Quantum Chromodynamics

- Strong interactions are described by Quantum Chromodynamics (QCD): quarks come in 3 "colours" ($a = 1, 2, 3$) and the gauge transformations are represented by SU(3) matrices $\mathcal{U}$:

$$
\begin{aligned}
q_a(x) \to q'_a(x) &= \mathcal{U}_{ab}(x)\, q_b(x) \\
\mathcal{U}_{ab}(x) &= e^{i\, \boldsymbol{t} \cdot \boldsymbol{\theta}(x)} \\
\boldsymbol{t} \cdot \boldsymbol{\theta} &\equiv \sum_{A=1}^{8} t^A \theta^A
\end{aligned}
$$

- The generators of SU(3), $t^A = \frac{1}{2}\lambda_A$, are given in terms of the *Gell-Mann* matrices λ_A:

$$
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}
$$

$$
\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

N.B. All λ_i are hermitian and traceless; λ_3 and λ_8 are diagonal.

- $\boldsymbol{t} \cdot \boldsymbol{t} = C_F\, I$ where $C_F = \frac{4}{3}$ is the "quark colour charge squared"

- $[t^A, t^B] = i f^{ABC} t^C$ where the structure constants of the group SU(3), f^{ABC} are totally antisymmetric and

$$\begin{aligned} &f^{123} = 1 \\ &f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \tfrac{1}{2} \\ &f^{458} = f^{678} = \sqrt{3}/2 \end{aligned}$$

- $\sum_{A,B} f^{ABC} f^{ABD} = C_A\, \delta_{CD}$ where $C_A = 3$ is the "gluon colour charge squared"

- The QCD Lagrangian density is

$$\mathcal{L}_{\text{QCD}} = -\tfrac{1}{4} F^A_{\mu\nu} F^{A\mu\nu} + \sum_{\text{flavours } f} \bar{q}^f_a \left(i \not{D}_{ab} - m_f\, \delta_{ab} \right) q^f_b$$

where

$$F^A_{\mu\nu} = \partial_\mu \mathcal{A}^A_\nu - \partial_\nu \mathcal{A}^A_\mu - g_{\,\mathrm{s}} f^{ABC} \mathcal{A}^B_\mu \mathcal{A}^C_\nu$$

and the covariant derivative is

$$(D_\mu)_{ab} = \partial_\mu \, \delta_{ab} + i g_{\,\mathrm{s}} \, (\boldsymbol{t} \cdot \boldsymbol{\mathcal{A}}_\mu)_{ab}$$

$\mathcal{A}^A_\mu$ are the gluon fields $(A = 1, \ldots, 8)$ and $g_{\,\mathrm{s}}$ is the (bare) QCD coupling, with

$$\alpha_{\,\mathrm{s}} = \frac{g^2_{\mathrm{s}}}{4\pi}$$

- For gauge invariance of $\mathcal{L}_{\,\mathrm{QCD}}$ we need

$$D_\mu q \;\rightarrow\; D'_\mu q' = \mathcal{U} D_\mu q$$

where $\mathcal{U} \in$ SU(3). Now

$$\begin{aligned} D'_\mu q' &= (\partial_\mu + i g_{\,\mathrm{s}} \boldsymbol{t} \cdot \boldsymbol{\mathcal{A}}'_\mu) \mathcal{U} q \\ &= (\partial_\mu \mathcal{U}) q + \mathcal{U} \partial_\mu q + i g_{\,\mathrm{s}} \boldsymbol{t} \cdot \boldsymbol{\mathcal{A}}'_\mu \mathcal{U} q \end{aligned}$$

and

$$\mathcal{U} D_\mu q = \mathcal{U} \partial_\mu q + i g_{\,\mathrm{s}} \mathcal{U} \boldsymbol{t} \cdot \boldsymbol{\mathcal{A}}_\mu q$$

Hence we need

$$
\begin{aligned}
& ig_{\,\text{s}}\boldsymbol{t}\cdot\boldsymbol{\mathcal{A}}'_\mu\mathcal{U} = ig_{\,\text{s}}\mathcal{U}\boldsymbol{t}\cdot\boldsymbol{\mathcal{A}}_\mu - (\partial_\mu\mathcal{U}) \\
\Rightarrow \qquad & \boldsymbol{t}\cdot\boldsymbol{\mathcal{A}}'_\mu = \mathcal{U}\boldsymbol{t}\cdot\boldsymbol{\mathcal{A}}_\mu\mathcal{U}^\dagger + \frac{i}{g_{\,\text{s}}}(\partial_\mu\mathcal{U})\mathcal{U}^\dagger
\end{aligned}
$$

- The field strength tensor $F^A_{\mu\nu}$ then transforms according to

$$\boldsymbol{t}\cdot\boldsymbol{F}_{\mu\nu} \to \boldsymbol{t}\cdot\boldsymbol{F}'_{\mu\nu} = \mathcal{U}\boldsymbol{t}\cdot\boldsymbol{F}_{\mu\nu}\mathcal{U}^\dagger$$

Note that $\text{Tr}\left(t^A t^B\right) = \frac{1}{2}\delta_{AB}$, so that

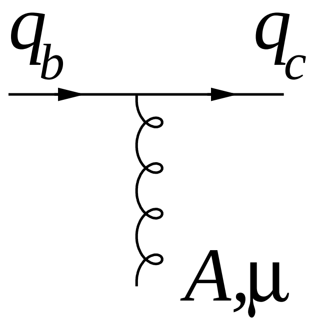
$$\text{Tr}\left(\boldsymbol{t}\cdot\boldsymbol{F}_{\mu\nu}\boldsymbol{t}\cdot\boldsymbol{F}^{\mu\nu}\right) = \tfrac{1}{2}F^A_{\mu\nu}F^{A\mu\nu}$$

- Thus under gauge transformation

$$
\begin{aligned}
F^A_{\mu\nu}F^{A\mu\nu} &= 2\text{Tr}\left(\boldsymbol{t}\cdot\boldsymbol{F}_{\mu\nu}\boldsymbol{t}\cdot\boldsymbol{F}^{\mu\nu}\right) \;\to \\
F'^A_{\mu\nu}F'^{A\mu\nu} &= 2\text{Tr}\left(\mathcal{U}\boldsymbol{t}\cdot\boldsymbol{F}_{\mu\nu}\mathcal{U}^\dagger\mathcal{U}\boldsymbol{t}\cdot\boldsymbol{F}^{\mu\nu}\mathcal{U}^\dagger\right) \\
&= 2\text{Tr}\left(\mathcal{U}^\dagger\mathcal{U}\boldsymbol{t}\cdot\boldsymbol{F}_{\mu\nu}\mathcal{U}^\dagger\mathcal{U}\boldsymbol{t}\cdot\boldsymbol{F}^{\mu\nu}\right) \\
&= F^A_{\mu\nu}F^{A\mu\nu}
\end{aligned}
$$

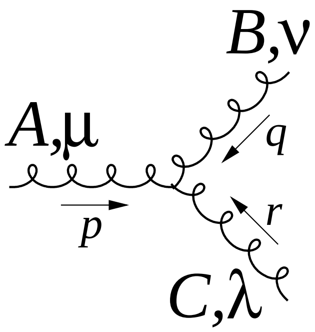
This confirms that the QCD Lagrangian in invariant w.r.t. SU(3) gauge transformations.

- In addition to the quark-gluon coupling



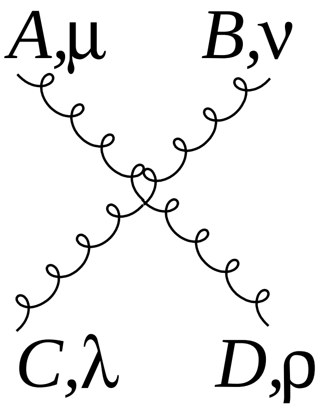
$$-ig_{\,\mathrm{S}}(t^A)_{cb}\gamma^\mu$$

- there is also the 3-gluon self-coupling



$$-g_{\,\mathrm{S}} f^{ABC}\left[g^{\mu\nu}(p-q)^\lambda + g^{\nu\lambda}(q-r)^\mu + g^{\lambda\mu}(r-p)^\nu\right]$$

❖ and the 4-gluon self-coupling



$$
\begin{aligned}
-ig_s^2 \quad [\quad & f^{XAC} f^{XBD} (g^{\mu\nu} g^{\lambda\rho} - g^{\mu\rho} g^{\nu\lambda}) \\
+ \quad & f^{XAD} f^{XBC} (g^{\mu\nu} g^{\lambda\rho} - g^{\mu\lambda} g^{\nu\rho}) \\
+ \quad & f^{XAB} f^{XCD} (g^{\mu\lambda} g^{\nu\rho} - g^{\mu\rho} g^{\nu\lambda})]
\end{aligned}
$$

- In addition to $\bar{q}_f q_{f'}$ systems (mesons), there is another class of gauge-invariant systems of quarks:

$$
\begin{aligned}
\epsilon_{abc}\, q_a q_b q_c \quad \to \quad & \epsilon_{abc}\, q'_a q'_b q'_c \\
= \quad & \epsilon_{abc} \mathcal{U}_{aa'} \mathcal{U}_{bb'} \mathcal{U}_{cc'}\, q_{a'} q_{b'} q_{c'} \\
= \quad & \epsilon_{a'b'c'} \det \mathcal{U}\, q_{a'} q_{b'} q_{c'} \\
= \quad & \epsilon_{abc}\, q_a q_b q_c
\end{aligned}
$$

since $\det \mathcal{U} = 1$ for any $\mathcal{U} \in$ SU(3).

- Similarly

$$\epsilon_{abc}\, \bar{q}_a \bar{q}_b \bar{q}_c \to \epsilon_{abc}\, \det \mathcal{U}^\dagger\, q_a q_b q_c$$

where $\det \mathcal{U}^\dagger = \det \mathcal{U} = 1$. We recognise these as the baryon and antibaryon systems.

- The above meson and baryon states are only invariant w.r.t. local gauge transformations if all the quark and antiquark fields are at the same point. Otherwise, gauge transformation gives

$$\bar{q}(y) q(x) \to \bar{q}(y) \mathcal{U}^\dagger(y) \mathcal{U}(x) q(x) \neq \bar{q}(y) q(x)$$

since in general $\mathcal{U}(x) \neq \mathcal{U}(y)$ for $x \neq y$. To restore gauge invariance we need to introduce the gluon field. This is done with the colour string operator

$$\begin{aligned} S(y,x) &= \mathcal{P} \exp \left[-i g_{\,\mathrm{S}} \int_x^y dz^\mu\, \boldsymbol{t} \cdot \boldsymbol{\mathcal{A}}_\mu(z) \right] \\ &= \prod_{z=x}^{y} \left[1 - i g_{\,\mathrm{S}}\, dz^\mu\, \boldsymbol{t} \cdot \boldsymbol{\mathcal{A}}_\mu(z) \right] \end{aligned}$$

where $\mathcal{P}$ stands for path ordering of the matrix product.

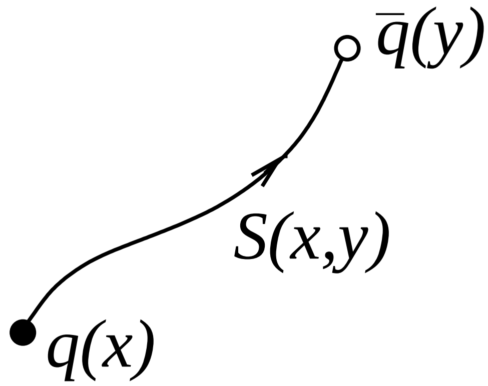
- Under gauge transformation

$$1 - i g_{\,\mathrm{S}}\, dz^\mu \boldsymbol{t} \cdot \boldsymbol{\mathcal{A}}_\mu(z) \;\to$$

$$
\begin{aligned}
& 1 - ig_{\,\mathrm{s}}\, dz^\mu \left[\mathcal{U}(z) \boldsymbol{t} \cdot \boldsymbol{\mathcal{A}}_\mu \mathcal{U}^\dagger(z) + \frac{i}{g_{\,\mathrm{s}}} (\partial_\mu \mathcal{U}(z))\, \mathcal{U}^\dagger(z) \right] \\
&= \; [\mathcal{U}(z) + dz^\mu \partial_\mu \mathcal{U}(z)]\, [1 - ig_{\,\mathrm{s}}\, dz^\mu \boldsymbol{t} \cdot \boldsymbol{\mathcal{A}}_\mu(z)]\, \mathcal{U}^\dagger(z) \\
&= \; \mathcal{U}(z + dz)\, [1 - ig_{\,\mathrm{s}}\, dz^\mu \boldsymbol{t} \cdot \boldsymbol{\mathcal{A}}_\mu(z)]\, \mathcal{U}^\dagger(z)
\end{aligned}
$$

and therefore $S(y,x) \to \mathcal{U}(y) S(y,x) \mathcal{U}^\dagger(x)$.

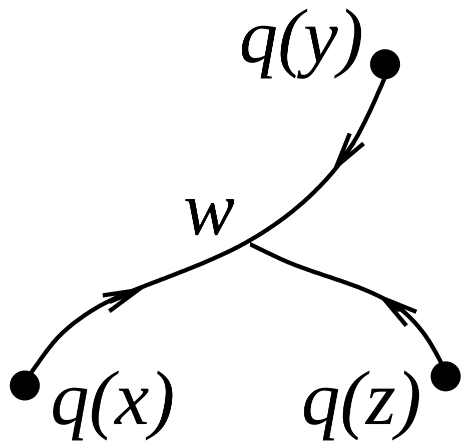
- Hence the gauge invariant object is a quark and an antiquark joined by a colour string



$$
\begin{aligned}
\bar{q}(y) S(y,x) q(x) \quad &\to \quad \bar{q}(y) \mathcal{U}^\dagger(y) \mathcal{U}(y) S(y,x) \mathcal{U}^\dagger(x) \mathcal{U}(x) q(x) \\
&= \quad \bar{q}(y) S(y,x) q(x)
\end{aligned}
$$

- This is similar to QED, where the field between an electron and positron can be regarded as a superposition of strings representing flux lines.

- However, the gauge invariant baryon state has no analogue in QED, since it involves a string junction at some (arbitrary) space-time point w^μ

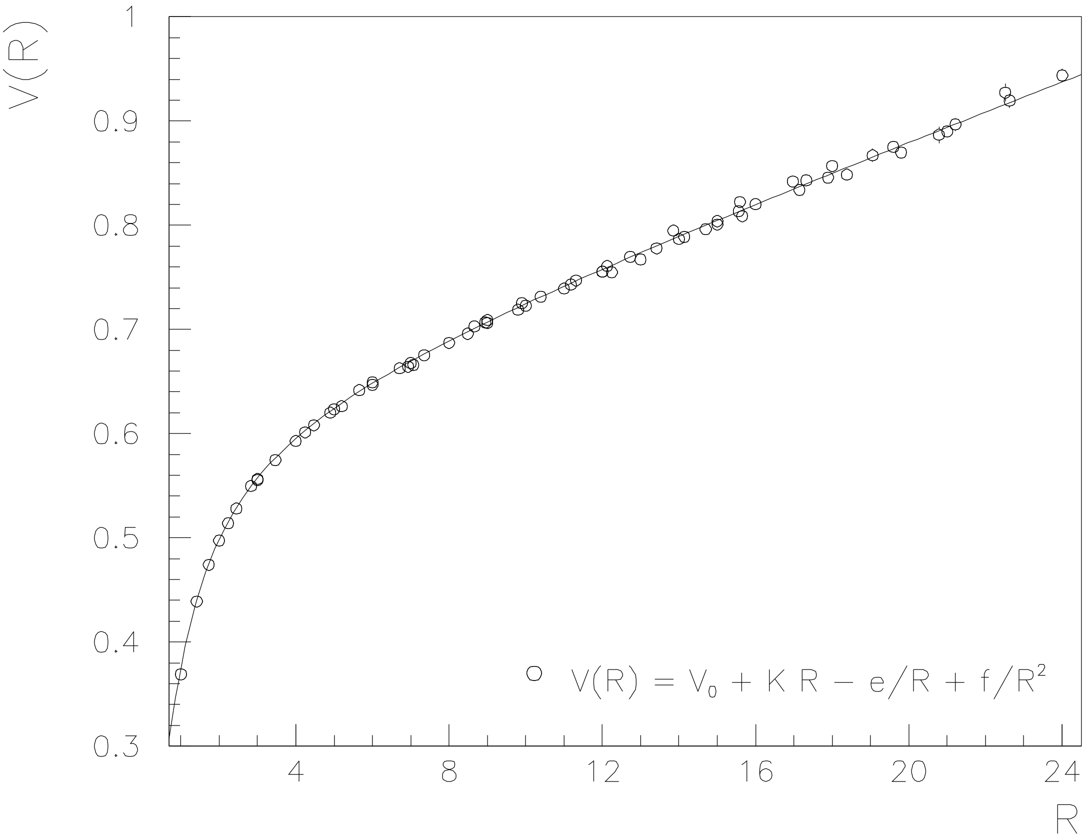


Putting in colour indices explicitly, this is

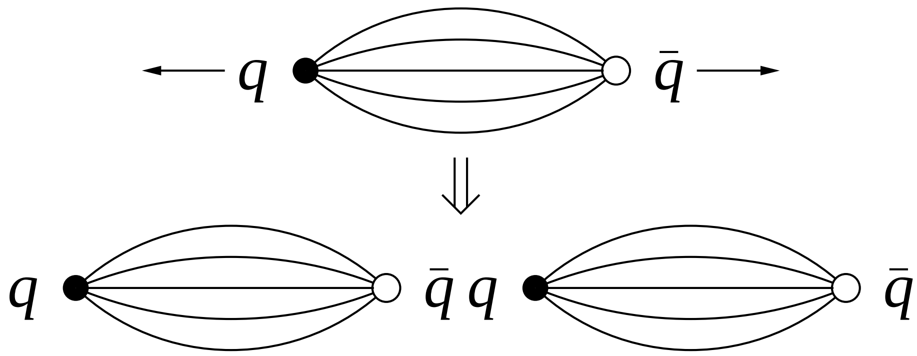
$$\epsilon_{abc} S_{aa'}(w,x) q_{a'}(x) S_{bb'}(w,y) q_{b'}(y) S_{cc'}(w,z) q_{c'}(z)$$

which is gauge invariant, by the argument used above for three quarks at the same point.

- The force between two colour charges (e.g. quarks) becomes strong at large distances, and is believed to give rise to a linear potential.

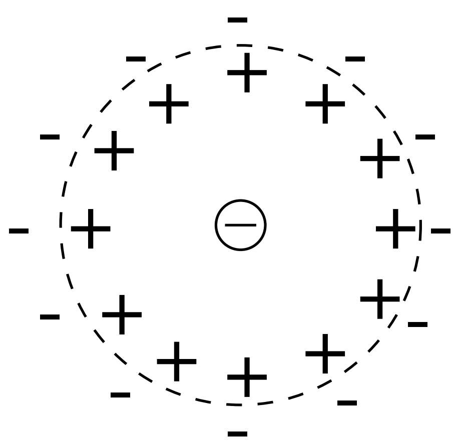


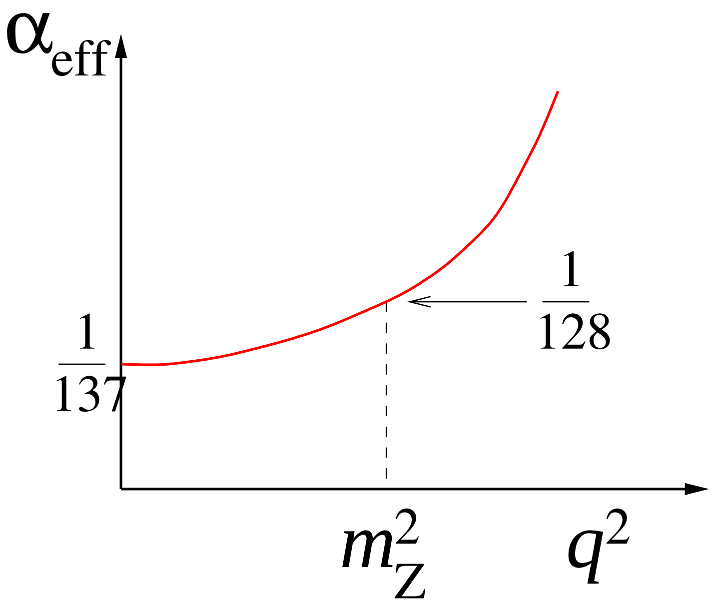
- Thus quarks (and gluons) cannot be observed as isolated objects and exist only in bound states (mesons and baryons). This mechanism is called confinement.



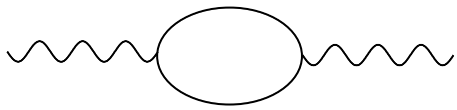
Running Coupling

- In QED the observed electron charge is distance-dependent ($\Rightarrow$ momentum transfer dependent) due to vacuum polarization:

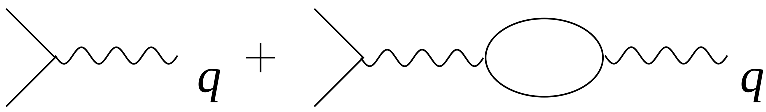




- The one-loop vacuum polarization diagram



is log-divergent and can be regularized by introducing a cut-off Λ. Then the observed coupling at momentum transfer q^μ is



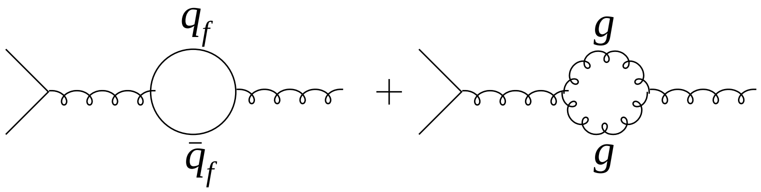
$$\alpha(q^2) = \alpha_{\text{bare}} \left(1 + \frac{\alpha}{3\pi} \ln \frac{q^2}{\Lambda^2} + \cdots \right)$$

Thus

$$q^2 \frac{d\alpha}{dq^2} = \frac{\alpha^2}{3\pi} + \cdots \equiv \beta_{\text{QED}}(\alpha)$$

where the QED β-function $\beta_{\text{QED}}(\alpha) > 0$. Therefore $\alpha \to \infty$ at large q^2 (short distances).

- In QCD there is an additional diagram:



As a consequence, the β-function changes. Now

$$\alpha_{\text{S}}(q^2) = \alpha_{\text{S,bare}} \left(1 + \frac{N_f}{6\pi} \alpha_{\text{S}} \ln \frac{q^2}{\Lambda^2} - \frac{11}{4\pi} \alpha_{\text{S}} \ln \frac{q^2}{\Lambda^2} - \cdots \right)$$

where N_f is the number of quark flavours. Hence $\beta_{\text{QCD}}(\alpha_{\text{S}}) = -\beta_0 \alpha_{\text{S}}^2 - \cdots < 0$ where

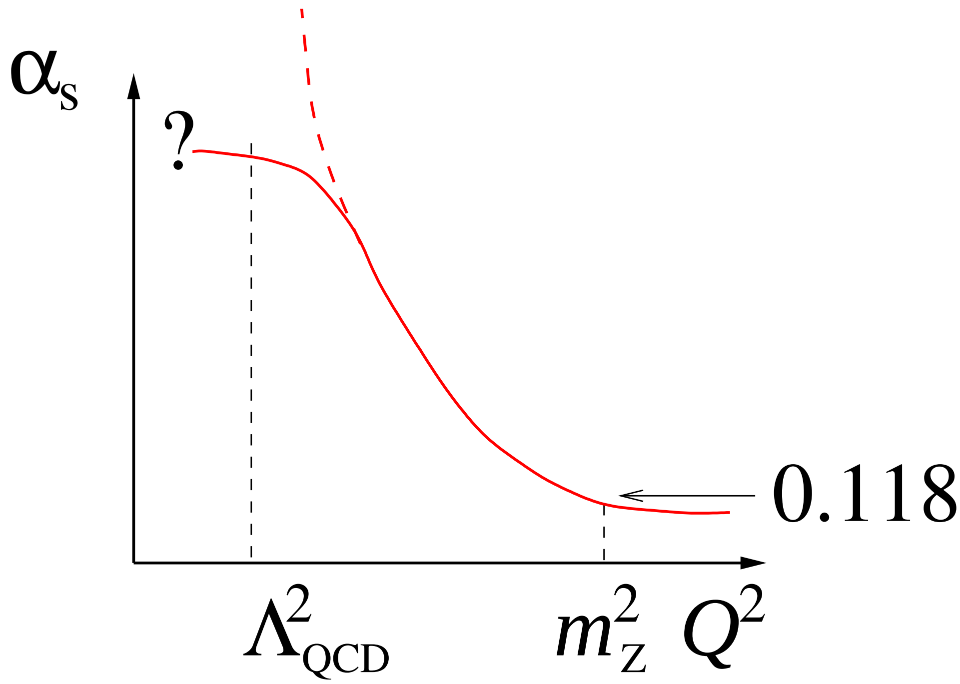
$$\beta_0 = \frac{1}{12\pi}(33 - 2N_f)$$

- The gluonic contribution to the vacuum polarization reverses the sign of the β-function, so that the strong coupling $\alpha_{\,\mathrm{S}}$ decreases at large q^2 (short distances). This is called asymptotic freedom. It implies that quarks behave as (almost) free particles at short distances, and perturbation theory can be used for hard processes, i.e. processes involving large momentum transfers.

- From
$$q^2 \frac{d\alpha_{\,\mathrm{S}}}{dq^2} = \beta_{\,\mathrm{QCD}}(\alpha_{\,\mathrm{S}})$$
we find
$$\int_{\alpha_{\,\mathrm{S}}(\mu^2)}^{\alpha_{\,\mathrm{S}}(Q^2)} \frac{d\alpha_{\,\mathrm{S}}}{\beta_{\,\mathrm{QCD}}(\alpha_{\,\mathrm{S}})} = \int_{\mu^2}^{Q^2} \frac{dq^2}{q^2} = \ln \frac{Q^2}{\mu^2}$$
In lowest order, $\beta_{\,\mathrm{QCD}}(\alpha_{\,\mathrm{S}}) = -\beta_0 \alpha_{\,\mathrm{S}}^2$ and thus
$$\begin{aligned} \frac{1}{\alpha_{\,\mathrm{S}}(Q^2)} - \frac{1}{\alpha_{\,\mathrm{S}}(\mu^2)} &= \beta_0 \ln \frac{Q^2}{\mu^2} \\ \Rightarrow \quad \alpha_{\,\mathrm{S}}(Q^2) &= \frac{\alpha_{\,\mathrm{S}}(\mu^2)}{1 + \alpha_{\,\mathrm{S}}(\mu^2)\beta_0 \ln(Q^2/\mu^2)} \end{aligned}$$

- We had to introduce a dimensionful parameter μ to specify the boundary conditions. Alternatively, we can define $\Lambda_{\,\text{QCD}}$ (not to be confused with the cut-off Λ introduced earlier) as the scale at which the (perturbative) solution diverges:

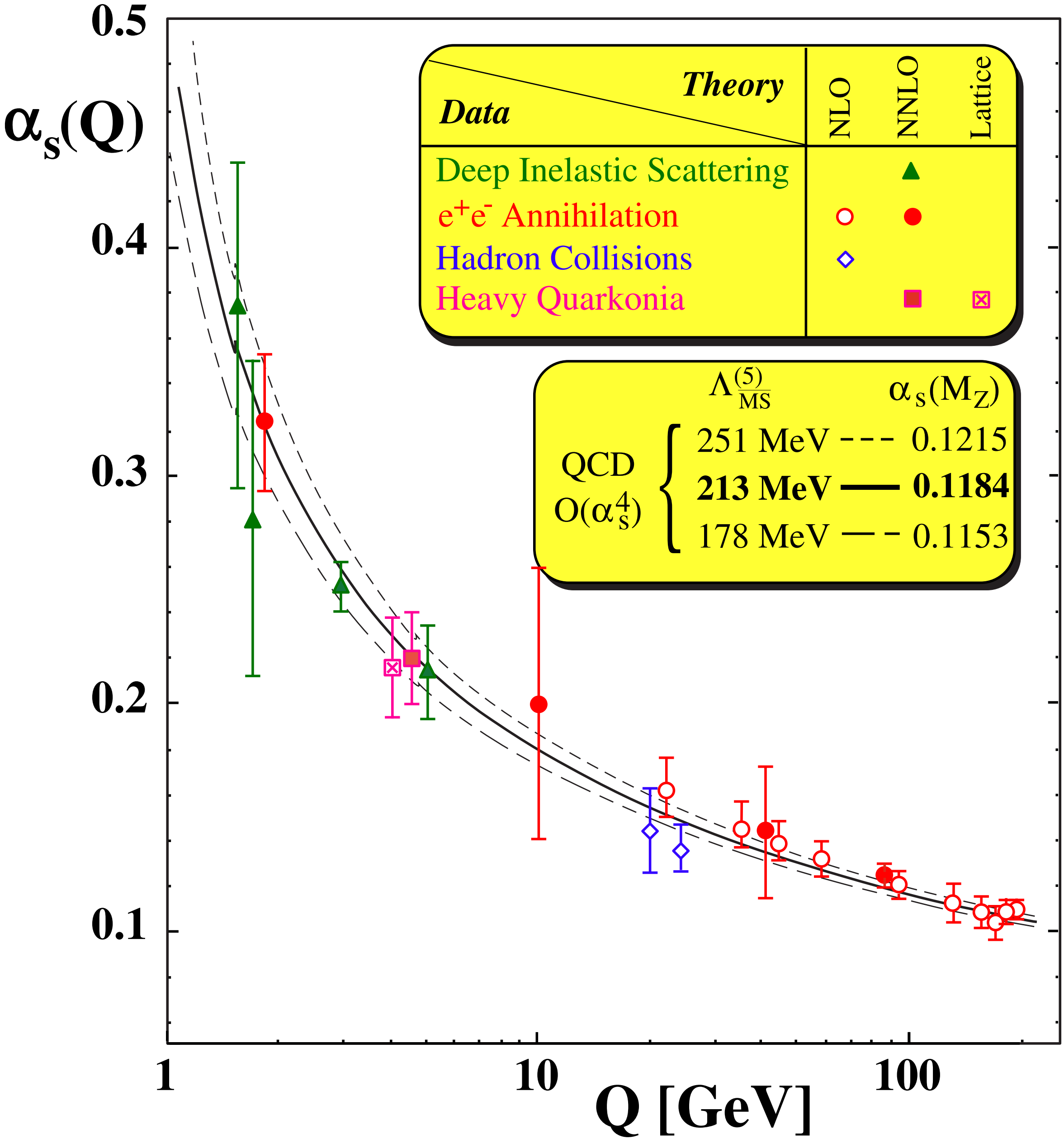
$$\begin{aligned} \frac{1}{\alpha_{\,\text{S}}(Q^2)} - 0 &= \beta_0 \ln \frac{Q^2}{\Lambda^2_{\text{QCD}}} \\ \Rightarrow \quad \alpha_{\,\text{S}}(Q^2) &= \frac{1}{\beta_0 \ln(Q^2/\Lambda^2_{\text{QCD}})} \end{aligned}$$



- Experimentally the fundamental scale of QCD is

$$\Lambda_{\,\text{QCD}} \simeq 210 \pm 40 \text{ MeV}$$

- Measurements at different scales show clear evidence that α_s does run



- Using the formula for the running of α_s, measurements at different scales can be expressed as measurements of $\alpha_s(M_Z^2)$