The scale depends on size $L$ of the extra dimension! That is drastically different from the case of flat extra dim!

Thus $\omega$ in 4d remains large $\propto M_{*}$, and therefore gravity is weak!

The 4d hierarchy problem is resolved in this model!

3) Radius stabilization in RS1

(Goldberger-Wise mechanism)

Problem is that in RS1 as discussed so far there is no potential for the size parameter $L$, implying the existence of a massless scalar field (radion).

Why is the size $k \cdot L \approx 3$ natural, and how could it be stable?

In the Goldberger-Wise mechanism this problem is fixed by introducing a massive scalars field in the bulk in a suitable way.
To obtain a potential for $L$ with a minimum one needs two "forces" pulling in opposite directions (like for stable orbits in a two-body problem). Here the mass of the scalar field will tend to make the six smaller. In addition, one introduces a non-trivial profile of the scalar field $VEV$ along the $y$-direction which tends to make the six larger (in order to have a smaller slope).

Technically, the latter is done by introducing $brane$ potentials which fix the $VEV$ at 0 and $L$ at two different values.

The corresponding actions are

$$S_{bulk}^{\phi} = \frac{1}{2} \int d^4x \sqrt{\text{det} G} \left( \partial_{\mu} \phi \partial^\mu \phi - m^2 \phi^2 \right)$$

$$S_{brane}^{\phi} = \int d^4x \sqrt{\text{det} \gamma} \Lambda_T \left( \phi^2 - v_T^2 \right)^2$$

$$S_{brane}^{\phi} = \int d^4x \sqrt{\text{det} \gamma} \Lambda_T \left( \phi^2 - v_T^2 \right)^2$$
The general solution in the bulk is found as
\[ \phi(y) = e^{2ky} \left( A e^{ky} + B e^{-ky} \right) \]
with \[ V = \sqrt{y + \frac{w^2}{k^2}} \]
The brane potentials are chosen such that the situation looks like

Then one can write a potential for \( L \), \( V(L) \), and minimize it. One finds for the minimum
\[ k \cdot L = \frac{q}{\pi} \frac{k^2}{w^2} \ln \left( \frac{V_{\text{eq}}}{V_{\text{inv}}} \right) \]
For \( k \cdot L < 35 \) one needs \( k > w \), but not by much, which is a simple requirement without fine-tuning.

Note that due to the potential for \( L \) (that is due to stabilization) the radion acquires a mass. → important for radion phenomenology!
The Goldberger–Wise mechanism was later improved by including the back-reaction of the energy density on the space-time curvature.

4) Gravity in RS1

Now study KK-decomposition of graviton in AdS background.

Usual expectation from flat extra dim. or $S^1$ is:

- two modes: \( \) graviton
  \( \) vector ("graviphoton")
  \( \) scalar ("graviscalar")

+ massive graviton at higher KK levels (having extra scalar + vector)

But here situation is different due to orbifold! Consider generic form of metric with fluctuations:

\[
d s^2 = e^{-2k(y)} \left[ g_{\mu\nu} \, dx^\mu \, dx^\nu + A_\mu \, dx^\mu \, dy - b_\mu \, dy \right] + \text{terms containing graviton, vector, and scalar fluctuation (see below)}
\]
Since $ds^2$ is symmetric under $y \to -y$, $A_\mu$ has to change sign under that transform and hence cannot have a two-mode. Therefore there will only be a scalar and the graviton two-mode, plus massive gravitons at higher k\ell levels.

We first concentrate only on the graviton and set $\kappa \to 2\kappa$.

To find the $\kappa\kappa$ expansion of the graviton, go to the conformal frame,

$$ds^2 = e^{-A(z)} \left[ (g_{\mu\nu} + 2\mu(x,z)) dx^\mu dx^\nu - dz^2 \right]$$

with

$$e^{-A(z)} = \frac{1}{(1+k|z|)^2}$$

or

$$A(z) = 2 \log (k|z| + 1)$$

The relation between $y$ and $z$ is

$$\frac{1}{1+k|z|} = e^{-k|y|}$$
We look for linearized fluctuations around the background satisfying ("perturbed") Einstein equation,

\[ \delta G_{MN} = \frac{\Lambda}{M_*^3} \delta T_{MN} \]

We fix the so-called RS-gauge

\[ \delta h^\mu = \partial_\mu \delta h^\nu = 0 \]

and expand \( \delta G_{MN} \) keeping only linear terms in order to get linearized

Einstein eq. in warped background,

\[ -\frac{1}{2} \partial^\rho \partial_\rho \delta h^\mu_\nu + \frac{3}{4} \partial^\rho A \partial_\rho \delta A_{\mu_\nu} = 0 \]

It is convenient to use the rescaling

\[ \delta h^\mu_\nu = e^{\frac{5A}{3\Lambda}} \tilde{\delta h}^\mu_\nu \]

giving

\[ -\frac{1}{2} \partial^\rho \partial_\rho \tilde{\delta h}^\mu_\nu + \left[ \frac{9}{32} \partial^\rho A \partial_\rho A - \frac{3}{8} \partial^\rho A \partial_\rho A \right] \tilde{\delta h}^\mu_\nu = 0 \]

which has the form of a one-dimensional Schrödinger equation.

\( \text{(for} \quad \partial^\rho \partial_\rho = -\partial_\mu - \partial_\nu^2) \)
Now separate variables

$$\tilde{\phi}_{\mu \nu} (x, \tau) = \hat{\phi}_{\mu \nu} (x) \hat{f} (\tau)$$

and require \( \hat{\phi}_{\mu \nu} \) to be a Tolman mass eigenstate with

$$\Box \hat{\phi}_{\mu \nu} = m^2 \hat{\phi}_{\mu \nu}.$$

Then the Schrödinger equation for the KK modes becomes

$$- \frac{\partial^2}{\partial \tau^2} \hat{f} + \left( \frac{9}{16} A'^2 - \frac{3}{4} A'' \right) \hat{f} - m^2 \hat{f} = V(\tau) \quad \text{Schrödinger potential}$$

We have \( A = 2 \log \left( k/\ell + 1 \right) \), so

$$V(\tau) = \frac{15}{4} \frac{m^2}{(1 + k/\ell)^2} - \frac{3k}{1 + k/\ell} \delta (\tau)$$

the so-called volcano potential

The fall-off is

$$V(\tau) \sim \frac{1}{\tau^2} \quad \text{at large } \tau$$

and

$$\delta - fct. \text{ at } \tau = 0.$$
As known from quantum mechanics the 5-plet potential has a single bound state, here: the massless graviton (which was expected since 4d Lorentz invariance is unbroken).

Explicitly, one finds solution

\[ f^{(0)} (z) = e^{-\frac{3}{4} A(z)} \]

\[ = \frac{1}{(1 + k |z|)^{3/4}} \]

or in \( y \)-coordinates with

\[ ds^2 = e^{-2 k |y|} (dy^a + d\phi^a) - dy^2 \]

(confirming the ansatz made earlier)

one has

\[ f^{(n)} (y) = e^{-\frac{3}{4} k |y|} \]

This means that the graviton 50-mode is localized at the Planck brane \((y = 0)\) but exponentially suppressed \((\sim e^{-\frac{3}{4} k L}, kL \approx 35)\) at the TeV brane, without introducing small parameters.
Gravity is localized around the positive tension brane.

Here the weakness of gravity at the TeV-brane is due to the localization of the graviton wave function at the other brane!

(Note that this mechanism is completely different from flat (large) extra dimensions!)

Higher modes, on the other hand, are pushed to large $|\ell|$ by the barrier in the volcano potential and cannot easily get to the Planck brane (only by tunneling - estimate possible with WKB methods).

To find wave functions of higher KK modes, impose suitable boundary conditions obtained from orbifold conditions under $\gamma \rightarrow -\gamma$. For gravitons that implies

$$\partial_y \phi_{\ell \nu} = 0 \quad \text{at} \quad y=0 \quad \text{and} \quad L,$$

or

$$\partial_z \phi_{\ell \nu} = 0 \quad \text{in} \quad z\text{-coord. at branes}.$$
That gives
\[ \frac{\partial^2 \psi}{\partial x^2} = -\frac{3}{2} k \psi \quad | \quad \xi = \xi_{\text{PC}} = 0 \]
\[ \frac{\partial^2 \psi}{\partial x^2} = -\frac{3}{2} \frac{k}{\sqrt{1 + k \xi^2}} \psi \quad | \quad \xi = \xi_{\text{UV}} = \frac{1}{k} e^{k \xi} \]

Then in the bulk
\[ -\frac{\partial^2 \psi}{\partial x^2} + \frac{15}{4} \left( 1 + k \xi^2 \right) \psi = m^2 \psi \]

with general solution
\[ \psi(x) = \frac{1}{\sqrt{1 + k \xi^2}} \left[ a \text{ Y}_2 \left( \frac{m \xi}{k} \right) + b \text{ J}_2 \left( \frac{m \xi}{k} \right) \right] \]

where \( \text{Y}_2 \) and \( \text{J}_2 \) are Bessel functions and \( a \) and \( b \) are determined by boundary conditions.

The mass spectrum found from this is
\[ m_j = x_j \sqrt{1 + k \xi^2} \]

with \( x_j \) the roots of the Bessel function \( \text{J}_2 \), \( \text{J}_2(x_j) = 0 \). Approximately, they are

<table>
<thead>
<tr>
<th>j</th>
<th>x_j</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.8</td>
</tr>
<tr>
<td>2</td>
<td>7.0</td>
</tr>
<tr>
<td>3</td>
<td>10.2</td>
</tr>
<tr>
<td>4</td>
<td>16.5</td>
</tr>
</tbody>
</table>
Note that the modes are not evenly spaced. Since $k e^{-kL} \sim O(\text{TeV})$, the massive gravitons have masses of $O(\text{TeV})$. Due to the barrier in the volcano potential, these massive gravitons have wave functions peaked near the TeV-brane. Therefore, their coupling to TeV-brane matter is exponentially enhanced.

This can also be seen from

$$\frac{4}{\alpha'(\text{TeV})} \sim e^{-kL}$$

$4(\text{TeV})$ is the inverse warp factor

Therefore, the interaction of TeV-brane matter with the graviton zero-mode and with the massive graviton modes is very different:

$$\mathcal{L}_{\text{grav. matter}} = -\frac{1}{\mathcal{M}_{\text{Pl}}} T^{\alpha\beta} h_{\alpha\beta}^{(0)}$$

$$-\frac{1}{\mathcal{M}_{\text{Pl}} e^{-kL}} T^{\alpha\beta} \sum_{a=1}^{\infty} \tilde{h}_{a\alpha\beta}^{(a)} \sim O(\text{TeV})$$
5) Phenomenology of RS I

The massive gravitons couple strongly to matter on the TeV-brane. As we have seen, this holds even for individual modes. Therefore they decay quickly.

Due to the discrete spectrum with spacings $\Delta m \sim \Theta(\text{TeV})$ we can expect to see single resonances.

In hadron-hadron collisions, possible channels will be for example

\[
\begin{align*}
\gamma \gamma, q\bar{q} &\rightarrow \ell^+\ell^- \\
\gamma \gamma, q\bar{q} &\rightarrow \ell^+\ell^- + \ell^+\ell^- \\
\end{align*}
\]

(corresponding to lepton pairs or 2-jet events with resonances in the invariant mass spectrum.

With these events one can test a large region in the parameter space (warped Planck scale / AdS curvature) at LHC, see figure.
LHC reach for discovery/exclusion of first massive KK graviton in RS1

Parameter space to the left of the curves is covered.
An even cleaner signal can be expected at a future e⁺e⁻ linear Collider where broad resonances should occur in e⁺e⁻ → ℓ⁺ℓ⁻, see figure.

After radius stabilization the radius acquires a mass

\[ m_\phi = E \frac{1}{\sqrt{6kL}} M_{\text{pe}} e^{-kL} \]

and the radion couplings are found to be

\[ \frac{\phi}{\sqrt{6M_{\text{pe}} e^{-kL}}} \]

Many effects of the radion are very similar to those of a Higgs boson since they have similar couplings to matter. Radion and Higgs are therefore difficult to distinguish in RS1. Gluons, however, couple much stronger (50 times) to radion than to Higgs, since the radion coupling is direct.

[Diagram of radion and Higgs interactions]

radion

Higgs
6) **RS2 model**

In the RS1 model the physical space was an interval \([0, L]\),

\[
\begin{array}{c}
| \quad L \quad | \\
\text{Planck} \quad \text{TeV} \\
\text{brane} \quad \text{brane}
\end{array}
\]

We found for the (effective) 4d Planck scale

\[
\frac{\langle M^2 \rangle_{\text{Pl}}}{M_{\text{Pl}}} = \frac{M_{\text{Pl}}^3}{k} \left(1 - e^{-2kL}\right)
\]

Note that this depends only very little on \(L\), and is even finite for \(L \to \infty\). One could therefore consider the limit \(L \to \infty\), taking the TeV brane to infinity - thus removing it completely.

Then there is only one (positive tension) brane left (the Planck brane), and we have an infinite extra dimension.

\[ \text{Matter lives on this brane} \]

\[ \text{Planck brane} \]

\[ \text{Warped metric in bulk} \]
Naturally, matter now has to be put on the Planck brane.
This is the RS2 model.

Obviously, in contrast to the RS1 model, it is no longer designed to resolve the hierarchy problem. But it is still an interesting and simple setup.

In fact, even the volume of the extra dimension remains finite:

\[ V_5 = 2 \int d^4x \int_0^{\infty} dy \sqrt{g} \]

\[ = V_4 \cdot 2 \int_0^{\infty} dy \ e^{-ky} \]

\[ = V_4 \frac{e}{2} \]

Hence \( l = \frac{1}{k} \), and not \( L \), plays the role of the compactification radius \( R_{\text{comp}} \) (compare to flat extra dimensions).

Close to the brane, everything remains as it was in the RS1 model.
In particular, the graviton two-mode remains localized to the Planck brane, since it remains normalizable:

We had \( \psi^{(2)} = e^{-\frac{3}{2}\lambda A(t)} \)

\[-1 \int_0^{t_0} 4^{(2)} \psi^{(2)} \psi^{(2)*} = \int_0^{t_0} dt e^{-3\lambda A(t)} \]

\[= \int_0^{t_0} dt \frac{1}{(A + k(t))^2} \]

which is convergent in the limit \( t_0 \to \infty \) \((\frac{A}{L} \to \infty)\).

[Recall that in flat extra dimensions the two-mode would become non-normalizable for \( R_{\text{comp}} \to \infty \) due to the finite volume of the flat extra dim and would then decouple.]

Again, the massive gravitons will stay away from the brane.

Therefore, normal 4d gravity is recovered on the Planck brane.
More precisely, corrections to the Newton potential in the effective 4d
theory on the brane are of the form
\[ V(r) = G_N \frac{M_1 M_2}{r} \left( 1 + \frac{C}{(kr)^2} \right) \]
with \( C \approx O(1) \). Since \( k \ll M_P \), these corrections will be tiny.

One can even show that not only the Newton potential but also full 4d
Einstein gravity is recovered on the brane in RS2.

Experimentally, it will be very difficult
to distinguish RS2 from the SM.
On the brane everything looks like
the SM plus gravity. Therefore
there is probably no chance to find
limits for RS2 at colliders.

But there might be realistic possibilities to find limits for RS2
in astrophysics and (more likely)
in cosmology.
7) \textit{AdS/CFT}

It has been conjectured by Maldacena (and subsequently been confirmed in many tests) that gravity in AdS\textsubscript{5} space is equivalent to a certain gauge theory on 4d Minkowski space. More precisely, type \textit{II}B superstring theory on AdS\textsubscript{5} is equivalent to maximally supersymmetric Yang-Mills theory in 4d, according to that conjecture. This supersymmetric theory in 4d is scale invariant and hence a \textit{Conformal Field Theory} (CFT).

The Maldacena conjecture has been extended in various ways and indeed seems to indicate much deeper relations between theories in different dimensions, in particular it relates theories with strong coupling to theories with weak coupling.
As the Randall - Sundrum models live exactly in AdS, the conjecture can be used to study these models. What emerges is an interesting holographic picture of warped extra dimensions, in which the physics in the bulk is `holographically' represented by the physics on the brane.

This holographic point of view has by now become very common in studies of RS models.

Details of the AdS/CFT correspondence are beyond the scope of the present lectures, but are highly recommended for further private study.