}_{a'}\gamma^\mu u_a)(\bar{u}_{b'}\gamma_\mu u_b)$$

Hence

$$|\mathcal{M}_{fi}|^2 = \frac{e_a^2 e_b^2}{t^2} L_a^{\mu\nu} L^b_{\mu\nu}$$

where

$$\begin{aligned} L_a^{\mu\nu} &= (\bar{u}_{a'}\gamma^\mu u_a)(\bar{u}_{a'}\gamma^\nu u_a)^* \\ L^b_{\mu\nu} &= (\bar{u}_{b'}\gamma_\mu u_b)(\bar{u}_{b'}\gamma_\nu u_b)^* \end{aligned}$$

- For given spin states of a, b, a' and b', we can evaluate this explicitly using the above expressions for free-particle spinors. However, we often consider unpolarized scattering, when we have to average over initial and sum over final spin states. Then

$$L_a^{\mu\nu} = \tfrac{1}{2}\sum_{\text{spins}}(\bar{u}_{a'}\gamma^\mu u_a)(\bar{u}_{a'}\gamma^\nu u_a)^*$$

and similarly for $L^b_{\mu\nu}$. This can be evaluated using the algebra of the γ-matrices.

Gamma Matrix Algebra

- The tensor

$$L_a^{\mu\nu} = \tfrac{1}{2} \sum_{\text{spins}} (\bar{u}_{a'} \gamma^\mu u_a)(\bar{u}_{a'} \gamma^\nu u_a)^*$$

can be expressed in terms of traces of products of γ-matrices, using

$$\sum_{\text{spins}} u\bar{u} = u^\uparrow \bar{u}^\uparrow + u^\downarrow \bar{u}^\downarrow = \gamma^\mu p_\mu + m$$

We also have

$$(\bar{u}_{a'} \gamma^\nu u_a)^* = (u^\dagger_{a'} \gamma^0 \gamma^\nu u_a)^* = u^\dagger_a \gamma^{\nu\dagger} \gamma^0 u_{a'} = \bar{u}_a \gamma^\nu u_{a'}$$

since $\gamma^{\nu\dagger}\gamma^0 = \gamma^0 \gamma^\nu$.

- Thus

$$L_a^{\mu\nu} = \tfrac{1}{2} \sum_{a'} \sum_{\text{spins}} \bar{u}_{a'} \gamma^\mu (\not{p}_a + m_a) \gamma^\nu u_{a'}$$

where we use Feynman's notation $\not{p} = \gamma^\mu p_\mu$. Putting in Dirac matrix indices, $\bar{u}_\alpha \Gamma_{\alpha\beta} u_\beta = \mathrm{Tr}\,(u\bar{u}\Gamma)$. Hence

$$L_a^{\mu\nu} \quad = \quad \tfrac{1}{2}\mathrm{Tr}\, \{(\not{p}'_a + m_a)\gamma^\mu(\not{p}_a + m_a)\gamma^\nu\}$$

$$
\begin{aligned}
k_\mu k'_\nu L_a^{\mu\nu} &= \tfrac{1}{2}\mathrm{Tr}\,\{(\not{p}'_a + m_a)\not{k}(\not{p}_a + m_a)\not{k}'\} \\
&= \tfrac{1}{2}\mathrm{Tr}\,\{\not{p}'_a\not{k}\not{p}_a\not{k}'\} + \tfrac{1}{2}m_a^2\mathrm{Tr}\,\{\not{k}\not{k}'\}
\end{aligned}
$$

$$= 2(p'_a \cdot k p_a \cdot k' + p_a \cdot k p'_a \cdot k' - p_a \cdot p'_a k \cdot k' + m_a^2 k \cdot k')$$

(see examples sheet for last step).

- Removing the arbitrary vectors k_μ and k'_ν,

$$L_a^{\mu\nu} = 2\left[p_a^\mu p_a'^\nu + p_a'^\mu p_a^\nu - (p_a \cdot p'_a - m_a^2)g^{\mu\nu}\right]$$

and similarly

$$L^b_{\mu\nu} = 2\left[p_{b\mu}p'_{b\nu} + p'_{b\mu}p_{b\nu} - (p_b \cdot p'_b - m_b^2)g_{\mu\nu}\right]$$

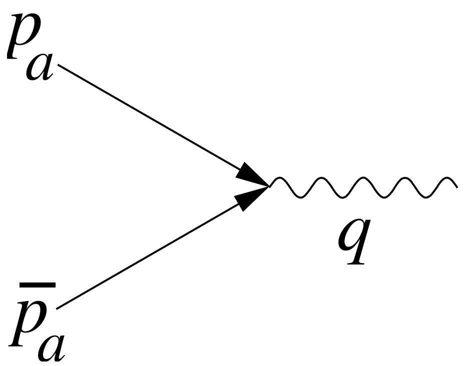
so

$$
\begin{aligned}
&L_a^{\mu\nu}L^b_{\mu\nu} = 8(p_a \cdot p_b\, p'_a \cdot p'_b + p_a \cdot p'_b\, p'_a \cdot p_b \\
&-m_a^2 p_b \cdot p'_b - m_b^2 p_a \cdot p'_a + 2m_a^2 m_b^2)
\end{aligned}
$$

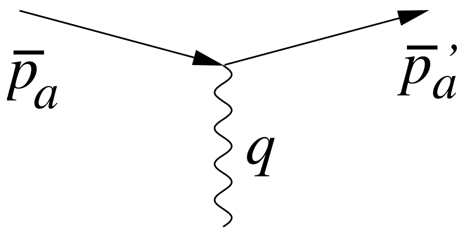
- Expressing this in terms of the Mandelstam invariants s, t and u, we find an invariant differential cross section

$$
\begin{aligned}
&\frac{d\sigma}{dt} = \frac{e_a^2 e_b^2}{32\pi s t^2 (p_a^*)^2} \times \\
&\left[s^2 + u^2 - 4(m_a^2 + m_b^2)(s+u) + 6(m_a^2 + m_b^2)^2\right]
\end{aligned}
$$

- For processes involving Dirac antiparticles, we should use the v-spinors in place of u's:



"u_i"$=u_a$, "$\bar{u}_f$"$=\bar{v}'_a$ $\Rightarrow$ vertex factor $-ie\bar{v}'_a\gamma^\mu u_a$.



"u_i"$=v'_a$, "$\bar{u}_f$"$=\bar{v}_a$ $\Rightarrow$ vertex factor $-ie\bar{v}_a\gamma^\mu v'_a$.

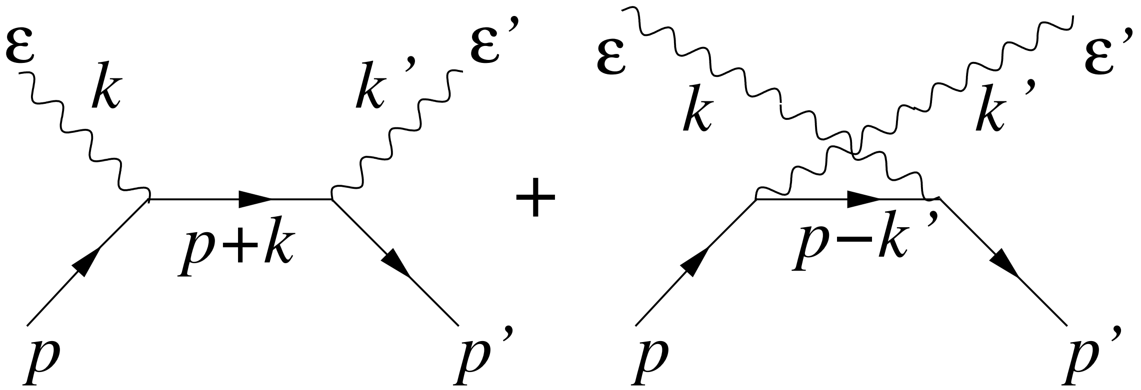
- We also need

$$\sum_{\text{spins}} v\bar{v} = v^\uparrow\bar{v}^\uparrow + v^\downarrow\bar{v}^\downarrow = \not{p} - m$$

N.B. Different sign of m!

- Note, however, that the tensor $L_a^{\mu\nu}$ only involves m_a^2. Replacing a by $\bar{a}$ reverses sign of m_a, which does not affect the (unpolarized) scattering cross section. Hence ab, $\bar{a}b$, $a\bar{b}$ and $\bar{a}\bar{b}$ scattering (by single photon exchange) are all the same.

Compton Scattering



- In the Compton scattering process $\gamma + e \to \gamma + e$, we need the propagator factor for a virtual Dirac particle. This is

$$\frac{i}{q^2 - m^2} \sum_{\text{spins}} u\bar{u} = \frac{i(\not{q} + m)}{q^2 - m^2}$$

- Compare with photon propagator

$$\frac{i}{q^2} \sum_{\text{spins}} \varepsilon_\mu \varepsilon_\nu^* \text{ ``}=\text{''} \frac{i(-g_{\mu\nu})}{q^2}$$

Thus the two graphs give $\mathcal{M}_{fi} = \mathcal{M}_1 + \mathcal{M}_2$ where

$$\begin{aligned} \mathcal{M}_1 &= \varepsilon_\nu' \bar{u}'(-ie\gamma^\nu) \frac{i(\not{p} + \not{k} + m)}{(p+k)^2 - m^2} (-ie\gamma^\mu) u \varepsilon_\mu \\ \mathcal{M}_2 &= \varepsilon_\mu \bar{u}'(-ie\gamma^\mu) \frac{i(\not{p} - \not{k}' + m)}{(p-k')^2 - m^2} (-ie\gamma^\nu) u \varepsilon_\nu' \end{aligned}$$

- The relative phase is $+1$ because the graphs differ by exchange of identical bosons.

- For the unpolarized case, we want to average over initial spin states and sum over final ones. We know how to do this for the electrons. For the photons, consider the incoming polarization ε_μ. We can write schematically

$$\sum_{\text{spins}} |\mathcal{M}_1 + \mathcal{M}_2|^2 = \sum_{\varepsilon=\varepsilon_x,\varepsilon_y} \varepsilon_\mu \varepsilon_\lambda^* M^{\mu\lambda}$$

  where the tensor $M^{\mu\lambda}$ is to be determined. However, we know that it must have the properties $k_\mu M^{\mu\lambda} = k_\lambda M^{\mu\lambda} = 0$ to ensure gauge invariance, which allows us to replace $\varepsilon_\mu \to \varepsilon_\mu + a k_\mu$ for any a.

  Choose z-axis along $\boldsymbol{k}$: $k_\mu = |\boldsymbol{k}|(1,0,0,-1)$. Then above property implies $M^{00} = M^{03} = M^{30} = M^{33}$, while

$$\begin{aligned} \sum_{\varepsilon=\varepsilon_x,\varepsilon_y} \varepsilon_\mu \varepsilon_\lambda^* M^{\mu\lambda} &= M^{11} + M^{22} \\ &= M^{11} + M^{22} + M^{33} - M^{00} \\ &= -M_\mu^\mu = -g_{\mu\lambda} M^{\mu\lambda} \end{aligned}$$

- Thus, due to gauge invariance, we can replace photon polarization sum by $-g_{\mu\lambda}$.

- Applying the same trick to the outgoing photon polarization (ε'_ν) sum, we find a contribution from first diagram of

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_1|^2 = \frac{e^4}{4(s-m^2)^2} \times$$
$$\text{Tr}\,\{(\not p' + m)\gamma^\nu(\not p + \not k + m)\gamma^\mu(\not p + m)\gamma_\mu(\not p + \not k + m)\gamma_\nu\}$$

- In extreme relativistic limit ($s, |t|, |u| \gg m^2$), this becomes (using results on examples sheet)

$$\begin{aligned}\frac{e^4}{s^2}\text{Tr}\,\{\not p(\not p + \not k)\not p'(\not p + \not k)\} &= 8\frac{e^4}{s^2}(p\cdot k)(p'\cdot k) \\ &= -2e^4\frac{u}{s}\end{aligned}$$

- Other diagram and interference terms are left as an exercise.