Gauge Symmetry in QED

- The Lagrangian density for the free e.m. field is

$$\mathcal{L}_{\text{em}} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where $F^{\mu\nu}$ is the field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Thus $\mathcal{L}_{\text{em}} = \frac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2)$

- In $A^0 = 0$ gauge the momentum density is

$$\boldsymbol{\pi} = \frac{\partial \boldsymbol{A}}{\partial t} = -\boldsymbol{E}$$

hence

$$\mathcal{H}_{\text{em}} = \boldsymbol{\pi} \cdot \dot{\boldsymbol{A}} - \mathcal{L}_{\text{em}} = \tfrac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2)$$

- Gauge symmetry: $F^{\mu\nu}$ and hence $\mathcal{L}_{\text{em}}$ are invariant under transformation

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \chi$$

N.B. A photon mass term $\frac{1}{2}m_\gamma^2 A^\mu A_\mu$ is forbidden by gauge symmetry.

- The full Lagrangian for charged Dirac fermions interacting with the e.m. field is $\mathcal{L} = \mathcal{L}_{\text{em}} + \mathcal{L}_{\text{D}}$ where

$$\mathcal{L}_{\text{D}} = i\bar{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi - m\bar{\psi}\psi$$

- Under a phase transformation

$$\psi \to \psi' = e^{i\phi(x)}\psi$$

we have

$$\mathcal{L}_{\text{D}} \to i\bar{\psi}'\gamma^\mu(\partial_\mu + ieA_\mu - i\partial_\mu\phi)\psi' - m\bar{\psi}'\psi'$$

- Invariance under phase transformations thus requires a gauge transformation of the e.m. field

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\chi = A_\mu - \partial_\mu\phi/e$$

i.e. $\chi(x) = -e\phi(x)$. Conversely, gauge invariance of the whole Lagrangian will be preserved provided we simultaneously change the phase of the fermion field

$$\psi \to \psi' = e^{-ie\chi(x)}\psi$$

- Gauge transformations of e.m. field $\Rightarrow$ phase transformation of (charged) Dirac field.

- Conversely, if we demand symmetry of $\mathcal{L}$ under local (x^μ-dependent) phase transformations of ψ, this requires the existence of a (massless) vector field to cancel the term involving $\partial_\mu \phi(x)$.

- The gauge symmetry occurs because the derivative ∂_μ only appears in the combination called the covariant derivative

$$D_\mu = \partial_\mu + ieA_\mu$$

Then $D_\mu \psi$ transforms in the same way as ψ itself:

$$D'_\mu \psi' = [\partial_\mu + ieA_\mu + ie(\partial_\mu \chi)]e^{-ie\chi}\psi = e^{-ie\chi} D_\mu \psi$$

- Note that $\mathcal{L}_{\text{em}}$ also involves only D_μ since

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = D^\mu A^\nu - D^\nu A^\mu$$

- Recall that e.m. phase (gauge) symmetry $\Rightarrow$ conservation of electric current and charge (Noether)

- Successive gauge transformations commute:

$$e^{-ie\chi_1}e^{-ie\chi_2} = e^{-ie\chi_2}e^{-ie\chi_1} = e^{-ie(\chi_1+\chi_2)}$$

  This is an Abelian gauge symmetry, with gauge group U(1). Thus quantum electrodynamics is a U(1) gauge theory.

- It is believed that all fundamental interactions are described by some form of gauge theory.

Non-Abelian Gauge Symmetry

- Suppose the Lagrangian involves two fermion fields (e.g. ν_e and e^-) and we demand symmetry under transformations that mix them together while preserving normalization and orthogonality:

$$\begin{aligned} \psi_1 &\rightarrow \psi_1' = \alpha\psi_1 + \beta\psi_2 \\ \psi_2 &\rightarrow \psi_2' = \gamma\psi_1 + \delta\psi_2 \end{aligned}$$

We require

$$\begin{aligned} \alpha\alpha^* + \beta\beta^* &= \gamma\gamma^* + \delta\delta^* = 1 \\ \alpha\gamma^* + \beta\delta^* &= \gamma\alpha^* + \delta\beta^* = 0 \end{aligned}$$

Hence

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \rightarrow \Psi' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \equiv U\Psi$$

where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

i.e. $UU^\dagger = I$. Hence the matrix of coefficients U is a unitary matrix.

- U has 4 complex elements satisfying 4 constraints $\Rightarrow$ 4 real parameters. It can be written as

$$U = \exp[i\alpha_0 + i\alpha_1\tau_1 + i\alpha_2\tau_2 + i\alpha_3\tau_3]$$

where $\tau_{1,2,3}$ are the Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\alpha_{0,1,2,3}$ are the real parameters.

- The exponential of a matrix A can be defined as

$$\exp[A] = I + A + \tfrac{1}{2}A^2 + \ldots$$

Note that in general

$$\exp[A_1]\exp[A_2] \neq \exp[A_2]\exp[A_1] \neq \exp[A_1 + A_2]$$

$\Rightarrow$ the gauge symmetry group [U(2)] is non-Abelian.

- We can write $U = e^{i\alpha_0}V$ where $e^{i\alpha_0} \in$ U(1), the Abelian symmetry group, and $V = e^{i\,\boldsymbol{\alpha}\cdot\,\boldsymbol{\tau}} \in$ SU(2), the non-Abelian group of 2×2 unitary matrices with unit determinant.

Check:

$$\begin{aligned} \det U &= \exp[i\mathrm{Tr}\,(\boldsymbol{\alpha}\cdot\boldsymbol{\tau})] = \exp[0] = 1 \\ U^\dagger &= \exp[-i\boldsymbol{\alpha}^*\cdot\boldsymbol{\tau}^\dagger] = \exp[-i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}] = U^{-1} \end{aligned}$$

- The matrices $\tau_{1,2,3}$ are the generators of the group SU(2). An infinitesimal gauge transformation can be written as

$$U = e^{i\,\boldsymbol{\alpha}\cdot\,\boldsymbol{\tau}} \simeq I + i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}$$

- In fact we can define the exponential for matrices like this:

$$\begin{aligned} e^A &= \lim_{N\to\infty}\left(I + \frac{A}{N}\right)^N \\ e^{i\,\boldsymbol{\alpha}\cdot\,\boldsymbol{\tau}} &= \lim_{\epsilon\to 0}\left(I + i\,\epsilon\,\boldsymbol{\alpha}\cdot\boldsymbol{\tau}\right)^{1/\epsilon} \end{aligned}$$

- For simplicity we shall usually consider infinitesimal gauge transformations. If necessary, we can then build up finite ones as above.

- Consider a small SU(2) gauge transformation of the form

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \Psi' = \left(I + i\frac{g}{2}\boldsymbol{\omega}(x)\cdot\boldsymbol{\tau}\right)\Psi$$

i.e. $\alpha_{1,2,3} = \frac{1}{2} g\, \omega_{1,2,3}$ small. By analogy with QED, we expect gauge invariance to require the presence of vector fields W_1^μ, W_2^μ, W_3^μ, coupling to the fermions via the covariant derivative

$$\begin{aligned} D^\mu &= \partial^\mu + i\frac{g}{2}\boldsymbol{W}^\mu \cdot \boldsymbol{\tau} \\ \mathcal{L}_{\text{D}} &= i\overline{\Psi}\gamma^\mu D_\mu \Psi \end{aligned}$$

N.B. Mass term $\propto \overline{\Psi}\Psi$ to be discussed later.

- What transformation law must $W^\mu_{1,2,3}$ have to make $\mathcal{L}_{\text{D}}$ gauge invariant? We need $D_\mu\Psi$ to transform just like Ψ itself. Then

$$\begin{aligned} D'_\mu \Psi' &= \left(1 + i\frac{g}{2}\boldsymbol{\omega} \cdot \boldsymbol{\tau}\right) D_\mu \Psi \\ \overline{\Psi}' &= \overline{\Psi}\left(1 - i\frac{g}{2}\boldsymbol{\omega} \cdot \boldsymbol{\tau}\right) \\ \Rightarrow & \qquad \overline{\Psi}'\gamma^\mu D'_\mu \Psi' = \overline{\Psi}\gamma^\mu D_\mu \Psi \end{aligned}$$

(up to terms of order ω^2, which we are neglecting).

- This implies a more complicated gauge transformation law for $W^\mu_{1,2,3}$. With

$$D^\mu \to D'^\mu = \partial^\mu + i\frac{g}{2}\boldsymbol{W}'^\mu \cdot \boldsymbol{\tau}$$

we have

$$D'^{\mu}\Psi' = \left(\partial^{\mu} + i\frac{g}{2}\boldsymbol{W}'^{\mu} \cdot \boldsymbol{\tau}\right) \left(1 + i\frac{g}{2}\boldsymbol{\omega} \cdot \boldsymbol{\tau}\right) \Psi$$

As we have seen, this should equal

$$\begin{aligned} &\left(1 + i\frac{g}{2}\boldsymbol{\omega} \cdot \boldsymbol{\tau}\right) D^{\mu}\Psi \\ &= \left(1 + i\frac{g}{2}\boldsymbol{\omega} \cdot \boldsymbol{\tau}\right) \left(\partial^{\mu} + i\frac{g}{2}\boldsymbol{W}^{\mu} \cdot \boldsymbol{\tau}\right) \Psi \end{aligned}$$

Thus

$$\begin{aligned} &\partial^{\mu} + i\frac{g}{2}\boldsymbol{W}'^{\mu} \cdot \boldsymbol{\tau} \\ &= \left(1 + i\frac{g}{2}\boldsymbol{\omega} \cdot \boldsymbol{\tau}\right) \left(\partial^{\mu} + i\frac{g}{2}\boldsymbol{W}^{\mu} \cdot \boldsymbol{\tau}\right) \left(1 + i\frac{g}{2}\boldsymbol{\omega} \cdot \boldsymbol{\tau}\right)^{-1} \\ &= \partial^{\mu} + i\frac{g}{2}\boldsymbol{W}^{\mu} \cdot \boldsymbol{\tau} \;\; - i\frac{g}{2}(\partial^{\mu}\boldsymbol{\omega}) \cdot \boldsymbol{\tau} \\ &-\frac{g^2}{4}\left(\omega_j\tau_j W_k^{\mu}\tau_k - W_k^{\mu}\tau_k\omega_j\tau_j\right) + \mathcal{O}(\omega^2) \end{aligned}$$

Now

$$\tau_j\tau_k - \tau_k\tau_j = 2i\,\epsilon_{jkl}\,\tau_l$$

Hence

$$W_l'^{\mu} = W_l^{\mu} - \partial^{\mu}\omega_l - g\,\epsilon_{jkl}\,\omega_j W_k^{\mu}$$

- To preserve gauge invariance we have introduced gauge fields $W^{\mu}_{1,2,3}$ via the covariant derivative. Clearly we also have to add a gauge field part $\mathcal{L}_{\mathrm{G}}$ to the Lagrangian, to allow propagation of the gauge fields.
  - We might expect $\mathcal{L}_{\mathrm{G}}$ to be $-\frac{1}{4}F^{\mu\nu}_j F_{j\mu\nu}$ where $F^{\mu\nu}_j = \partial^\mu W^\nu_j - \partial^\nu W^\mu_j$, by analogy with QED.
    But this is not gauge invariant!
  - From transformation law for W^μ_j we have

$$\begin{aligned} F'^{\mu\nu}_j &= F^{\mu\nu}_j - g\epsilon_{jkl}\omega_k F^{\mu\nu}_l \\ &\quad -g\,\epsilon_{jkl}\left[(\partial^\mu\omega_k)W^\nu_l - (\partial^\nu\omega_k)W^\mu_l\right] \end{aligned}$$

and so

$$\begin{aligned} -\tfrac{1}{4}F'^{\mu\nu}_j F'_{j\mu\nu} &= -\tfrac{1}{4}F^{\mu\nu}_j F_{j\mu\nu} \\ &\quad +\frac{g}{2}\epsilon_{jkl}F^{\mu\nu}_j\left[(\partial_\mu\omega_k)W_{l\nu} - (\partial_\nu\omega_k)W_{\mu_l}\right] \end{aligned}$$

using antisymmetry of ϵ_{jkl}, neglecting terms of order ω^2.

- To get rid of the extra term we must define

$$\mathcal{L}_{\mathrm{G}} = -\tfrac{1}{4}G^{\mu\nu}_j G_{j\mu\nu}$$

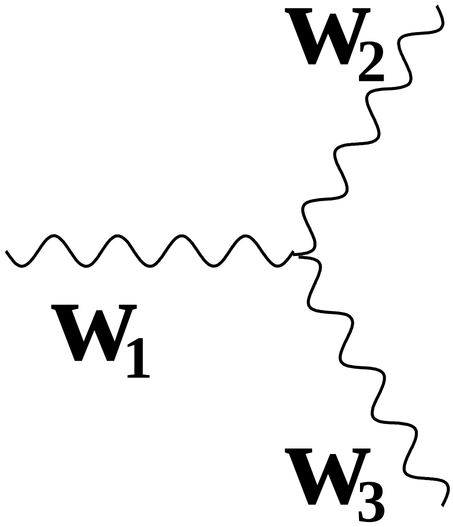
where

$$G^{\mu\nu}_j = \partial^\mu W^\nu_j - \partial^\nu W^\mu_j - g\,\epsilon_{jkl}W^\mu_k W^\nu_l$$

- Notice that $\mathcal{L}_{\mathrm{G}}$ now contains terms that represent self-interactions of the gauge fields

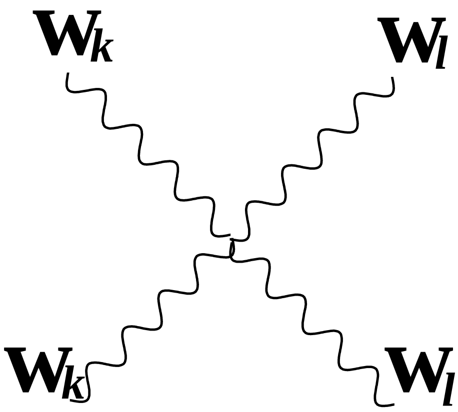
$$-\tfrac{1}{2} g \epsilon_{jkl} \left(\partial^\mu W_j^\nu - \partial^\nu W_j^\mu \right) W_{k\mu} W_{l\nu}$$

$\Rightarrow$ $W_1 W_2 W_3$ vertex.



$$-\tfrac{1}{4} g^2 \epsilon_{jkl} \epsilon_{jmn} W_k^\mu W_l^\nu W_{m\mu} W_{n\nu}$$

$\Rightarrow$ $W_k W_l W_k W_l$ vertices.



Weak Interactions

- We can use a two-component notation to represent the two charge states of a given species of lepton or quark:

$$\Psi_e = \begin{pmatrix} \psi_{\nu_e} \\ \psi_e \end{pmatrix} , \qquad \Psi_q = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}$$

- Then the weak interaction is described by an SU(2) gauge theory: the gauge invariance w.r.t. $\Psi \to \Psi' = U\Psi$ is weak isospin symmetry.

- The interaction term is

$$\frac{g}{2}\overline{\Psi}\gamma^\mu \boldsymbol{W}_\mu \cdot \boldsymbol{\tau}\Psi$$

where

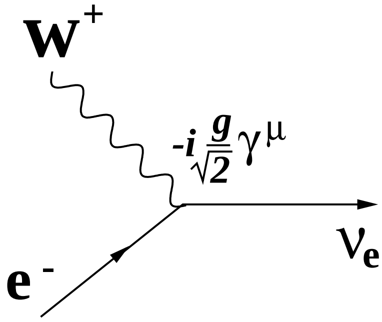
$$\begin{aligned} \Psi_e &= \begin{pmatrix} \psi_{\nu_e} \\ \psi_e \end{pmatrix} , \quad \overline{\Psi}_e = (\bar{\psi}_{\nu_e}, \bar{\psi}_e) \\ \boldsymbol{W}^\mu \cdot \boldsymbol{\tau} &= \begin{pmatrix} W_3^\mu & W_1^\mu - iW_2^\mu \\ W_1^\mu + iW_2^\mu & -W_3^\mu \end{pmatrix} \end{aligned}$$

Defining

$$W^\pm = \frac{1}{\sqrt{2}}(W_1 \mp iW_2)$$

we get a term

$\frac{g}{\sqrt{2}}\bar{\psi}_{\nu_e}\gamma^\mu W_\mu^+\psi_e \Rightarrow$



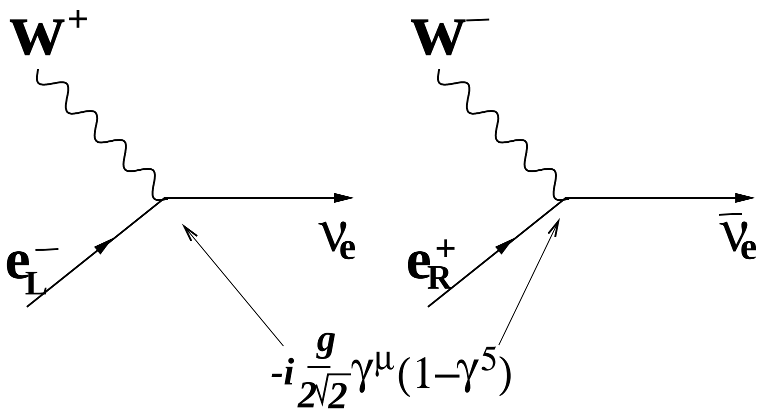
- We know from experiment that the W$^\pm$ in fact only interact with the left-handed fermion states $\psi_L = \frac{1}{2}(1-\gamma^5)\psi$ and correspondingly with right-handed antifermions:

$$(\bar{\psi})_R = \overline{\psi_L} = \tfrac{1}{2}\bar{\psi}(1+\gamma^5)$$

- Thus by Ψ we really mean Ψ_L:

$$\begin{aligned}
\overline{\Psi}_L\gamma^\mu \boldsymbol{W}_\mu \cdot \boldsymbol{\tau}\Psi_L &= \tfrac{1}{4}\overline{\Psi}(1+\gamma^5)\gamma^\mu \boldsymbol{W}_\mu \cdot \boldsymbol{\tau}(1-\gamma^5)\Psi \\
&= \tfrac{1}{4}\overline{\Psi}\gamma^\mu(1-\gamma^5)^2 \boldsymbol{W}_\mu \cdot \boldsymbol{\tau}\Psi \\
&= \tfrac{1}{2}\overline{\Psi}\gamma^\mu(1-\gamma^5) \boldsymbol{W}_\mu \cdot \boldsymbol{\tau}\Psi
\end{aligned}$$

using $(1-\gamma^5)^2 = 1 + (\gamma^5)^2 - 2\gamma^5 = 2(1-\gamma^5)$.



- We say that the left-handed fermions have weak isospin $I_w = \frac{1}{2}$ ($\Rightarrow 2I_w + 1 = 2$ states, e.g. ν_e, e_L^-), transforming under weak isospin gauge transformations as

$$\Psi_L = \begin{pmatrix} \psi_{L1} \\ \psi_{L2} \end{pmatrix} \rightarrow e^{ig\frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{\tau}}\Psi_L$$

- The right-handed fermions have $I_w = 0$, i.e. only 1 state (e_R^-), invariant under weak isospin transformations:

$$\psi_R \rightarrow e^0\psi_R = \psi_R$$

- This is all fine except that it implies that fermions cannot have mass! The mass term in the Lagrangian is

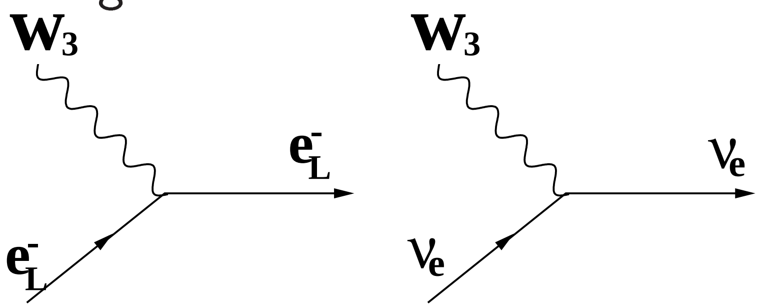
$$\begin{aligned} m\bar{\psi}\psi &= \tfrac{1}{4}m\bar{\psi}(1-\gamma^5)(1-\gamma^5)\psi + \tfrac{1}{4}m\bar{\psi}(1+\gamma^5)(1+\gamma^5)\psi \\ &= m\overline{\psi_R}\psi_L + m\overline{\psi_L}\psi_R \end{aligned}$$

- Since ψ_R and $\overline{\psi_R}$ do not transform while ψ_L and $\overline{\psi_L}$ do, this is clearly not gauge invariant (unless $m = 0$). We postpone this problem for the moment. . .

Electroweak Interactions

S. Glashow, S. Weinberg, A. Salam, 1961–68

- We interpreted $(W_1 \mp iW_2)/\sqrt{2}$ as W$^{\pm}$ boson fields, but what about W_3? Could it represent the Z^0 boson?
  - It has the right kind of vertices:



  - However, the Z^0 also interacts with right-handed electrons. Also is has a different mass from W$^{\pm}$, so they cannot belong in an exact symmetry multiplet (a weak isospin triplet, $I_w = 1$).

- But there is another neutral gauge boson, the photon. Therefore we suppose that W_3 may be a mixture:

$$W_3^{\mu} = \cos\theta_w Z^{\mu} + \sin\theta_w A^{\mu}$$

Here θ_w is the Weinberg angle or *weak mixing angle*

$$\sin^2\theta_w = 0.23117(16)$$

- The combination orthogonal to W_3 is

$$B^\mu = -\sin\theta_w Z^\mu + \cos\theta_w A^\mu$$

In the Standard Model, B^μ is the gauge boson field for an additional Abelian gauge symmetry. Thus the overall electroweak symmetry is SU(2)×U(1).

- The coupling constant for the U(1) gauge interaction is $g' \neq g$. The coupling of any fermion to B^μ is proportional to its weak hypercharge, Y, defined by

$$Y = Q - I_3$$

where Q is the charge (in units of $|e|$) and I_3 is the third component of the weak isospin, i.e. $\pm\frac{1}{2}$ for the upper/lower component of a weak isospin doublet ($I_w = \frac{1}{2}$) and 0 for a singlet ($I_w = 0$).

Electroweak Quantum Numbers

Particles	Q	I_3	Y
ν_e, ν_μ, ν_τ	0	$\frac{1}{2}$	$-\frac{1}{2}$
e_L, μ_L, τ_L	-1	$-\frac{1}{2}$	$-\frac{1}{2}$
e_R, μ_R, τ_R	-1	0	-1
u_L, c_L, t_L	$+\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{6}$
d_L, s_L, b_L	$-\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{6}$
u_R, c_R, t_R	$+\frac{2}{3}$	0	$\frac{2}{3}$
d_R, s_R, b_R	$-\frac{1}{3}$	0	$-\frac{1}{3}$

N.B. Hypercharge is the average charge of the weak isospin multiplet.

- Weak isospin doublets are

$$\begin{pmatrix} \nu_e \\ e_L \end{pmatrix}, \quad \begin{pmatrix} \nu_\mu \\ \mu_L \end{pmatrix}, \quad \begin{pmatrix} \nu_\tau \\ \tau_L \end{pmatrix}$$
$$\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \quad \begin{pmatrix} t_L \\ b_L \end{pmatrix}$$

- The electroweak Lagrangian becomes

$$\mathcal{L}_{\text{EW}} = -\tfrac{1}{4} G_j^{\mu\nu} G_{j\mu\nu} - \tfrac{1}{4} B^{\mu\nu} B_{\mu\nu} + i\overline{\Psi}\gamma^\mu D_\mu \Psi$$

where $B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$ and now

$$D_\mu = \partial_\mu + ig\boldsymbol{W}_\mu \cdot \boldsymbol{I} + ig' B_\mu Y$$

with $\boldsymbol{I} = \frac{1}{2}\boldsymbol{\tau}$ for doublets (0 for singlets), and Y is as given in the table.

- General gauge transformation is

$$\Psi \to \Psi' = \exp\left(ig\boldsymbol{\omega} \cdot \boldsymbol{I} + ig'\omega_0 Y\right)\Psi$$

where $\omega_{0,\ldots,3}$ are 4 arbitrary real functions of the space-time coordinates x^μ.

- The terms in $\mathcal{L}_{\text{EW}}$ involving the neutral gauge bosons are

$$\begin{aligned} &-\overline{\Psi}\gamma^\mu [g(\cos\theta_w\, Z_\mu + \sin\theta_w\, A_\mu) I_3 \\ &+ g'(-\sin\theta_w\, Z_\mu + \cos\theta_w\, A_\mu) Y]\Psi \end{aligned}$$

Hence the coupling to the photon is

$$g\sin\theta_w\, I_3 + g'\cos\theta_w\, Y$$

which must be equal to $eQ = e\,I_3 + e\,Y$

$$\Rightarrow \qquad g\sin\theta_w = g'\cos\theta_w = e$$

- Note that since Q is the same for left- and right-handed states, the photon couples to the currents

$$\begin{aligned}&\bar\psi_L\gamma^\mu\psi_L + \bar\psi_R\gamma^\mu\psi_R\\ &= \tfrac{1}{2}\bar\psi\gamma^\mu(1-\gamma^5)\psi + \tfrac{1}{2}\bar\psi\gamma^\mu(1+\gamma^5)\psi\\ &= \bar\psi\gamma^\mu\psi\end{aligned}$$

Hence photon interactions are parity conserving (no γ^5).

- The coupling of the Z^0, on the other hand,

$$\begin{aligned}&g\cos\theta_w\,I_3 - g'\sin\theta_w\,Y\\ &= \frac{e}{\sin\theta_w\cos\theta_w}\left(\cos^2\theta_w\,I_3 - \sin^2\theta_w\,Y\right)\\ &= \frac{2e}{\sin 2\theta_w}\left(I_3 - \sin^2\theta_w\,Q\right)\end{aligned}$$

involves the current

$$\bar\psi_L\gamma^\mu(I_{3L} - \sin^2\theta_w\,Q)\psi_L + \bar\psi_R\gamma^\mu(-\sin^2\theta_w\,Q)\psi_R$$

$$
\begin{aligned}
&= \tfrac{1}{2}\bar{\psi}\gamma^\mu[(I_{3L} - \sin^2\theta_w\, Q)(1-\gamma^5) - \sin^2\theta_w\, Q(1+\gamma^5)]\psi \\
&= \tfrac{1}{2}\bar{\psi}\gamma^\mu(I_{3L} - 2Q\sin^2\theta_w - I_{3L}\gamma^5)\psi
\end{aligned}
$$

The presence of γ^5 indicates parity violation.

The Higgs Mechanism

- The electroweak theory we have discussed so far is perfectly self-consistent but is cannot be correct because all the fermions and gauge bosons have to be massless to preserve gauge-invariance of the Lagrangian.

- Recall that a fermion mass term would convert left-handed particles into right-handed ones and vice-versa:

$$m\bar{\psi}\psi = m\overline{\psi_R}\psi_L + m\overline{\psi_L}\psi_R$$

- Since the left-handed fermions are weak isospin doublets and the right-handed ones are singlets, we need to replace m by a doublet ($I_w = \frac{1}{2}$) type of quantity to restore gauge invariance, e.g. a Yukawa interaction

$$g_f \Phi^\dagger \overline{\psi_R} \Psi_L + g_f \overline{\Psi_L} \psi_R \Phi$$

where $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ is a weak isospin doublet of scalar fields.

- For first-generation leptons, for example, we find

$$g_e \phi_1^\dagger \bar{\psi}_{e_R} \psi_{\nu_e} + g_e \phi_2^\dagger \bar{\psi}_{e_R} \psi_{e_L} + \text{ h.c.}$$

and thus for an electron mass we need

$$\Phi_0 = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$$

with $g_e v/\sqrt{2} = m_e$.

- Φ_0 is the vacuum expectation value of the Higgs field Φ. Note that ϕ_2 must be neutral, with

$$Q = 0\,, \quad I_3 = -\tfrac{1}{2} \quad \Rightarrow \quad Y_{\text{Higgs}} = \tfrac{1}{2}$$

Then ϕ_1 must have $Q = +1$, so in general

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

- Of course, the particular choice of $\Phi_0 = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$ breaks gauge invariance, so we do not seem to have achieved much. But we shall see that this type of symmetry breaking is 'natural', and does not spoil the good properties of gauge theories.

- The interactions between the Higgs field and the gauge fields are generated by the usual 'kinetic' term in the

Higgs (Klein-Gordon) part of the Lagrangian

$$\mathcal{L}_{\mathsf{H}} = \left(D^\mu \Phi^\dagger\right)\left(D_\mu \Phi\right) + \ldots \text{(see later)}$$

where D^μ is the electroweak covariant derivative,

$$D^\mu = \partial^\mu + ig\boldsymbol{W}^\mu \cdot \boldsymbol{I} + ig' B^\mu Y$$

- For the vacuum Higgs field we have explicitly (since v is constant)

$$D^\mu \Phi_0 = \frac{iv}{2\sqrt{2}} \begin{pmatrix} \sqrt{2} g W^{+\mu} \\ g' B^\mu - g W_3^\mu \end{pmatrix}$$

Now

$$\begin{aligned} g' B^\mu - g W_3^\mu &= \frac{g}{\cos\theta_w}\left(\sin\theta_w\, B^\mu - \cos\theta_w\, W_3^\mu\right) \\ &= -\frac{g}{\cos\theta_w} Z^\mu \end{aligned}$$

Hence

$$\mathcal{L}_{\mathsf{H}} = \frac{v^2 g^2}{8}\left(2W^{-\mu}W_\mu^+ + \frac{1}{\cos^2\theta_w} Z^\mu Z_\mu\right) + \ldots$$

which corresponds to $W^{\pm}$ and Z^0 mass terms

$$m_W = \tfrac{1}{2} v g = m_Z \cos\theta_w$$

N.B. $\sin^2\theta_w = 1 - \frac{m_W^2}{m_Z^2}$.

Parameters of the Standard Model

- The Standard (Glashow-Weinberg-Salam) Model describes all electroweak interactions at present energies in terms of three basic parameters

$$g\ ,\quad g'(\text{or }\theta_w)\ ,\quad v$$

- These are most accurately measured from
  - the fine structure constant

$$\alpha = \frac{e^2}{4\pi} = \frac{g^2 \sin^2\theta_w}{4\pi} = \{137.03599976(50)\}^{-1}$$

  - the Fermi weak coupling constant

$$G_F = \frac{g^2\sqrt{2}}{8m_W^2} = \frac{1}{\sqrt{2}v^2} = 1.16639(1) \times 10^{-5}\ \text{GeV}^{-2}$$

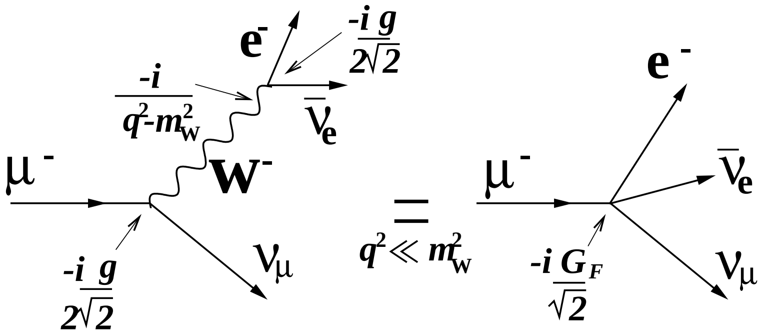
  - the Z^0 boson mass

$$M_Z = \frac{gv}{2\cos\theta_w} = 91.1882(22)\ \text{GeV}$$

N.B. These relations are subject to higher-order (electroweak and strong) corrections.

- In addition there are further parameters of the Higgs sector, including Yukawa couplings.

- The Fermi constant is the effective 4-fermion coupling for the decay $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$



$$G_F = 1.166 \times 10^{-5} \text{ GeV}^{-2} \quad \Rightarrow \quad v = 246 \text{ GeV}$$

- The Yukawa couplings can be deduced from the fermion masses:

$$g_f = m_f \frac{\sqrt{2}}{v}$$

The numerical values of the Yukawa couplings (and thus the fermion masses) are presently not understood (1 parameter per fermion).

N.B. $m_t = 175$ GeV $\Rightarrow$ $g_t = 1.0$.

Is this accidental??

Spontaneous Symmetry Breaking

- The advantage of the Higgs mechanism for mass generation is that the gauge-symmetry-breaking vacuum Higgs field Φ_0 can arise 'spontaneously', even though the Lagrangian is exactly gauge-invariant.

- Let's study first for simplicity the spontaneous breaking of a global (i.e. space-time independent) Abelian symmetry. Consider a classical complex scalar field ϕ with Lagrangian density

$$\mathcal{L} = (\partial^\mu \phi^*)(\partial_\mu \phi) - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2$$

  Clearly $\mathcal{L}$ is invariant w.r.t.

$$\phi \to e^{i\alpha} \phi \,, \quad \phi^* \to e^{-i\alpha} \phi^*$$

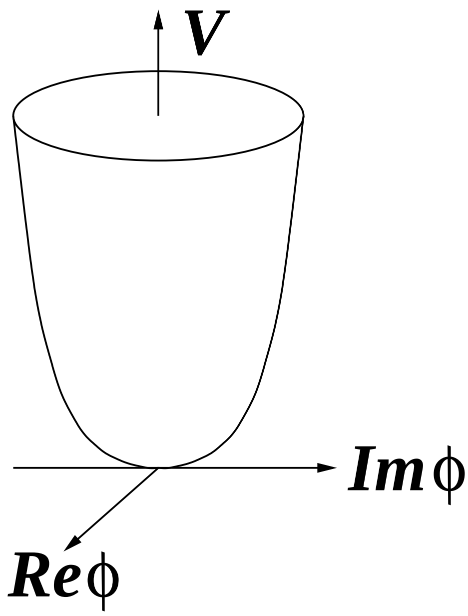
  for any real constant α.

- The Hamiltonian density is

$$\mathcal{H} = \left| \frac{\partial \phi}{\partial t} \right|^2 + \nabla \phi^* \cdot \nabla \phi + V(\phi)$$

  where the 'potential energy' $V(\phi)$ is

$$V(\phi) = \mu^2 |\phi|^2 + \lambda |\phi|^4$$



- This potential has a minimum at the origin and hence the minimum-energy (vacuum) state has $\phi = 0$ (classically).
- After second quantization there will be zero-point fluctuations, but the vacuum expectation value $\phi_0 = \langle 0|\hat{\phi}|0\rangle$ will still be zero.
- The curvature of $V(\phi)$ at the minimum tells us the mass of the scalar bosons created and annihilated by the field operator (c.f. Klein-Gordon equation):

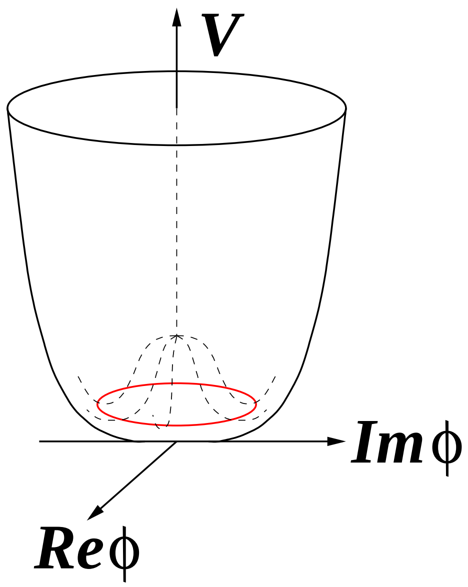
$$m^2 = \frac{1}{2} \frac{d^2V}{d\phi^2}\bigg|_{\phi=\phi_0} = \mu^2$$

- The term $\lambda|\phi|^4$ represents a 4-boson interaction that can be treated as a perturbation (coupling constant $\propto \lambda$).

- Now suppose we change the sign of the first term in

$V(\phi)$:

$$V = -\mu^2|\phi|^2 + \lambda|\phi|^4$$



❖ The potential has a 'hump': the minimum now lies anywhere on the circle

$$\phi = \sqrt{\frac{\mu^2}{2\lambda}}\,e^{i\theta} \qquad (0 \le \theta \le 2\pi)$$

❖ The physical vacuum can be any one of these (infinitely many) degenerate vacua. But choosing a particular one (a value of θ) breaks the U(1) symmetry, since $e^{i\alpha}\phi$ will be a different vacuum (c.f. a falling rod or ferromagnet). Since a particular vacuum is realized, there is spontaneous symmetry breaking.

❖ The two principal curvatures of $V(\phi)$ at the vacuum solution are now different. One is zero (around the circle), indicating a massless boson (a Goldstone boson), in agreement with Goldstone's theorem (1961): Spontaneous breaking of global symmetry $\Rightarrow$ massless boson.

N.B. We shall see this is not true in gauge theories.

- The other curvature (radial) is non-zero, indicating that a massive boson is also present.

- Without loss of generality (because of gauge invariance) we can define the vacuum expectation value to be real:

$$\phi_0 = \frac{v}{\sqrt{2}}\,, \quad v = \sqrt{\frac{\mu^2}{\lambda}}$$

Then setting

$$\phi = \frac{1}{\sqrt{2}}[v + \sigma(x) + i\eta(x)]$$

we expect σ and η to represent massive and massless boson fields, respectively.

- Substituting in the Lagrangian, we find

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}[(\partial^\mu\sigma)(\partial_\mu\sigma) + (\partial^\mu\eta)(\partial_\mu\eta)] \\ &\quad +\tfrac{1}{2}\mu^2[(v+\sigma)^2 + \eta^2] - \tfrac{1}{4}\lambda[(v+\sigma)^2 + \eta^2]\end{aligned}$$

$$= \quad \tfrac{1}{2}[(\partial^\mu\sigma)(\partial_\mu\sigma) - m_\sigma^2\sigma^2] + \tfrac{1}{2}(\partial^\mu\eta)(\partial_\mu\eta)$$
$$+ \text{ const. } + \text{ interaction terms}$$

where $m_\sigma = \sqrt{2}\mu$.

- Now suppose our model scalar field theory is a U(1) gauge theory (Higgs model, 1964), i.e. the Lagrangian is invariant w.r.t.

$$\phi \to e^{i\alpha(x)}\phi \,, \quad \phi^* \to e^{-i\alpha(x)}\phi^*$$

for any real function $\alpha(x)$. We know this means there must be a gauge field $B^\mu(x)$ and the Lagrangian becomes

$$\mathcal{L} = -\tfrac{1}{4}B^{\mu\nu}B_{\mu\nu} + D^\mu\phi^* D_\mu\phi + \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2$$

where

$$\begin{aligned} B^{\mu\nu} &= \partial^\mu B^\nu - \partial^\nu B^\mu \\ D^\mu &= \partial^\mu + ig'YB^\mu \end{aligned}$$

Y being the corresponding hypercharge.

- The spontaneous symmetry breaking in the Higgs field ϕ works as before, giving a vacuum with $B^\mu = 0$,

$\phi = \phi_0 \neq 0$. Choosing $\phi_0 = v/\sqrt{2}$ (real), this generates a mass term for the gauge boson:

$$\tfrac{1}{2}(D^\mu v)(D_\mu v) = \tfrac{1}{2}(g'Yv)^2 B^\mu B_\mu$$

corresponding to $m_B = g'Yv$.

- We seem to have created a degree of freedom: a massive vector field (S=1) has 3 polarization states in contrast to the 2 of a massless field. Longitudinal polarization (helicity 0) is possible, as well as transverse (helicity ± 1).

- This is because one degree of freedom of the Higgs field has disappeared. In a gauge theory we can always set the massless field $\eta(x)$ to zero by a gauge transformation: choosing

$$\tan \alpha(x) = -\frac{\eta(x)}{v + \sigma(x)}$$

we have

$$\begin{aligned} \phi' &= e^{i\alpha}\phi = (\cos\alpha + i\sin\alpha)\frac{1}{\sqrt{2}}(v + \sigma + i\eta) \\ &= \frac{1}{\sqrt{2}}\sqrt{(v+\sigma)^2 + \eta^2} = \frac{1}{\sqrt{2}}(v + \sigma') \end{aligned}$$

where $\quad \sigma' = \sigma + \dfrac{\eta^2}{2v} + \ldots =$ real

(This called the 'unitary gauge'.)

- Thus we can say that the gauge field has ‘eaten’ the Goldstone boson in order to create its extra polarization state.

- The situation in the electroweak case is a little more complicated. The original $SU(2)\times U(1)$ symmetry (with 4 generators) is spontaneously broken down to $U(1)_{\rm em}$ [not the original $U(1)$], with 1 generator. There should thus be 3 Goldstone bosons, but these are eaten by the W^+, W^- and Z^0 to produce their longitudinal polarization states. This leave one massless gauge boson – the photon – and one massive, neutral scalar boson – the Higgs boson.

Properties of the Higgs Boson

- Note that the mass of the Higgs boson H^0, represented by the field σ, is

$$M_H = m_\sigma = \sqrt{2}\mu \; .$$

  This is not determined directly by current data, which fixes only the combination of Higgs parameters

$$v = \frac{\mu}{\sqrt{\lambda}} = 246 \text{ GeV}$$

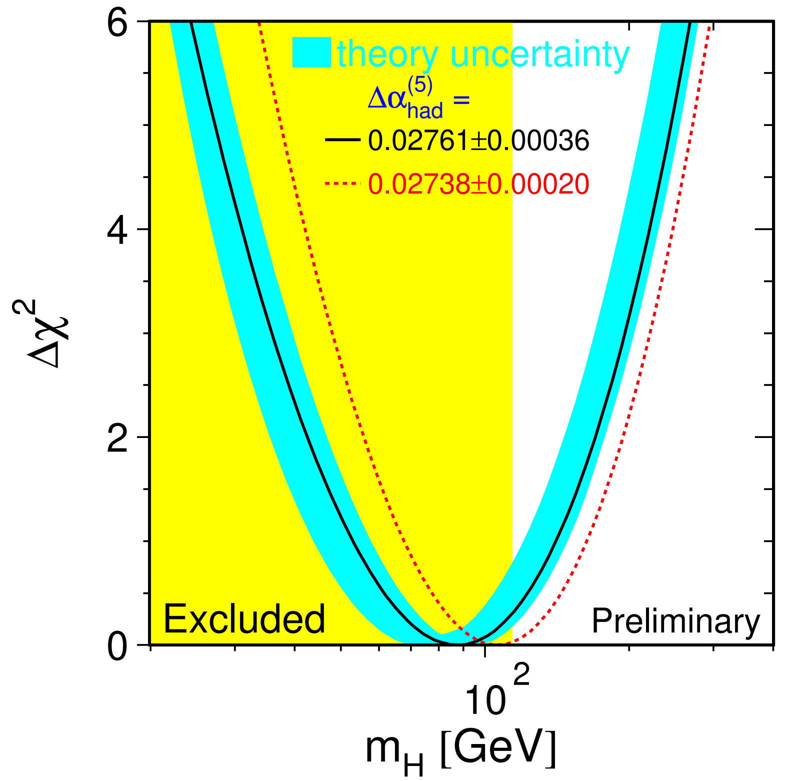
- The fact that Higgs bosons were not (?) produced at LEP implies

$$M_H > 114.4 \text{ GeV (95\% C.L.)}$$

- Higher-order corrections depend (weakly) on μ and these suggest that

$$M_H <\sim 200 \text{ GeV}$$

- Latest results from the LEP Electroweak Working Group:



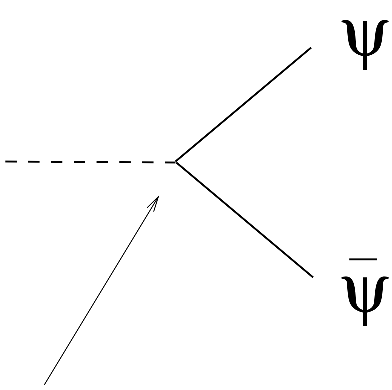
- The couplings of the Higgs boson to fermions are all given by the corresponding Yukawa couplings, which are fixed by the fermion masses:

$$g_f \frac{1}{\sqrt{2}}(v+\sigma)\bar{\psi}\psi$$

$$\Rightarrow \quad m_f = g_f \frac{v}{\sqrt{2}}$$

$\Rightarrow$ vertex factor

$$-i\frac{g_f}{\sqrt{2}} = -i\frac{m_f}{v}$$

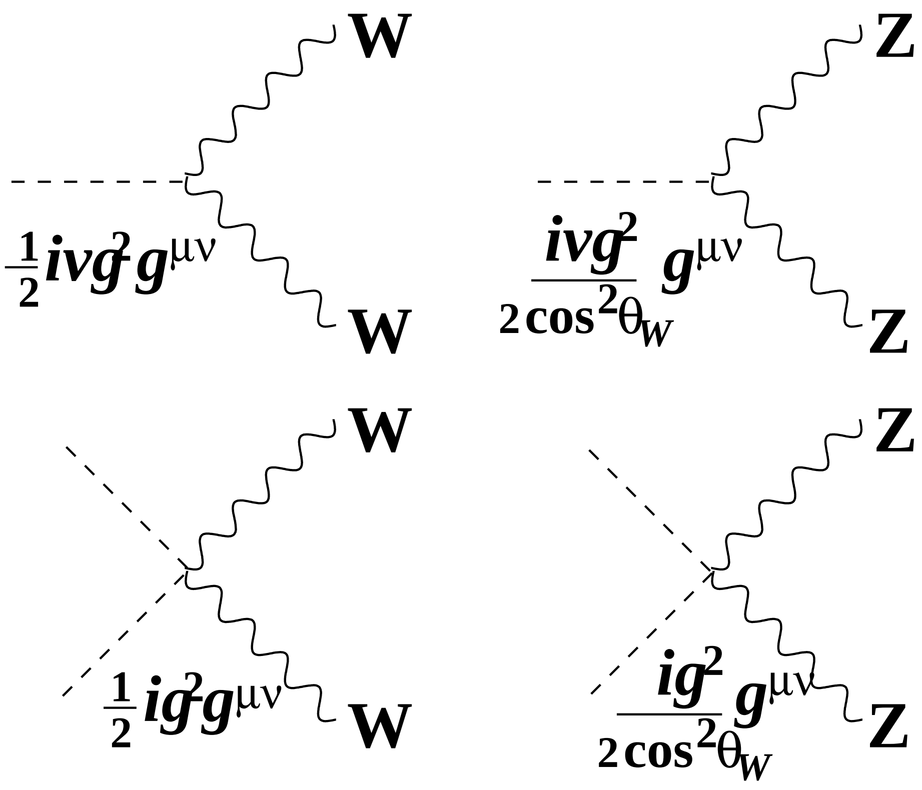


- Thus the Higgs boson likes to decay into (and be produced by) the heaviest available fermion, $H^0 \to b\bar{b}$ if $M_H < 2m_t$.

- The coupling to gauge bosons is obtained by replacing v by $(v + \sigma)$ in the term that produced the gauge boson masses:

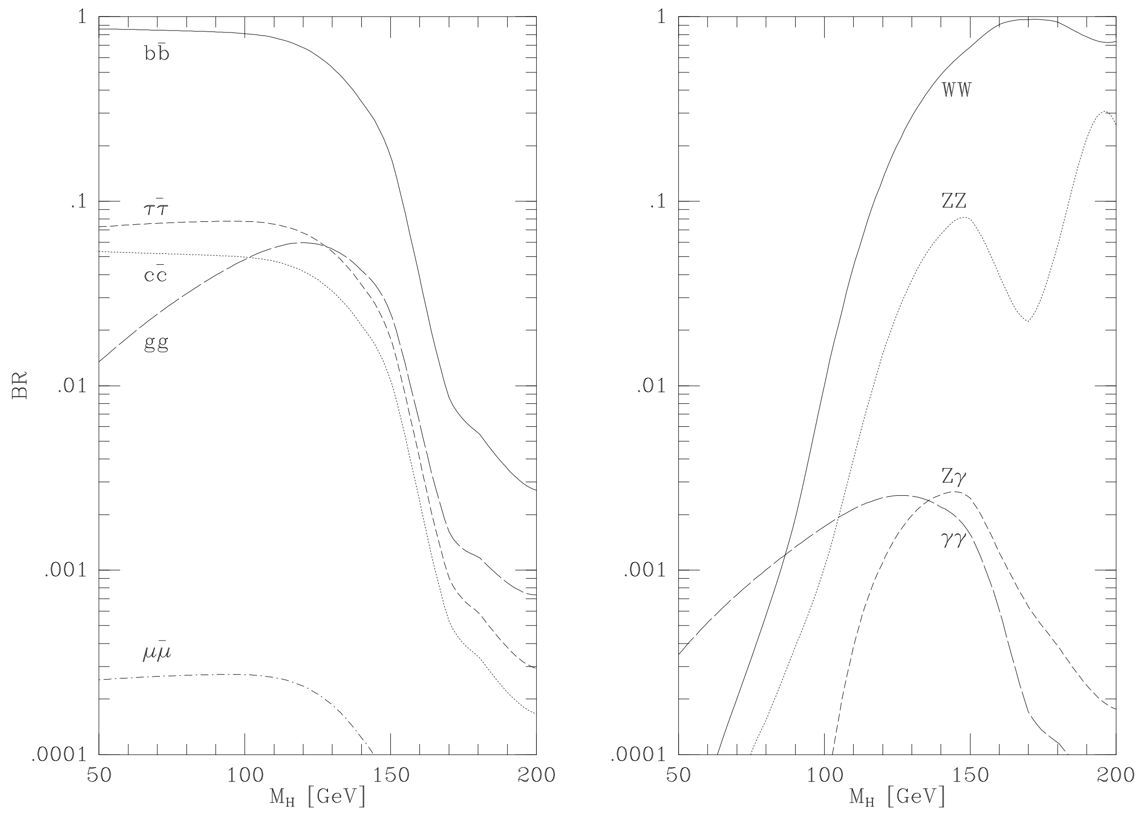
$$
\begin{aligned}
\mathcal{L}_{\mathrm{H}} &= \frac{(v+\sigma)^2}{8} g^2 \left(2W^{-\mu} W^{+}_{\mu} + \frac{1}{\cos^2\theta_w} Z^{\mu} Z_{\mu} \right) \\
&= \left(M_W^2 + \tfrac{1}{2} v g^2 \sigma + \tfrac{1}{4} g^2 \sigma^2 \right) W^{-\mu} W^{+}_{\mu} \\
&\quad + \tfrac{1}{2} \left(M_Z^2 + \frac{v g^2}{2\cos^2\theta_w} \sigma + \frac{g^2}{4\cos^2\theta_w} \sigma^2 \right) Z^{\mu} Z_{\mu}
\end{aligned}
$$

corresponding to the vertices:



- Thus decays to W^+W^- and Z^0Z^0 are expected if M_H is large enough.

- Branching ratios of the Higgs boson to various final states:



Quark Mixing

- In general the quark fields entering the doublets of the electroweak interaction need not be mass eigenstates. We should thus write the doublets as

$$\begin{pmatrix} u_L \\ d'_L \end{pmatrix}, \quad \begin{pmatrix} c_L \\ s'_L \end{pmatrix}, \quad \begin{pmatrix} t_L \\ b'_L \end{pmatrix}$$

with the primed down-type quarks q' differing from the mass eigenstates q by a unitary transformation, V:

$$\begin{pmatrix} d'_L \\ s'_L \\ b'_L \end{pmatrix} = V \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix}$$

The matrix V is known as the Cabibbo-Kobayashi-Maskawa (CKM) matrix.

- The CKM matrix implies a mixing of mass eigenstates in the electroweak interactions.

- Note that without quark mixing s and b quarks would be stable — contrary to observation.

- It is only a convention to shift the mixing to the down-type quarks.

- Writing the CKM matrix V as

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

the constraint of unitarity ($V^\dagger V = 1$) implies 9 conditions

$$\sum_{j=1}^{3} |V_{ij}|^2 = \sum_{i=1}^{3} |V_{ij}|^2 = 1$$

$$\sum_{k=1}^{3} V_{ki}^* V_{kj} = 0 \qquad (i \neq j)$$

- A unitary 3×3 matrix hence has 9 independent real parameters. The phases of 5 quark fields can be redefined. This leaves 4 real parameters:
  3 independent mixing angles θ_{ij}
  $+\ 1$ phase δ.

- The standard parametrizaiton of the CKM matrix is

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

with

$$s_{ij} = \sin\theta_{ij}, \qquad c_{ij} = \cos\theta_{ij}$$

- Note that the Feynman rules for the electroweak interaction of quarks have to be modified accordingly. The vertices hence include the CKM matrix elements V_{ij}.

- In the Standard Model the phase δ is responsible for CP violation arising from the mixing matrix.

- Note that if we had only two generations there would be one mixing angle (the Cabibbo angle θ_{12} and no phases. Hence there would be no possibility for CP violation from the mixing matrix.

- The experimental limits (taking into account unitarity)

$$\begin{pmatrix} 0.9741-0.9756 & 0.219-0.226 & 0.0025-0.0048 \\ 0.219-0.226 & 0.9732-0.9748 & 0.038-0.044 \\ 0.004-0.014 & 0.037-0.044 & 0.9990-0.9993 \end{pmatrix}$$

imply that the mixing angles are small, and satisfy a hierarchy:

$$1 \gg \theta_{12} \gg \theta_{23} \gg \theta_{13}$$

- A simplified (small-angle) parametrization for the CKM matrix V was introduced by Wolfenstein: Setting

$$\lambda = \sin\theta_{12} = s_{12}$$

one obtains with $V_{cb} \simeq s_{23} = A\lambda^2$ and $V_{ub} = s_{13}e^{i\delta} = A\lambda^3(\rho - i\eta)$, $A \simeq 1$, and $|\rho - i\eta| < 1$ finally

$$\begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)$$

- Note that mixing is smallest between the first and third generation.