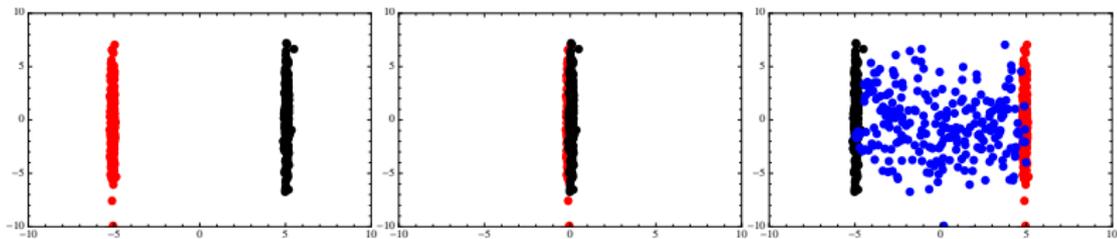


*Fluctuations in the hydrodynamical description  
of relativistic heavy ion collisions*

Stefan Flörchinger (CERN)

Santiago de Compostela, 06/06/2013

# Heavy Ion Collisions



- ions are strongly Lorentz-contracted
- *some* medium is produced after collision
- medium expands in longitudinal direction and gets diluted

# *Evolution in time*

- Non-equilibrium evolution at early times
  - initial state at from QCD? Color Glass Condensate? ...
  - thermalization via strong interactions, plasma instabilities, particle production, ...
- Local thermal and chemical equilibrium
  - strong interactions lead to short thermalization times
  - evolution from relativistic fluid dynamics
  - expansion, dilution, cool-down
- Chemical freeze-out
  - for small temperatures one has mesons and baryons
  - inelastic collision rates become small
  - particle species do not change any more
- Thermal freeze-out
  - elastic collision rates become small
  - particles stop interacting
  - particle momenta do not change any more

## *Fluid dynamic regime*

- assumes strong interaction effects leading to local equilibrium
- fluid dynamic variables
  - thermodynamic variables: e.g.  $T(x)$ ,  $\mu(x)$
  - fluid velocity  $u^\mu(x)$
- can be formulated as derivative expansion for  $T^{\mu\nu}$
- hydrodynamics is universal!
- ideal hydro: needs equation of state  $p = p(T, \mu)$  from thermodynamics
- first order hydro: needs also transport coefficients like viscosity  $\eta = \eta(T, \mu)$  from linear response theory
- second order hydro: needs also relaxation times,...

## *Bjorken boost invariance*

- How does the fluid velocity look like?
- Bjorkens guess:  $v_3(x^0, x^1, x^2, x^3) = \frac{x^3}{x^0}$
- leads to an invariance under Lorentz-boosts in the  $x^3$ -direction
- use coordinates

$$\tau = \sqrt{(x^0)^2 - (x^3)^2}, \quad x^1, \quad x^2, \quad y = \operatorname{arctanh}(x^3/x^0)$$

- fluid velocity

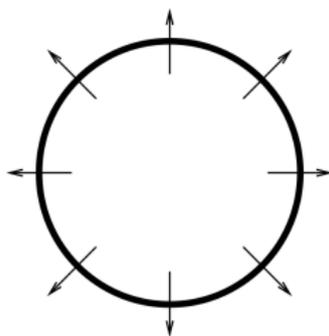
$$u^\mu = (u^\tau, u^1, u^2, u^y) = (\sqrt{1 - (u^1)^2 - (u^2)^2}, u^1, u^2, 0)$$

- thermodynamic scalars like energy density

$$\epsilon = \epsilon(\tau, x^1, x^2)$$

- remaining problem is 2+1 dimensional

## Transverse Expansion



- for central collisions

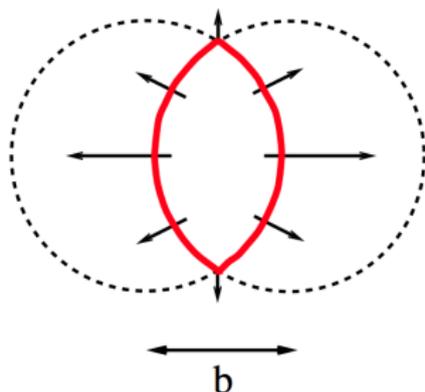
$$\epsilon = \epsilon(\tau, r)$$

- initial pressure gradient leads to radial flow

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} f(\tau, r)$$

- experimental signature not very distinct  
(particle momenta go in all directions)

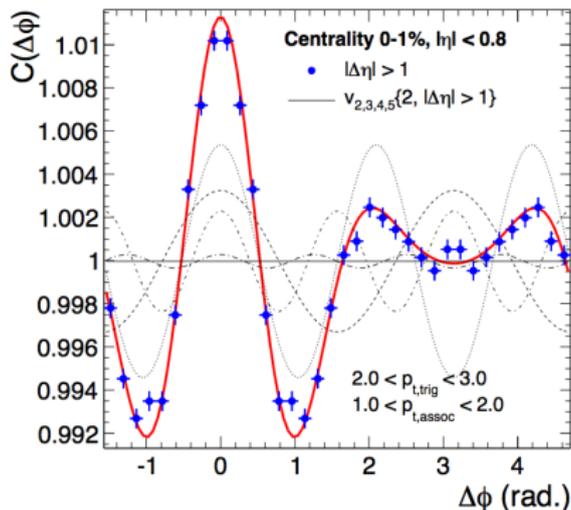
## *Elliptic flow*



- non-central collisions lead to deviations from rotation symmetry
- pressure gradients larger in one direction
- larger fluid velocity in this direction
- more particles will fly in this direction
- can be quantified in terms of elliptic flow  $v_2$

$$C(\Delta\phi) \sim 1 + 2 v_2 \cos(2 \Delta\phi)$$

## A puzzle: $v_3$ and $v_5$



(ALICE, arXiv:1105.3865, similar pictures also from CMS, ATLAS, Phenix)

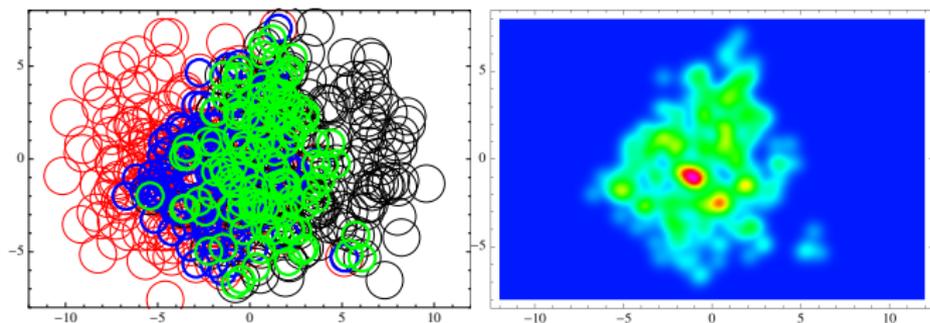
- quite generally, one can expand

$$C(\Delta\phi) \sim 1 + \sum_{n=2}^{\infty} 2 v_n \cos(n \Delta\phi)$$

- from symmetry reasons one expects naively  $v_3 = v_5 = \dots = 0$

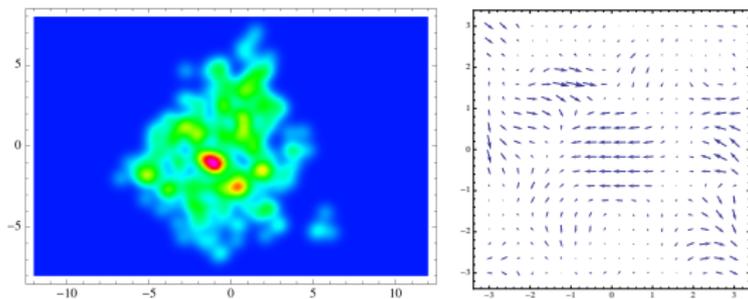
## *Event-by-event fluctuations*

- argument for  $v_3 = v_5 = 0$  is based on smooth energy density distribution
- there can be deviations from this due to event-by-event fluctuations
- for example using a Glauber model

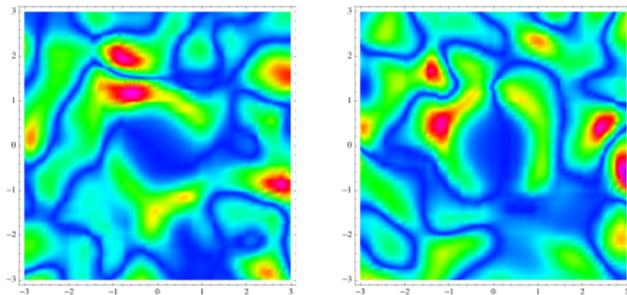


- this leads to sizeable  $v_3$  and  $v_5$

## Velocity fluctuations



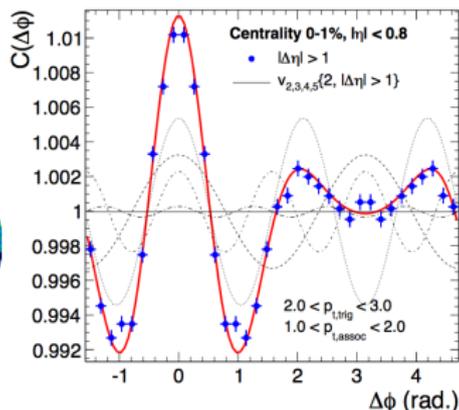
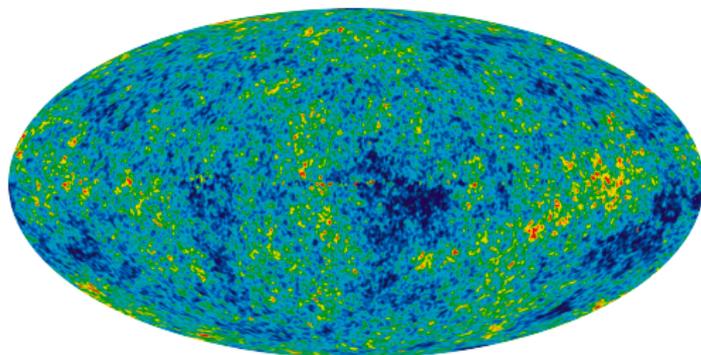
- also the velocity field will fluctuate at the initialization time  $\tau_0$
- take here transverse velocity for every participant to be Gaussian distributed with width  $0.1c$
- vorticity  $|\partial_1 u^2 - \partial_2 u^1|$  and divergence  $|\partial_1 u^1 + \partial_2 u^2|$



## *Why are fluctuations interesting?*

- **Hydrodynamic fluctuations:** Local and event-by-event perturbations around the average of hydrodynamical fields:
  - energy density  $\epsilon$
  - fluid velocity  $u^\mu$
  - shear stress  $\pi^{\mu\nu}$
  - more general also: baryon number density  $n_B$ , ...
- measure for deviations from equilibrium
- contain interesting information from early times
- can be used to constrain thermodynamic and transport properties
- might affect other phenomena, e.g. jet quenching

## Similarities to cosmic microwave background



- fluctuation spectrum contains info from early times
- many numbers can be measured and compared to theory
- detailed understanding of evolution needed
- could trigger precision era in heavy ion physics

## *A complete story about fluctuations*

- 1 Initial fluctuations at initialization time of hydro should be characterized and quantified completely.
- 2 Fluctuations have to be propagated through the hydrodynamical regime.
- 3 Contribution of fluctuations to the particle spectra at freeze-out must be understood and quantified.
- 4 Fluctuations generated from non-hydro sources (such as jets) have to be taken into account.

## Characterization of fluctuations

- traditional characterization based on eccentricities

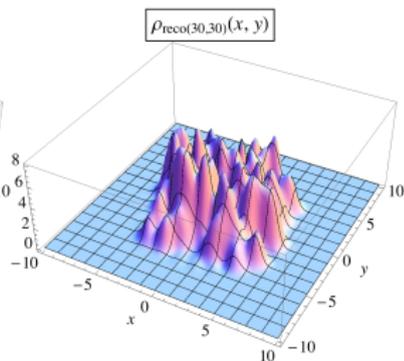
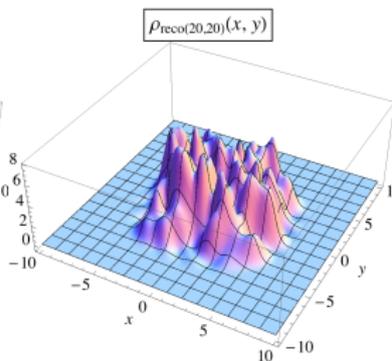
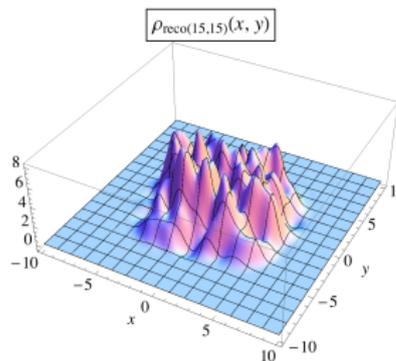
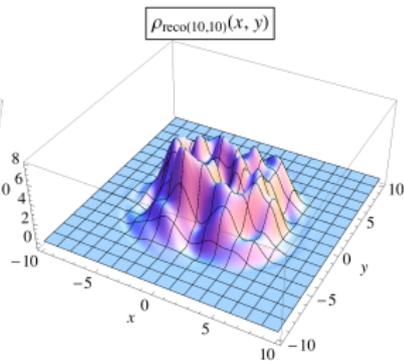
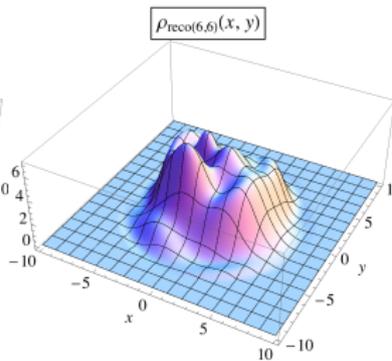
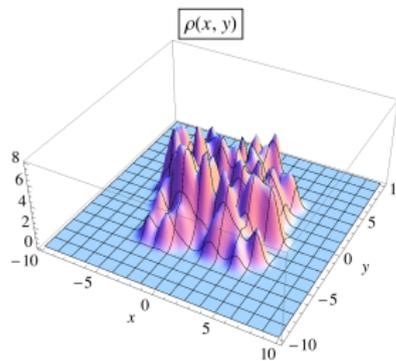
$$\epsilon_{n,m} = \frac{\int dr \int_0^{2\pi} d\varphi r^n e^{im\varphi} \rho(r, \varphi)}{\int dr \int_0^{2\pi} d\varphi r^n \rho(r, \varphi)}$$

- resolves radial dependence only poorly
- “inverse transform” ill defined
- use instead a more differential way based on Bessel functions

$$\rho(r, \varphi) = \sum_{m=-\infty}^{\infty} \sum_{l=1}^{\infty} \rho_l^{(m)} e^{im\varphi} J_m(k_l^{(m)} r)$$

- higher  $l$  correspond to smaller spatial resolution
- can be inverted
- single modes can be propagated
- generalizable to vectors (velocity) and tensors (shear stress)

# Reconstructing the transverse density



## *Background-fluctuation splitting*

- Background or ensemble average over many events with fixed impact parameter  $b$  is described by smooth hydrodynamical fields

$$\bar{\epsilon} = \langle \epsilon \rangle$$

$$\bar{u}^\mu = \langle u^\mu \rangle$$

- Fluctuations are added on top

$$\epsilon = \bar{\epsilon} + \delta\epsilon$$

$$u^\mu = \bar{u}^\mu + \delta u^\mu$$

- For background one can assume Bjorken boost and azimuthal rotation invariance

$$\bar{\epsilon} = \bar{\epsilon}(\tau, r)$$

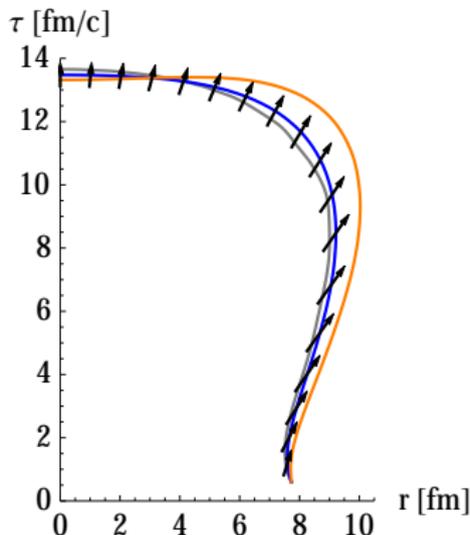
$$\bar{u}^\mu = (\bar{u}^\tau, \bar{u}^r, 0, 0)$$

# *Evolving fluctuations*

show Mathematica files...

## Freeze-out surface

Background and fluctuations are propagated until  $T_{fo} = 130$  MeV is reached.

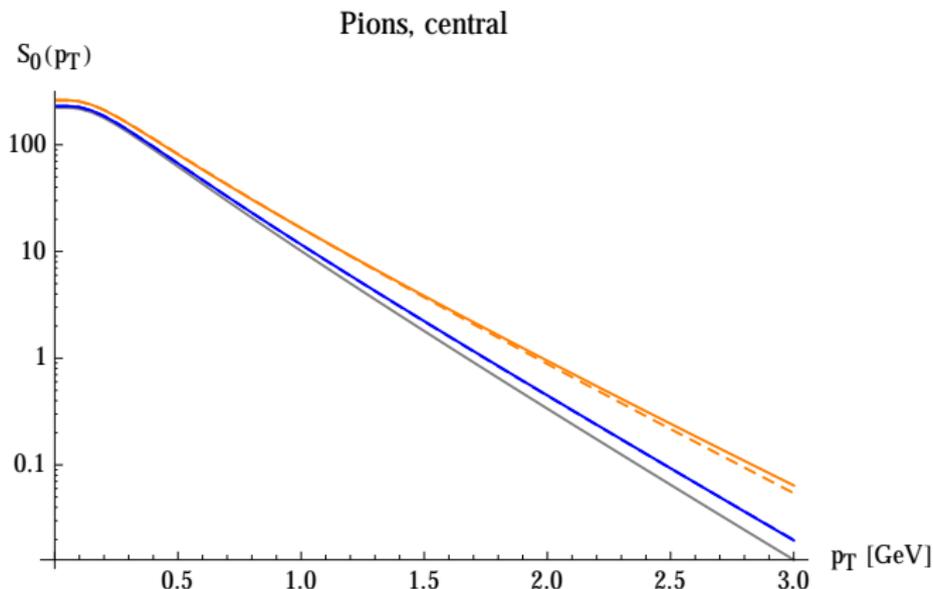


(gray: ideal hydro, blue:  $\eta/s = 0.08$ , orange:  $\eta/s = 0.3$ )

Distribution functions are determined on this surface and free streaming is assumed for later times (Cooper-Frye freeze out).

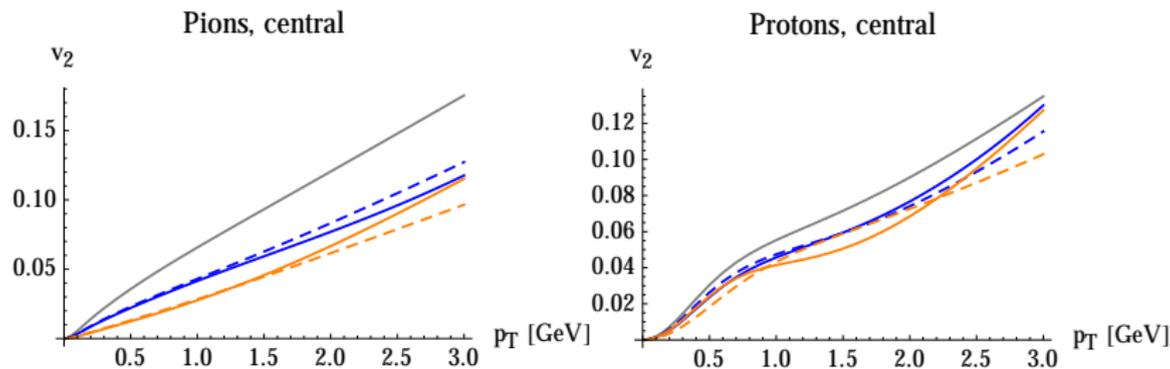
## *Background: one-particle spectrum*

Bjorken boost and azimuthal invariant background hydro fields lead to boost invariant and azimuthal symmetric one-particle spectrum.



## *Fluctuations: harmonic flow coefficients*

Fluctuation modes with  $m = 2$  lead to elliptic flow, modes with  $m = 3$  lead to triangular flow and so on.



These figures for central collisions are obtained for a Monte-Carlo Glauber model of initial conditions.

## *Velocity fluctuations on top of Bjorken expansion*

- So far we considered central collisions with radial expansion, realistic initial conditions, equation of state and transport properties
- This is ongoing work, more on this in September :)
- Consider now simpler case of infinitely extended transverse plane. No transverse expansion,

$$\bar{\epsilon} = \bar{\epsilon}(\tau), \quad \bar{u}^T = 1, \quad \bar{u}^r = 0.$$

and simple equation of state

$$\epsilon = 3p.$$

## Linearized equations for velocity fluctuations

- Consider first only terms linear in  $\delta\epsilon$ ,  $(u^1, u^2, u^y)$ .
- Decompose velocity field into
  - gradient term, described by divergence

$$\theta = \partial_1 u^1 + \partial_2 u^2 + \partial_y u^y$$

- rotation term, described by vorticity

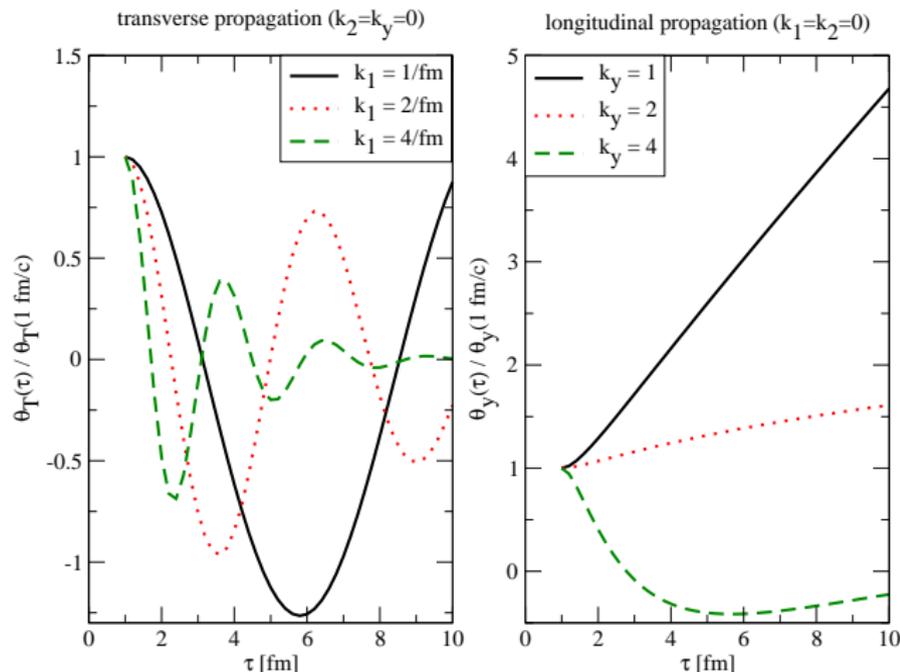
$$\omega_1 = \tau \partial_2 u^y - \frac{1}{\tau} \partial_y u^2$$

$$\omega_2 = \frac{1}{\tau} \partial_y u^1 - \tau \partial_1 u^y$$

$$\omega_3 = \partial_1 u^2 - \partial_2 u^1$$

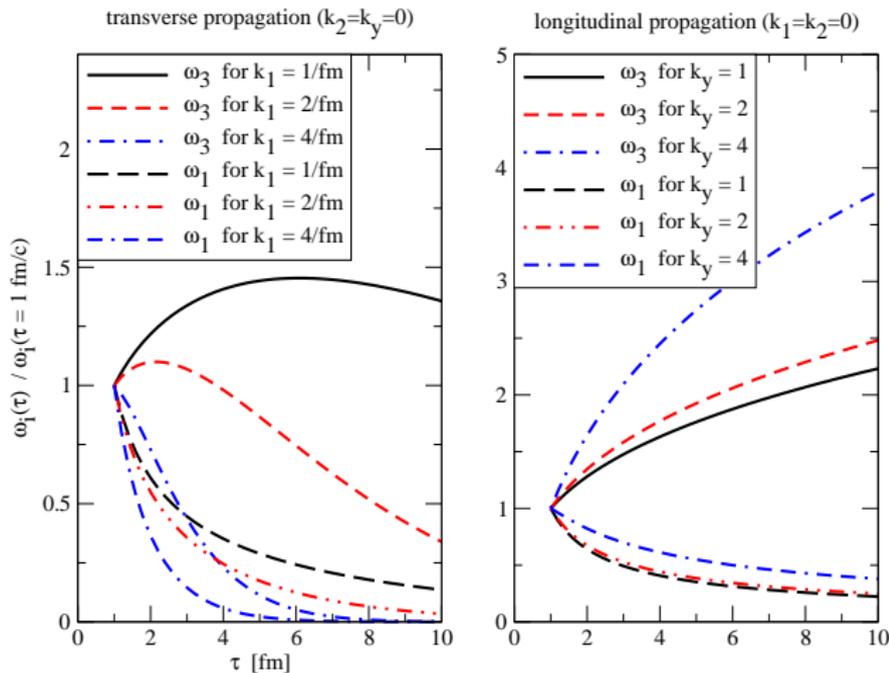
- $\theta$  and  $\delta\epsilon$  are coupled: density or sound waves.
- Vorticity modes decouple from  $\theta$  and  $\delta\epsilon$ .

# Sound modes



- oscillation for long times
- short wavelength modes get damped by viscosity
- also here some modes can grow!

# Vorticity modes



- solve equations in Fourier space  $\omega_j = \omega_j(\tau, k_1, k_2, k_y)$
- short wavelength modes get damped by viscosity
- some modes can grow, however!

## *Linear vs. non-linear evolution*

- for linearized theory one can easily determine two-point correlation function at late times from the one at early times
- more difficult for non-linear evolution of fluctuations
- evolution equation for  $n$ -point functions couple:  
“*Closure problem*” in fluid dynamics literature
- needs more elaborate tools:  
functional techniques, numerical simulation, ...
- ... but here we do something else:  
we map the problem to another one!

## *Limits of linearized theory*

linear approximation works for:

- energy density

$$\frac{\delta\epsilon}{\bar{\epsilon}} \ll 1$$

- velocity field

$$\text{Re} \ll 1$$

large Reynolds number  $\text{Re} \gg 1$  leads to turbulence!

typical numbers:  $T = 0.3 \text{ GeV}$ ,  $l = 5 \text{ fm}$ ,  $u_T = 0.1c$

$$\Rightarrow \text{Re} \approx \frac{1}{\eta/s} \approx \mathcal{O}(10)$$

## *Small Mach number*

$$\text{Ma} = \frac{\sqrt{u_1 u^1 + u_2 u^2 + u_y u^y}}{c_S} \ll 1$$

- turbulent motion can be described as “compression-less”

$$\theta = \partial_1 u^1 + \partial_2 u^2 + \partial_y u^y = 0$$

- sound modes decouple from vorticity, they are much faster
- this does not mean that there are no sound waves present

## Change of variables

- kinematic viscosity, essentially independent of time

$$\nu_0 = \frac{\eta}{s T_{Bj}(\tau_0)}$$

- new time variable (*not* laboratory time)

$$t = \frac{3}{4\tau_0^{1/3}} \tau^{4/3} \quad \partial_t = \left(\frac{\tau_0}{\tau}\right)^{1/3} \partial_\tau$$

- rescaled velocity field

$$v_j = \left(\frac{\tau_0}{\tau}\right)^{1/3} u_j$$

- temperature field

$$d = \left(\frac{\tau_0}{\tau}\right)^{2/3} \ln \left(\frac{T}{T_{Bj}(\tau)}\right)$$

## Compression-less flow

this leads to

$$\partial_t v_j + \sum_{m=1}^2 v_m \partial_m v_j + \frac{1}{\tau^2} v_y \partial_y v_j + \partial_j d - \nu_0 \left( \partial_1^2 + \partial_2^2 + \frac{1}{\tau^2} \partial_y^2 \right) v_j = 0.$$

- index  $j = 1, 2, y$
- solenoidal constraint

$$\partial_1 v_1 + \partial_2 v_2 + \frac{1}{\tau^2} \partial_y v_y = 0$$

- for large times  $\tau$  effectively *two-dimensional Navier-Stokes*!

# Reynolds numbers

assume

- typical velocity in transverse direction  $v_T$
- typical velocity in rapidity direction  $v_y$
- typical transverse length scale  $l$
- typical rapidity difference  $\Delta y$
- kinematic viscosity  $\nu_0 = \frac{\eta}{sT}$

define

$$\text{Re}^{(T)} = \frac{v_T l}{\nu_0}, \quad \text{Re}^{(y)} = \frac{v_y l^2}{\nu_0 \Delta y \tau^2} \quad \text{for} \quad \frac{l}{\tau \Delta y} \ll 1$$

and

$$\text{Re}^{(T)} = \frac{v_T \tau^2 \Delta y^2}{\nu_0 l}, \quad \text{Re}^{(y)} = \frac{v_y \Delta y}{\nu_0} \quad \text{for} \quad \frac{l}{\tau \Delta y} \gg 1$$

## Reynolds numbers 2

- depending on  $\text{Re}^{(T)}$  and  $\text{Re}^{(y)}$  there are different regimes
- for many initial conditions one has  $\text{Re}^{(T)} \gg 1$  at large  $\tau$
- this implies two-dimensional turbulent behavior!

## *Describing turbulence*

- turbulence is best described statistically
- hydrodynamic fluctuations in general have probability distribution

$$p_{\tau}[u^{\mu}(\tau, x_1, x_2, y), \epsilon(\tau, x_1, x_2, y)]$$

- assume this probability distribution to have the symmetries:
  - translational and rotational symmetries in transverse plane
  - Bjorken boost invariance
- implies for expectation values

$$\langle u^{\mu}(\tau, x_1, x_2, y) \rangle = (1, 0, 0, 0), \quad \langle \epsilon(\tau, x_1, x_2, y) \rangle = \bar{\epsilon}(\tau)$$

and for correlation functions

$$\langle u^i(\tau, x_1, x_2, y) u^j(\tau, x'_1, x'_2, y') \rangle = G_u^{ij}(\tau, |\mathbf{x} - \mathbf{x}'|, y - y')$$

## Velocity spectrum

in two-dimensional situation

$$\langle v_m(t, \mathbf{x}) v_n(t, \mathbf{x}') \rangle = \int \frac{d^2 k}{(2\pi)^2} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \left( \delta_{mn} - \frac{k_m k_n}{k_1^2 + k_2^2} \right) \frac{2\pi}{k} E(t, k)$$

the function  $E(t, k)$  describes how kinetic energy is distributed over different length scales

## Turbulence in $d = 3$

fully developed turbulence

$$\text{Re} \rightarrow \infty$$

dissipated energy per unit time

$$\frac{d}{dt} \langle \vec{v}^2 \rangle = -\nu_0 \langle (\vec{\nabla} \times \vec{v})^2 \rangle = -\varepsilon$$

RICHARDSON (1922):

*Big whorls have little whorls,  
Which feed on their velocity;  
And little whorls have lesser whorls,  
And so on to viscosity.*

KOLMOGOROV (1941):

$$E(k) \sim \varepsilon^{2/3} k^{-5/3}$$



L. DA VINCI (CA. 1500)

## Turbulence in $d = 2$



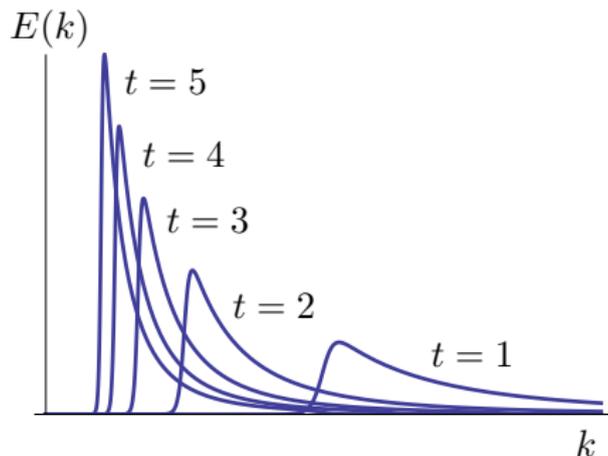
KRAICHNAN (1967):

- vorticity is conserved for  $\nu_0 \rightarrow 0$
- scaling theory of forced turbulence in  $d = 2$
- inverse cascade of energy to small wave numbers !
- cascade of vorticity to large wave numbers

$$E(k) \sim k^{-3}$$

- qualitatively different to  $d = 3$ , emerges here dynamically

## Decaying turbulence in $d = 2$



BATCHELOR (1969):

- scaling theory of decaying turbulence in  $d = 2$

$$E(t, k) = \lambda^3 t f(k \lambda t) \quad \text{with} \quad \lambda^2 = \langle \bar{v}^2 \rangle = \text{const.}$$

- turbulent motion goes to smaller and smaller wave numbers

## *Effects on macroscopic motion of fluid*

- turbulent fluctuations might affect macroscopic motion
  - modified equation of state
  - modified transport properties
- anomalous, turbulent or eddy viscosity
  - proposed by ASAKAWA, BASS, MÜLLER (2006) for plasma turbulence and ROMATSCHKE (2007) for fluid turbulence
  - could become negative in  $d = 2$  (KRAICHNAN (1976))
  - depends on detailed state of turbulence – not universal
  - gradient expansion needs separation of scales
- more work needed

# Summary

- Fluid fluctuations contain interesting information about:
  - Early time dynamics
  - Equation of state
  - Transport properties
- All kinds of fluid fluctuations should be propagated:
  - Density fluctuations
  - Velocity fluctuations
  - Baryon number fluctuations, Electric charge fluctuations,...
  - Deviations from smooth geometry in transverse plane
  - Deviations from Bjorken boost invariance
- Here we investigated mainly velocity fluctuations:
  - Vorticity modes can grow
  - New physics phenomenon: Onset of fluid turbulence
  - Interesting effects on one- and two-particle spectra
- Contribution of hydrodynamic fluctuations need not factorize in two-particle spectrum

# BACKUP

# Little Bang vs. Big Bang

## Heavy Ions

### Bjorken model

$$x^\mu = (\tau, x^1, x^2, y)$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \tau^2 \end{pmatrix}$$

$$\epsilon_0(\tau), \quad u_0^\mu = (1, 0, 0, 0)$$

+ hydrodyn. fluctuations

$$\epsilon = \epsilon_0(\tau) + \epsilon_1(\tau, x^1, x^2, y)$$

$$u^\mu = u_0^\mu + u_1^\mu(\tau, x^1, x^2, y)$$

## Cosmology

### Friedmann-Robertson-Walker

$$x^\mu = (t, x^1, x^2, x^3)$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a(t) & & \\ & & a(t) & \\ & & & a(t) \end{pmatrix}$$

$$\epsilon_0(t), \quad u_0^\mu = (1, 0, 0, 0)$$

+ hydrodyn. fluctuations

$$\epsilon = \epsilon_0(t) + \epsilon_1(t, x^1, x^2, x^3)$$

$$u^\mu = u_0^\mu + u_1^\mu(t, x^1, x^2, x^3)$$

+ gravity fluctuations