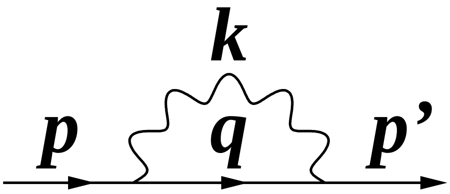
Ultraviolet Divergences

- In higher-order perturbation theory we encounter Feynman graphs with closed loops, associated with unconstrained momenta.

- For every such momentum k^μ, we have to integrate over all values, i.e.

$$\int \frac{d^4k}{(2\pi)^4}$$

E.g. "electron self-energy" in QED:



$$\begin{aligned}
\mathcal{A}_{fi} &= \int \frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\bar{u}(p')\gamma^\mu \frac{-ig_{\mu\nu}}{k^2}\frac{i(\not{q}+m)}{q^2-m^2}\gamma^\nu u(p) \\
&\times (-ie)(2\pi)^4\delta^4(p-q-k)(-ie)(2\pi)^4\delta^4(q+k-p') \\
&= -e^2(2\pi)^4\delta^4(p-p') \\
&\times \bar{u}(p)\int \frac{d^4k}{(2\pi)^4}\frac{\gamma^\mu(\not{p}-\not{k}+m)\gamma_\mu}{k^2[(p-k)^2-m^2]}u(p)
\end{aligned}$$

- $\int^{\infty} d^4k \frac{k}{k^2(p-k)^2}$ is divergent!

- We say that

$$\int d^4k\, k^{D-4}$$

has superficial degree of divergence D

$$\begin{aligned} D \quad = \quad & 0 \Rightarrow \text{log-divergent} \\ & 1 \Rightarrow \text{linearly divergent} \\ & 2 \Rightarrow \text{quadratically divergent} \end{aligned}$$

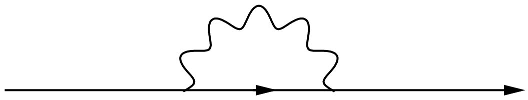
- The actual degree of divergence may be less, e.g. due to cancellations required by gauge invariance. For example, the electron self-energy is actually only log-divergent. Putting an upper cut-off Λ on the integral, one finds

$$\mathcal{A}_{fi} \sim -i(2\pi)^4 \, \delta^4(p-p') \frac{3\alpha}{2\pi} m \ln\left(\frac{\Lambda}{m}\right) + \ldots$$

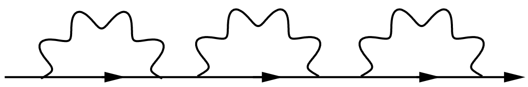
- If the theory has only a finite set of (classes of) divergent (i.e. cut-off dependent) diagrams, their contributions can be absorbed into redefinitions of the coupling constant(s) and masses. This is called renormalization.

- For example, iteration of the electron self-energy leads to renormalization of the electron mass. Defining $\Sigma = -\frac{3m}{8\pi^2}\ln\frac{\Lambda}{m} + \ldots$ we have

$$\frac{i(\not{p}+m)}{p^2-m^2} \equiv \frac{i}{\not{p}-m}$$



$$\frac{i}{\not{p}-m} ie^2\Sigma \frac{i}{\not{p}-m}$$



$$\frac{i}{\not{p}-m}\sum_n \left[ie^2\Sigma\frac{i}{\not{p}-m}\right]^n$$

$$= \frac{i}{\not{p}-m}\left[1 - ie^2\Sigma\frac{i}{\not{p}-m}\right]^{-1} = \frac{i}{\not{p}-m+e^2\Sigma}$$

- Hence $m \to m + \delta m$ where

$$\frac{\delta m}{m} = \frac{3\alpha}{2\pi}\ln\frac{\Lambda}{m} + \ldots$$

The real, observed mass is $m + \delta m$. The bare mass, i.e. the parameter in the Lagrangian, is not observable, and indeed depends on Λ if we keep the observed mass fixed.

Renormalizability

- How many classes of superficially divergent graphs are there in QED? We have
  - $\int d^4k$ for every loop (unconstrained momentum)
  - $\frac{i}{\not{k}-m}$ for every internal fermion line (electron)
  - $\frac{-ig^{\mu\nu}}{k^2}$ for every internal boson line (photon)

$$\Rightarrow \qquad D = 4L - F_I - 2B_I$$

  where

$$\begin{aligned} L &= \text{number of unconstrained momenta} \\ F_I &= \text{number of internal fermion lines} \\ B_I &= \text{number of internal boson lines} \end{aligned}$$

- But if V is the number of vertices,

$$L = F_I + B_I - V + 1$$

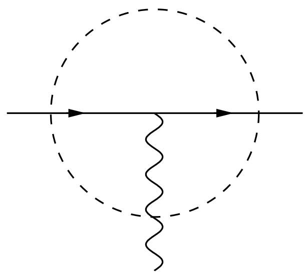
- If vertex V involves F_V fermions and B_V bosons, we have by ‘conservation of ends’:

$$\sum_V F_V = 2F_I + F_E$$
$$\sum_V B_V = 2B_I + B_E$$

where

$$F_E = \text{number of external fermion lines}$$
$$B_E = \text{number of external boson lines}$$

- In QED, $F_V = 2$, $B_V = 1$



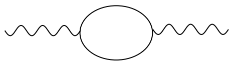
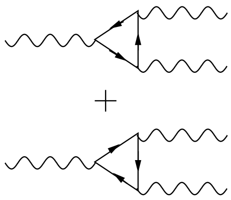
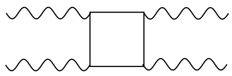
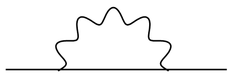
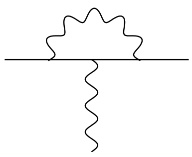
$$\Rightarrow \quad V = F_I + \tfrac{1}{2}F_E = 2B_I + B_E$$
$$\Rightarrow \quad D = 4 - \tfrac{3}{2}F_E - B_E$$

Note that D is independent of L and V.

- Thus there is only a finite number of classes of superficially divergent diagrams in QED, with

$$D = 4 - \tfrac{3}{2}F_E - B_E \geq 0$$

- There are only 5 classes of superficially divergent graphs in QED, of which 3 are actually (log) divergent.

F_E	B_E	D	Diagrams	Remarks
0	2	2		photon self-energy log-divergent $\Rightarrow$ charge renorm'n
0	3	1		$=$ 0 to all orders
0	4	0		light-by-light scattering actually convergent
2	0	1		electron self-energy log-divergent $\Rightarrow$ mass & charge renorm'n
2	1	0		vertex correction log-divergent $\Rightarrow$ charge renorm'n

N.B. In QED, charge renormalization from electron self-energy and vertex correction cancel, so it can be ascribed entirely to photon self-energy (vacuum polarization).

Dimensions of Fields and Couplings

- In natural units we have only mass (equivalently, energy or momentum) dimensions:

$$x \sim ct \sim \hbar c/E \sim \hbar/mc$$

$$\hbar = c = 1 \Rightarrow [L] = [T] = [E]^{-1} = [M]^{-1}$$

- Hence action S (units $\hbar$) is dimensionless, and

$$S = \int \mathcal{L}\, d^4x \qquad \Rightarrow \qquad [\mathcal{L}] = [x]^{-4} = [M]^4$$

Furthermore $[\partial^\mu] = [p^\mu] = [M]$. From this we can deduce dimensions of fields and couplings:

$$\begin{aligned}
\mathcal{L}_{\text{KG}} = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi \quad &\Rightarrow \quad [\phi] = [M] \\
\mathcal{L}_{\text{D}} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \quad &\Rightarrow \quad [\psi] = [M]^{3/2} \\
\mathcal{L}_{\text{em}} = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} \quad &\Rightarrow \quad [F^{\mu\nu}] = [M]^2 \\
F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad &\Rightarrow \quad [A^\mu] = [M]
\end{aligned}$$

Higgs self-coupling:

$$\lambda(\phi^\dagger\phi)^2 \quad \Rightarrow \quad [\lambda] = [M]^0$$

Gauge couplings:

$$D^\mu = \partial^\mu + ieA^\mu(+igW^\mu) \quad \Rightarrow \quad [e] = [g] = [M]^0$$

Fermi coupling:

$$G_F(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi) \quad \Rightarrow \quad [G_F] = [M]^{-2}$$

Yukawa coupling:

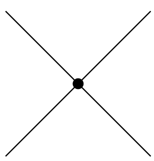
$$g_f\phi\bar{\psi}\psi \quad \Rightarrow \quad [g_f] = [M]^0$$

- Thus in any theory we can associate dimension 4 with any vertex, as follows

$$4 = \tfrac{3}{2}F_V + B_V + P_V + g_V$$

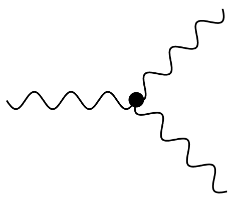
where P_V = number of momentum factors, g_V = dimension of coupling. For example

Fermi



$$4 = \tfrac{3}{2}(4) + 0 + 0 + (-2)$$

3-gauge-boson



$$4 = \tfrac{3}{2}(0) + 3 + 1 + 0$$

- Now we can derive superficial degree of divergence in any theory:

$$D = 4L - F_I - 2B_I + \sum_V P_V$$

Recall that

$$\begin{aligned} L &= F_I + B_I - V + 1 \\ \sum_V F_V &= 2F_I + F_E \\ \sum_V B_V &= 2B_I + B_E \\ D &= 4 - 4V + 3F_I + 2B_I + \sum_V P_V \\ &= 4 - 4V - \tfrac{3}{2}F_E - B_E + \\ &\quad + \sum_V (\tfrac{3}{2}F_V + B_V + P_V = 4 - g_V) \end{aligned}$$

$$\Rightarrow \qquad D = 4 - \tfrac{3}{2}F_E - B_E - \sum_V g_V$$

- Standard Model couplings are all dimensionless, so $\sum_V g_V = 0$ and the situation is similar to QED:

- Finite number of divergent sub-graphs ('primitive divergences')
- Can absorb cut-off dependence in bare parameters of Lagrangian
- Hence theory is renormalizable

N.B. Lots of work needed to *prove* this ('t Hooft and Veltman $\Rightarrow$ Nobel prize).

- Non-standard vertices have $g_V < 0$, so D gets larger and larger in higher orders of perturbation theory $\Rightarrow$ theory becomes unrenormalizable. For example

$$\begin{array}{rcl} \text{6-Higgs coupling:} & & \\ \lambda_6(\phi^\dagger\phi)^3 & \Rightarrow & [\lambda_6] = [M]^{-2} \\ \text{Fermi coupling:} & & \\ G_F(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma_\mu\psi) & \Rightarrow & [G_F] = [M]^{-2} \\ \text{2-boson Yukawa coupling:} & & \\ \lambda_f\phi^\dagger\phi\bar\psi\psi & \Rightarrow & [\lambda_f] = [M]^{-1} \end{array}$$

- Is it surprising that Nature provides only renormalizable interactions? Maybe not, because unrenormalizability $\Rightarrow$ bad (divergent) high-energy behaviour.

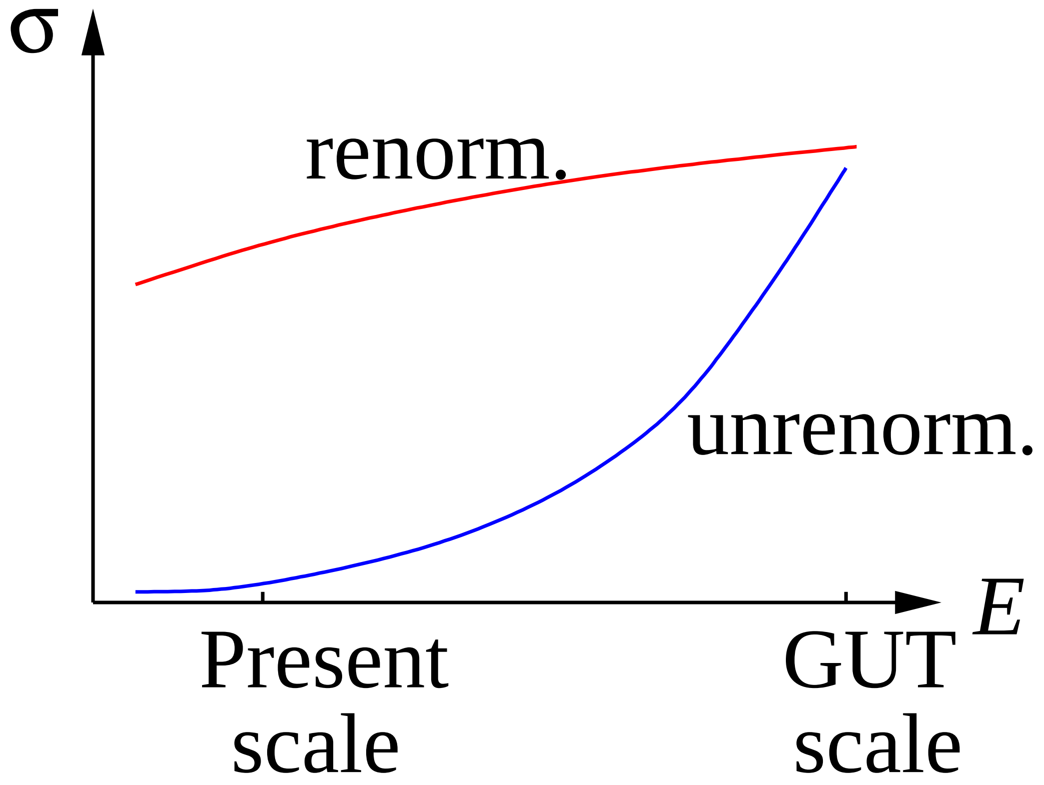
  E.g. Fermi theory:

$$\sigma(\nu_e e) \sim G_F^2$$

$$[G_F] = [M]^{-2}\,, \quad [\sigma] = [M]^{-2}$$

$$\Rightarrow \quad \sigma(\nu_e e) \sim G_F^2 E^2 \to \infty$$

- Thus if we suppose there exists a finite theory at very high energies (GUT? SUSY? Strings?), all unrenormalizable interactions will have shrunk to negligible values in going from that high scale to present energies:



Ultraviolet Divergences

- Feynman diagrams in gauge theories typically have divergences when they contain closed loops. In gauge theories, for example in QED, a cross section is expressed as

$$\sigma = A_1\alpha + A_2\alpha^2 + \cdots$$

where the coefficients A_i are divergent sums obtained from Feynman diagrams. The treatment of these divergences requires a regularization.

- Nowadays, most fixed-order calculations are performed in the modified minimal subtraction ($\overline{MS}$) scheme. This involves regularizing the UV loop divergences by reducing the number of space-time dimensions to $n < 4$. Then in the loop integrations we have

$$\frac{d^4k}{(2\pi)^4} \quad \longrightarrow \quad \mu^{2\epsilon}\frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}}$$

where $\epsilon = 2 - \frac{n}{2}$. This procedure is called dimensional regularization.

- The renormalization scale μ is introduced to preserve dimensions of couplings and fields. Loop integrals of the form

$$\int \frac{d^n k}{(k^2+m^2)^2}$$

then lead to poles at $\epsilon = 0$.

- The minimal subtraction renormalization prescription is to subtract off these poles and to replace bare by renormalized coupling, $\alpha(\mu^2)$. In practice, these poles always appear in the combination

$$\frac{\Gamma(1+\epsilon)}{\epsilon}(4\pi)^\epsilon = \frac{1}{\epsilon} + \log 4\pi - \gamma_E + \mathcal{O}(\epsilon)$$

with the Euler Gamma function $\Gamma(z) = \int_0^\infty dt e^{-t} t^{z-1}$ ($\mathrm{Re} z > 0$), and the Euler constant $\gamma_E = 0.5772\ldots$. In the $\overline{MS}$ scheme one subtracts

$$\frac{1}{\epsilon} + \log 4\pi - \gamma_E$$

- In this way one obtains finite coefficients A_i. These depend implicitly on the scheme and explicitly on the renormalization scale μ. Hence a value for μ needs to be chosen. What is the best choice? This is a difficult (and typical) problem in higher order calculations. In general, there is always some theoretical uncertainty coming from the dependence of a fixed-order result on the scale μ.