Lectures on quantum field theory 1

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Abstract: Notes for lectures that introduce students of physics to quantum field theory with applications to high energy physics, condensed matter and statistical physics. Prepared for a course at Heidelberg University in the winter term 2020/2021.

Contents

Literature

There is a large amount of literature on different aspects of quantum field theory. Here is only a fine selection.

Relativistic quantum field theory

- Mark Srednicki, *Quantum field theory* (2007)
- Michael Peskin & Daniel Schroeder, *An introduction to quantum field theory* (1995)
- Steven Weinberg, *The quantum theory of fields I & II* (1998)

Statistical field theory / renormalization group

- John Cardy, *Scaling and renormalization in statistical physics* (1996)
- Giorgio Parisi, *Statistical field theory* (1998)
- Jean Zinn-Justin, *Quantum field theory and critical phenomena* (2002)
- Andreas Wipf, *Statistical approach to quantum field theory* (2013)
- Crispin Gardiner, *Handbook of stochastic methods* (1985)

Non-relativistic quantum field theory / condensed matter

- Alexander Altland & Ben Simons, *Condensed matter field theory* (2010)
- Lev Pitaevskii & Sandro Stringari, *Bose-Einstein condensation* (2003)
- Crispin Gardiner & Peter Zoller, *The quantum world of ultra-cold atoms and light* (2014)

Group theory

- Anthony Zee, *Group theory in a nutshell for physicists* (2016)

Typos

Please send typos to floerchinger@thphys.uni-heidelberg.de.

Welcome

Video: **Lecture00Video01.mp4**

Video: **Lecture00Video02.mp4**

1 What is quantum field theory?

Video: **Lecture01Video01.mp4**

Historically, quantum field theory (QFT) has been developed as quantum mechanics for many (in fact infinitely many) degrees of freedom. For example, the quantum mechanical description for electromagnetic fields (light) and its excitations, the photons, leads to a quantum field theory. Quantum mechanics of photons, electrons and positrons is quantum electrodynamics (QED) and so one can go on.

In contrast to the transition from classical mechanics to quantum mechanics, the step from there to quantum field theory does not lead to a conceptually entirely new theoretical framework. Still, it was historically not an easy development and a lot of confusion was connected with notions like "second quantization" etc.

There are many new phenomena arising in a field theory setting. This includes collective effects of many degrees of freedom, e. g. spontaneous symmetry breaking. Particle number is not necessarily conserved and one can have particle creation and annihilation.

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Historically, quantum field theory has been developed as a *relativistic* theory, which combines quantum mechanics with Lorentz symmetry. This was necessary for quantum electrodynamics. Until today, Lorentz symmetry remains to be a key incredient for the quantum field theoretic description of elementary particle physics. It is not central for quantum field theory itself, however. Concepts of quantum field theory can also be used to describe the quantum theory of many atoms, for example ultra-cold quantum gases, or phonons in solids, or the spins composing magnets. These systems are treated by non-relativistic QFT.

Video: **Lecture01Video01c.mp4**

Probabilistic fields. One may characterize much of the content of the following lectures by two main ingredients

(i) *Fields* (degrees of freedom at every point x)

(ii) *Probabilistic theory* (as every quantum theory is one)

In this sense, one may speak of quantum field theory as a probabilistic theory of fields. The reader may note that "quantum" is missing in the above characterization. Indeed, in modern developments, all probabilistic field theories, be they "quantum" or "classical", are described with the same concepts and methods based on the *functional integral*. The key element here is the one of fluctuating fields as one has it in many situations. Something as tangible as the surface of an ocean is already an example. The concepts are useful in many areas, ranging from statistical mechanics to particle physics, quantum gravity, cosmology, biology, economics and so on. The common view on all these subjects, based on the functional integral, will be the guideline of these lectures.

PFT, *probabilistic field theory*, would be a more appropriate name. We will nevertheless use the traditional, historic name, QFT. Neither "quantum" nor "relativistic" are crucial conceptually. Relativistic quantum field theory is from this perspective an important "special case", to which we will pay much attention.

2 Functional integral

We start with a simple model, the one dimensional Ising model.

2.1 Ising model in one dimension

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Ising spin. An *Ising spin* has two possible values,

$$s = \pm 1.$$

One can also start somewhat more general with some two-level variable with possible values A_1 and A_2 and relate them to the Ising spins via a map,

$$A_1 \to s = +1, \qquad A_2 \to s = -1.$$

For example, a state could be occupied, $n = 1$, or empty, $n = 0$. These states can be mapped to Ising spins via $s = 2n - 1$. From an information theoretic point of view, each Ising spin carries one *bit of information.*

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Ising chain. Let us consider a chain of discrete points x and take them to be equidistant,

$$x \in \{x_{\rm in}, x_{\rm in} + \varepsilon, x_{\rm in} + 2\varepsilon, \ldots, x_{\rm f} - \varepsilon, x_{\rm f}\}.$$

The Ising chain contains a spin $s(x)$ at each point (or lattice site) x.

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Configuration. Now let us pose one Ising spin at each point or lattice site x. A set of of such spin values at all the possible points x will be called a *configuration* and denoted by $\{s(x)\}$. (This should be seen as an abbreviation for $\{s(x_{\rm in}), s(x_{\rm in} + \varepsilon), \ldots, s(x_{\rm f})\}$.) For example, the spin value $s(x)$, corresponding occupation number $n(x)$ and spin direction for a particular configuration of seven spins could be as follows.

1	1	-1	-1	-1	1	-1	spin value $s(x)$
1	1	0	0	0	1	0	occupation number $n(x)$
↑	↑	↓	↓	↓	↑	↓	spin direction

In general, for P points, or lattice sites, there are $N = 2^P$ possible configurations since each spin can be either up or down. We can label them by an index $\tau = 1, \ldots, N$.

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Euclidean action. We now introduce the concept of an *euclidean action* by assigning to each configuration a real number $S \in \mathbb{R}$,

$$\{s(x)\} \to S[s] = S(\{s(x)\}).$$

For example, one may have a next neighbor interaction and the action corresponds to

$$S[s] = -\sum_x \beta s(x+\varepsilon)s(x), \tag{2.1}$$

where we use the following abbreviation for a sum over lattice sites

$$\sum_x = \sum_{x=x_{\rm in}}^{x_{\rm f}-\varepsilon},$$

and β is a real parameter.

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Partition function. One can define a partition function as a sum over all configurations, weighted by the exponential of minus the action,

$$Z = \sum_{\{s(x)\}} e^{-S[s]} = \sum_\tau e^{-S_\tau}.$$

Note that the partition function is here a real and positive number, $Z > 0$.

Probability distribution. Let us now assign to each configuration a probability, $\{s(x)\} \to p[s] = p(\{s(x)\})$, or in another notation, $\tau \to p_\tau$. We will set

$$p[s] = \frac{1}{Z} e^{-S[s]}.$$

Note the following properties

(i) positivity $p[s] \geq 0$ (and $p[s] \to 0$ for $S[s] \to \infty$),

(ii) normalization $\sum_{\{s(x)\}} p[s] = \sum_\tau p_\tau = 1$.

These are the defining properties of probability distributions.

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Observables. We may construct an observable by assigning to every configuration $\{s(x)\}$ (also labeled by τ) a value $A[s] = A_\tau$,

$$\{s(x)\} \to A[s], \qquad \tau \to A_\tau.$$

In other words, the observable A has the value A_τ in the configuration τ.

Expectation value. The expectation value of an observable is defined by

$$\langle A \rangle = \sum_\tau p_\tau A_\tau = \frac{1}{Z} \sum_{\{s(x)\}} e^{-S[s]} A[s].$$

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Two-point correlation. A correlation function of two observables is given by the expression

$$\langle AB\rangle = \sum_\tau p_\tau A_\tau B_\tau = \frac{1}{Z}\sum_{\{s(x)\}} e^{-S[s]} A[s]B[s].$$

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Local action. Oftentimes one can write the action as a sum of the form

$$S[s] = \sum_x \mathscr{L}(x),$$

with $\mathscr{L}(x)$ depending only on the spins in some neighborhood of x. For our example (2.1) with next neighbor interaction one would have

$$\mathscr{L}(x) = -\beta s(x+\varepsilon)s(x).$$

In fact, the simplest version of the traditional *Ising model* has $\beta = \frac{J}{k_\mathrm{B}T}$ with interaction parameter J, temperature T and Boltzmann constant k_B. In this context, the Euclidean action corresponds in fact to the ratio $S = \frac{H}{k_\mathrm{B}T}$ of Energy or Hamiltonian H and temperature as it appears in the Boltzmann weight factor $\exp(-\frac{H}{k_\mathrm{B}T})$. The Hamiltonian is then obviously

$$H = -\sum_x Js(x+\varepsilon)s(x).$$

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Boundary terms. One must pay some attention to the boundaries of the Ising chain. Let us denote by $\mathscr{L}_\mathrm{in}$ a term that depends only on $s(x_\mathrm{in})$, the *initial spin* and similarly by $\mathscr{L}_\mathrm{f}$ a term that depends only on $s(x_\mathrm{f})$, the *final spin*. We write the action as

$$S = \sum_t \mathscr{L}(t) + \mathscr{L}_\mathrm{in} + \mathscr{L}_\mathrm{f}.$$

By choosing $\mathscr{L}_\mathrm{in}$ and $\mathscr{L}_\mathrm{f}$ appropriately one can pose different boundary conditions, in general probabilistic, or also deterministic as an approriate limit.

Typical problem. A typical problem one may encounter in the context of the Ising model in one dimension is: What is the expectation value $\langle s(x)\rangle$ or the two-point correlation function $\langle s(x_1)s(x_2)\rangle$ for given boundary conditions specified by $\mathscr{L}_\mathrm{in}$ and $\mathscr{L}_\mathrm{f}$?

Functional integral language. We now formulate the model in a language that is convenient for generalization. We write for expectation values

$$\langle A\rangle = \frac{1}{Z}\int Ds\, e^{-S[s]} A,$$

with the partition function

$$Z = \int Ds\, e^{-S[s]}.$$

The *functional measure* is here defined by

$$\int Ds = \sum_{\{s(x)\}} = \sum_\tau = \prod_x \sum_{s(x)=\pm 1}.$$

For a finite Ising chain, the functional integral is simply a finite sum over configurations.

2.2 Continuum functional integral

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Lattice functional integral. Let us now take a real, continuous variable $\phi(x) \in \mathbb{R}$ instead of the discrete Ising spins $s(x) \in \{+1, -1\}$. The position variable x is for the time being still labeling discrete points or lattice sites. We then define the *functional measure*

$$\int D\phi = \prod_x \int_{-\infty}^{\infty} d\phi(x).$$

This is now the continuum version of a sum over configurations. Indeed it sums over all possible functions $\phi(x)$ of the (discrete) position x. To realize that indeed every function appears in $\int D\phi$ one may go back to a discrete variable, $\phi(x) \in \{\phi_1, \ldots, \phi_M\}$ with M possible values and take $M \to \infty$.

Configuration. For every lattice site x we specify now a real number $\phi(x)$ which in total gives then one configuration. Obviously there are now infinitely many configurations even if the number of lattice sites is finite.

Path integral. At this point one can make the transition to a probabilistic *path integral*. To this end one would replace $x \to t$ and $\phi(x) \to \vec{x}(t)$, such that the sum over functions $\phi(x)$ becomes one over paths $\vec{x}(t)$. The functional measure would be $\int D\vec{x}$.

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Action. The Euclidean action can be written as

$$S = \sum_x \mathscr{L}(x) + \mathscr{L}_{\rm in} + \mathscr{L}_{\rm f},$$

where $\mathscr{L}(x)$ depends on $\phi(x')$ with x' in the vicinity of x. Similarly, $\mathscr{L}_{\rm in}$ depends on $\phi(x_{\rm in}) = \phi_{\rm in}$ and $\mathscr{L}_{\rm f}$ depends on $\phi(x_{\rm f}) = \phi_{\rm f}$.

Lattice ϕ^4 theory. Here we take the action local with

$$\mathscr{L}(x) = \frac{K}{8\varepsilon}\left[\phi(x+\varepsilon) - \phi(x-\varepsilon)\right]^2 + \varepsilon V(\phi(x)),$$

where the *potential* is given by

$$V(\phi(x)) = \frac{m^2}{2}\phi(x)^2 + \frac{\lambda}{8}\phi(x)^4.$$

The partition function is

$$Z = \int D\phi\, e^{-S[\phi]},$$

and a field expectation value is given by

$$\langle\phi(x)\rangle = \frac{1}{Z}\int D\phi\, e^{-S[\phi]}\phi(x).$$

The *functional integral* is here still a finite-dimensional integral where the dimension corresponds to the number of lattice points P. The action $S[\phi]$ is a function of P continuous variables $\phi(x)$.

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Continuum limit. Let us now take the limit $\varepsilon \to 0$ for $x_\mathrm{f} - x_\mathrm{in}$ fixed. This means that the number of lattice points P needs to diverge. The "lattice derivative"

$$\partial_x \phi(x) = \frac{1}{2\varepsilon} \left[\phi(x+\varepsilon) - \phi(x-\varepsilon)\right]$$

becomes a standard derivative, at least for sufficiently smooth configurations, where it exists. One also has

$$\sum_x \varepsilon \to \int dx,$$

and the Euclidean action becomes

$$S = \int dx \left\{\mathscr{L}(x) + \mathscr{L}_\mathrm{in} + \mathscr{L}_\mathrm{f}\right\},$$

where now

$$\mathscr{L}(x) = \frac{K}{2} \left[\partial_x \phi(x)\right]^2 + V(\phi(x)).$$

The first term is called the kinetic term, the second the potential. In the limit $\varepsilon \to 0$ the action is a *functional* of the functions $\phi(x)$.

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Physical observables. As *physical observables* one takes those $A[\phi]$ for which the limit $\langle A\rangle$, $\langle AB\rangle$ and so on exists in the limit $\varepsilon \to 0$. It will not always be easy to decide whether a given $A[\phi]$ is a physical observable, but the definition is simple. For $\varepsilon \to 0$ the expression $A[\phi]$ is again a functional.

Functional integral. The functional integral in the continuum theory is now defined as the "continuum limit" of the lattice functional integral for $\varepsilon \to 0$. By definition, this is well defined for "physical observables". One may ask: what are such physical observables? The answer to this question is not simple, in general. One should note here that also very rough functions $\phi(x)$ are included in the functional integral, although their contribution is suppressed. If the kinetic term in the Euclidean action $S_\mathrm{kin} = \sum_x \frac{K}{8\varepsilon} \left[\phi(x+\varepsilon) - \phi(x-\varepsilon)\right]^2$ diverges for $\varepsilon \to 0$, i. e. $S \to \infty$, then one has $e^{-S} \to 0$ and the probability of such configuration vanishes. The corresponding limits may not be trivial, however, because very many rough configurations exist.

Additive rescaling of action. Let us consider a change $S \to S' = S + C$ or $\mathscr{L}(x) \to \mathscr{L}'(x) = \mathscr{L}(x) + \tilde{c}$ where $C = (x_\mathrm{f} - x_\mathrm{in})\tilde{c}$ is a constant that is *independent of the fields*. The partition function changes then like $Z \to Z' = e^{-C} Z$. Similarly,

$$\int D\phi\, e^{-S} A[\phi] \to e^{-C} \int D\phi\, e^{-S} A[\phi].$$

This means that C drops out when one considers expectation values like $\langle A\rangle$! It can even happen that C diverges for $\varepsilon \to 0$ such that formally $Z \to 0$ or $Z \to \infty$. This is not a problem because the absolute value of Z is irrelevant. The probability distribution $p[\phi] = \frac{1}{Z} e^{-S[\phi]}$ is unchanged.

2.3 $O(N)$ models in classical statistical equilibrium

Video: **Lecture02Video01.mp4**

Classical thermal fluctuations. For the time being we are concerned with static (equilibrium) aspects of field theory models at non-zero temperature. These field theories can arise for example from a lattice model such as the Ising model if the latter is probed on distances that are large against the typical microscopic scale or inter-particle distance ε. Formally one can then take the limit $\varepsilon \to 0$ as discussed in the previous subsection. It turns out (and will become more clear later on), that in such a situation *classical* thermal fluctuations dominate over quantum fluctuations. We discuss here therefore *classical statistical field theories* in thermal equilibrium.

Such theories have a probabilistic description in terms of functional integrals with weight given by the Boltzmann factor $e^{-\beta H}$. Here $\beta = 1/T$ and we use now units where $k_B = 1$ such that temperature is measured in units of energy. In the following we will discuss possible forms of the field theory and in particular the Hamiltonian H.

Universality classes and models. In condensed matter physics, microscopic Hamiltonians are often not very well known and if they are, they are not easy to solve. However, in particular in the vicinity of second order phase transitions, there are some universal phenomena that are independent of the precise microscopic physics. This will be discussed in more detail later on, in the context of the *renormalization group*. Essentially, this arises as a consequence of thermal fluctuations and the fact that at a second order phase transition fluctuations are important on all scales. Roughly speaking, a theory changes in form when fluctuations are taken into account and can approach a largely universal *scaling form* for many different microscopic theories that happen to be in the same *universality class*.

In the following we will discuss a class of *model systems*. These are particularly simple field theories for which one can sometimes answer certain questions analytically, but one can also see them as representatives for their respective universality classes. In the context of quantum field theory, we will see that these field theory models gain a substantially deeper significance.

Scalar O(N) models in d dimensions. Let us consider models of the form

$$\beta H[\phi] = S[\phi] = \int d^d x \left\{ \frac{1}{2} \partial_j \phi_n \partial_j \phi_n + \frac{1}{2} m^2 \phi_n \phi_n + \frac{1}{8} \lambda \left(\phi_n \phi_n \right)^2 \right\}. \tag{2.2}$$

Here, $\phi_n = \phi_n(x)$ with $n = 1, \ldots, N$ are the fields. We use Einsteins summation convention which implies that indices that appear twice are summed over. We have formulated the theory in d spatial dimensions (where in practice $d = 3$, 2, 1 or even 0 for condensed matter systems and $d = 4$ will correspond to a quantum field theory after Wick rotation to Euclidean space). The index j is accordingly summed in the range $j = 1, \ldots, d$. Although not very precise, one sometimes calls $S[\phi]$ the *Euclidean microscopic action*. The square brackets indicate here that the action depends on the fields in a functional way, which means it depends not on single numbers but on the entire set of *functions* of space $\phi_n(x)$, with $x \in \mathbb{R}^d$ and $n = 1, \ldots, N$.

Fields as vectors. One can consider $\phi_n(x)$ as a vector in a vector space of infinite dimension where components are labeled by the spatial position x and the discrete index n. If in doubt, one can go back to a lattice model where x is discrete.

Applications. Models of the type (2.2) have many applications. For $N = 1$ they correspond in a certain sense to the continuum limit of the Ising model. For $N = 2$ the model can equivalently be described by complex scalar fields. It has then applications to Bose-Einstein condensates, for example. For $N = 3$ and $d = 3$ one can have situations where the rotation group and the internal symmetry group are coupled. This describes then vector fields, for example magnetization. Finally, for $N = 4$ and $d = 4$, the model essentially describes the Higgs field after a Wick rotation to Euclidean space.

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Engineering dimensions. In equation (2.2) we have rescaled the fields such that the coefficient of the derivative term is $1/2$. This is always possible. It is useful to investigate the so-called *engineering scaling dimension* of the different terms appearing in (2.2). The combination βH or the action S must be dimensionless. Derivatives have dimension of inverse length $[\partial] = L^{-1}$ and the fields must accordingly have dimension $[\phi] = L^{-\frac{d}{2}+1}$. One also finds $[m] = L^{-1}$ and $[\lambda] = L^{d-4}$. Note in particular that λ is dimensionless in $d = 4$ dimensions.

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Symmetries. It is useful to discuss the symmetries of the model (2.2). Symmetries are (almost) always very helpful in theoretical physics. In the context of statistical field theory, they are useful as a guiding principle in particular because they still survive (in a sense to be defined) when the effect of fluctuations is taken into account.

For the model (2.2) we have a space symmetry group consisting of rotations and translations, as well as a continuous, so-called internal symmetry group of global $O(N)$ transformations. We now discuss them step-by-step.

Rotations. Rotations in space are transformations of the form

$$x^j \to x'^j = R^{jk} x^k.$$

The matrices R fulfill the condition $R^T R = \mathbb{1}$ and we demand that they connect continuously to the unit matrix $R = \mathbb{1}$. This fixes $\det(R) = 1$. Matrices of this type in d spatial dimensions form a group, the *special orthogonal group* $SO(d)$. Mathematically, this is a *Lie group* which implies that all group elements can be composed of many infinitesimal transformations. An infinitesimal transformation can be written as

$$R^{jk} = \delta^{jk} + \frac{i}{2}\delta\omega_{mn}\, J^{jk}_{(mn)},$$

where $J^{jk}_{(mn)} = -i(\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})$ are the *generators of the Lie algebra* and $\delta\omega_{mn}$ are infinitesimal, anti-symmetric matrices. One may easily count that there are $d(d-1)/2$ independent components of an anti-symmetric matrix in d dimensions and as many generators. Finite group elements can be obtained as

$$R = \lim_{N\to\infty} \left(\mathbb{1} + \frac{i}{2}\frac{\omega_{mn}}{N} J_{(mn)}\right)^N = \exp\left(\frac{i}{2}\omega_{mn} J_{(mn)}\right).$$

Let us now work out how fields transform under rotations. We will implement them such that a field configuration with a maximum at some position x before the transformation will have a maximum at Rx afterwards. The field must transform as

$$\phi_n(x) \to \phi'_n(x) = \phi_n(R^{-1}x).$$

Note that derivatives transform as

$$\partial_j \phi_n(x) \to (R^{-1})_{kj}(\partial_k \phi_n)(R^{-1}x) = R_{jk}(\partial_k \phi_n)(R^{-1}x).$$

The brackets should denote that the derivatives are with respect to the full argument of ϕ_n and we have used the chain rule. The action in (2.2) is invariant under rotations acting on the fields, as one can confirm easily. Colloquially speaking, no direction in space is singled out.

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Translations. Another useful symmetry transformations are translations $x \to x + a$. The fields get transformed as

$$\phi_n(x) \to \phi'_n(x) = \phi_n(x - a).$$

One easily confirms that the action (2.2) is also invariant under translations. Colloquially speaking, this implies that no point in space is singled out.

Global internal $O(N)$ transformations. There is another useful symmetry of the action (2.2) given by rotations (and mirror reflections) in the "internal" space of fields,

$$\phi_n(x) \to O_{nm}\phi_m(x).$$

The matrices O_{nm} are here independent of the spatial position x (therefore this is a *global* and not a *local* transformation) and they satisfy $O^T O = \mathbb{1}$. Because we do not demand them to be smoothly connected to the unit matrix, they can have determinant $\det(O) = \pm 1$. These matrices are part of the *orthogonal group* $O(N)$ in N dimensions. It is an easy exercise to show that the action (2.2) is indeed invariant under these transformations.

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Partition function and functional derivatives. The partition function for the model (2.2) reads

$$Z[J] = \int D\phi \; e^{-S[\phi] + \int d^d x \{J_n(x)\phi_n(x)\}} \tag{2.3}$$

We have introduced here an external source term $\int d^d x \{J_n(x)\phi_n(x)\}$ which can be used to probe the theory in various ways. For example, one can take *functional derivatives* to calculate expectation values,

$$\langle \phi_n(x) \rangle = \frac{1}{Z[J]} \frac{\delta}{\delta J_n(x)} Z[J]\Big|_{J=0},$$

and correlation functions, e. g.

$$\langle \phi_n(x)\phi_m(y) \rangle = \frac{1}{Z[J]} \frac{\delta^2}{\delta J_n(x) \delta J_m(y)} Z[J]\Big|_{J=0} = \frac{\int D\phi \; \phi_n(x) \, \phi_m(y) \; e^{-S[\phi]}}{\int D\phi \; e^{-S[\phi]}}.$$

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Classical field equation. In the the functional integral the contribution of field configurations $\phi(x)$ is suppressed if the corresponding action $S[\phi]$ is large. In the partition function (2.3), large contributions come mainly from the region around the minima of $S[\phi] - \int_x J_n \phi_n$, which are determined by the equation

$$\frac{\delta}{\delta \phi(x)} \left(S[\phi] - \int d^d x \{J_n(x)\phi_n(x)\} \right) = \frac{\delta S[\phi]}{\delta \phi_n(x)} - J_n(x) = 0.$$

This equation is the field equation or equation of motion of a classical field theory. For the model (2.2) one has concretely

$$\frac{\delta S[\phi]}{\delta \phi_n(x)} = -\partial_j \partial_j \phi_n(x) + m^2 \phi_n(x) + \frac{1}{2} \lambda \phi_n(x) \phi_k(x) \phi_k(x) = J_n(x).$$

Note that this field equation is from a mathematical point of view a second order, semi-linear, partial differential equation. It contains non-linear terms in the fields ϕ_n, but the term involving derivatives is linear; therefore semi-linear. The equation involves the Euclidean Laplace operator $\Delta = \partial_j \partial_j$ and is therefore of elliptic type (as opposed to hyperbolic or parabolic). This field equation is the correspondence of Maxwells equations in electrodynamics for our scalar theory. The source J corresponds to the electromagnetic current in Maxwell's equations.

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The $O(N)$ symmetric potential. The model in (2.2) can be generalized somewhat to the action

$$S[\phi] = \int d^d x \left\{ \frac{1}{2} \partial_j \phi_n \partial_j \phi_n + V(\rho) \right\}, \tag{2.4}$$

where $\rho = \frac{1}{2} \phi_n \phi_n$ is an $O(N)$ symmetric combination of fields and $V(\rho)$ is the microscopic $O(N)$ symmetric potential. The previous case (2.2) can be recovered for $V(\rho) = m^2 \rho + \frac{1}{2} \lambda \rho^2$.

More general, $V(\rho)$ might be some function with a minimum at ρ_0 and a Taylor expansion around it,

$$V(\rho) = m^2 (\rho - \rho_0) + \frac{1}{2} \lambda (\rho - \rho_0)^2 + \frac{1}{3!} \gamma (\rho - \rho_0)^3 + \dots$$

If the minimum is positive, $\rho_0 > 0$, the linear term vanishes of course, and one takes $m^2 = 0$. In contrast, if the minimum is at $\rho_0 = 0$ one has in general $m^2 > 0$. In practice, one uses either ρ_0 or m^2 for a parametrization of $V(\rho)$. It costs a certain amount of energy for the field to move away from the minimum. In particular, for large λ such configurations are suppressed.

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Homogeneous solutions. It is instructive to discuss homogeneous solutions of the field equation, i.e. solutions that are independent of the space variable x. For vanishing source $J_n(x) = 0$, and the model (2.4) we need to solve

$$\frac{\partial}{\partial \phi_n} V(\rho) = \phi_n \frac{\partial}{\partial \rho} V(\rho) = 0.$$

This has always a solution $\phi_n = 0$ and for $\rho_0 = 0$ and positive m^2 this is indeed a minimum of the action $S[\phi]$. For positive ρ_0 the situation is more interesting, however. In that case, $\phi_n = 0$ is actually typically a maximum while the minimum is at $\phi_k \phi_k = 2\rho_0$, i. e. at a non-zero field value. One possibility is $\phi_1 = \sqrt{2\rho_0}$ with $\phi_2 = \ldots = \phi_n = 0$, but there are of course many more. But such a solution breaks the $O(N)$ symmetry! One says that the $O(N)$ symmetry is here *spontaneously broken on the microscopic level* which technically means that the action $S[\phi]$ is invariant, but the solution to the field equation (i. e. the minimum of $S[\phi]$) breaks the symmetry. It is an interesting and non-trivial question whether the symmetry breaking survives the effect of fluctuations. One has proper *macroscopic* spontaneous symmetry breaking if the field expectation value $\langle \phi_n \rangle$ is non-vanishing and singles out a direction in field space. An example for spontaneous symmetry breaking is the magnetization field in a ferromagnet.

2.4 Non-linear sigma models

Video: **Lecture02Video09.mp4**

Constrained fields. It is also interesting to consider models where $\rho = \rho_0$ is fixed. In fact, they arise naturally in the low energy limit of the models described above when the fields do not have enough energy to climb up the effective potential. Technically, this corresponds here to the limit $\lambda \to \infty$ with fixed ρ_0 and can be implemented as a constraint

$$\phi_n(x)\phi_n(x) = 2\rho_0. \tag{2.5}$$

Note that with this constraint, the field is now living on a manifold corresponding to the surface of an N-dimensional sphere, denoted by S_{N-1}. One can parametrize the field as

$$\phi_1 = \sigma, \quad \phi_2 = \pi_1, \quad \dots \quad \phi_N = \pi_{N-1},$$

where only the fields π_n are independent while σ is related to them via the non-linear constraint

$$\sigma = \sqrt{2\rho_0 - \vec{\pi}^2}.$$

Linear and non-linear symmetries. The symmetry group $O(N)$ falls now into two parts. The first consists of transformations $O(N-1)$ which only act on the fields π_n but do not change the field σ. Such transformations are realized in the standard, linear way

$$\pi_n \to O^{(N-1)}_{nm}\pi_m, \qquad \sigma \to \sigma.$$

In addition to this, there are transformations in the complement part of the group (rotations that also involve the first component σ). They act infinitesimally on the independent fields like

$$\delta\pi_n = \delta\alpha_n\sigma = \delta\alpha_n\sqrt{2\rho_0 - \vec{\pi}^2}, \qquad \delta\sigma = -\delta\alpha_n\pi_n,$$

where $\delta\alpha_n$ are infinitesimal parameters (independent of the fields). Note that this is now a non-linearly realized symmetry in the internal space of fields. This explains also the name *non-linear sigma model*.

Action. Let us now write an action for the non-linear sigma model. Because of the constraint (2.5), the effective potential term in (2.4) becomes irrelevant and only the kinetic term remains,

$$S[\pi] = \int d^dx \left\{ \frac{1}{2}\partial_j\phi_n\partial_j\phi_n \right\} = \int d^dx \left\{ \frac{1}{2}G_{mn}(\vec{\pi})\partial_j\pi_m\partial_j\pi_n \right\}.$$

In the last equation we rewrote the action in terms of the independent fields π_n and introduced the *metric in the field manifold*

$$G_{mn}(\vec{\pi}) = \delta_{mn} + \frac{\pi_m\pi_n}{2\rho_0 - \vec{\pi}^2}.$$

The second term originates from

$$\partial_j\sigma = \partial_j\sqrt{2\rho_0 - \vec{\pi}^2} = \frac{1}{\sqrt{2\rho_0 - \vec{\pi}^2}}\pi_m\partial_j\pi_m.$$

Functional integral. Note that also the functional integral is now more complicated. It must involve the determinant of the metric G_{mn} to be $O(N)$ invariant. For a single space point x one has

$$\int \prod_n d\phi_n \to \int \prod_n d\phi_n\, \delta(\phi_n\phi_n - 2\rho_0) = \text{const} \times \int \sqrt{\det(G(\vec{\pi}))} \prod_n d\pi_n.$$

Only in the presence of the determinant $\det(G(\vec{\pi}))$ the functional measure preserves the $O(N)$ symmetry. Accordingly, the functional integral for the non-linear sigma model must be adapted to contain the factor $\det(G(\vec{\pi}))$.

Ising model. Everything becomes rather simple again for $N = 1$. The constraint $\phi(x)^2 = 2\rho_0$ allows only the field values $\phi(x) = \pm\sqrt{2\rho_0}$. By a multiplicative rescaling of $\phi(x)$ one can obtain $2\rho_0 = 1$. On a discrete set of space points (a lattice), this leads us back to the Ising model.

2.5 Classical statistical thermodynamics

Video: **Lecture02Video10.mp4**

Hamiltonian and partition function. We have now all the ingredients for a microscopic formulation of thermodynamics. The well-known macroscopic thermodynamic laws all follow from this microscopic formulation. Furthermore, the behaviour of particular systems is encoded in the partition function which yields the "equation of state" of a given system.

The starting point is the classical Hamiltonian H for a given model. It is a functional of the microscopic variables ϕ, $H[\phi]$. For our example of $O(N)$-models, these variables are the fields $\phi_n(x)$, or for the non-linear σ-models (including the Ising model), the constrained fields $\pi_n(x)$. The Hamiltonian associates to each field configuration an energy $H[\phi]$. The classical action reads

$$S[\phi] = \beta H[\phi] = \frac{H[\phi]}{T},$$

with $\beta = 1/T$ the inverse temperature. The functional integral (2.3) yields for vanishing source $J = 0$ the partition function $Z(\beta)$. The mean energy is found as

$$E = \langle H \rangle = -\frac{\partial \ln Z(\beta)}{\partial \beta},$$

relating E to the temperature T. In this simplest version $Z(\beta)$ is the partition function of the canonical ensemble, and the entropy $\tilde{S}$ is defined as

$$\tilde{S} = \left(1 - \beta\frac{\partial}{\partial\beta}\right) \ln Z(\beta).$$

Particle number. In the case of systems with a preserved particle number N we can also include in the action a term $-\beta\mu N[\phi]$, with $N[\phi]$ the particle number and μ the chemical potential,

$$S = \beta H[\phi] - \beta\mu N[\phi].$$

In this case the partition function $Z(\beta, \mu)$ is the grand canonical partition function, with mean particle number N, mean energy E and entropy $\tilde{S}$ given by

$$N = \frac{1}{\beta}\frac{\partial}{\partial\mu} \ln Z(\beta, \mu), \qquad E = -\frac{\partial}{\partial\beta} \ln Z(\beta, \mu) + \mu N,$$
$$\tilde{S} = \left(1 - \beta\frac{\partial}{\partial\beta}\right) \ln Z(\beta, \mu).$$

All thermodynamic relations follow from this setting, and the particular form of the grand canonical potential or Gibbs free energy $\Omega = -\ln Z(\beta, \mu)/\beta$ yields the equation of state of the system.

Video: **Lecture02Video11.mp4**

Magnetization. Source terms such as a homogeneous magnetic field for the case where $\phi_n(x)$ describes magnetization, can be added. If we take $\vec{\phi}(x)$ to be a microscopic magnetization density and $\vec{B}$ a constant magnetic field, the action becomes

$$S = \beta H[\phi] - \beta\kappa \int_x \phi_n(x) B_n.$$

The macroscopic magnetization $\vec{M}$ as a function of $\vec{B}$ and temperature T obtains from $Z(\beta, \vec{B})$ as

$$M_n = \frac{T}{\kappa} \frac{\partial \ln Z}{\partial B_n}.$$

Video: **Lecture02Video12.mp4**

Pressure. If one wants to investigate questions related to volume and pressure, one has to confine the system in a box with volume V with suitable, for example periodic, boundary conditions for $\phi(x)$. The partition function depends in this case on V as an additional parameter, and the pressure p obeys

$$p = \frac{1}{\beta} \frac{\partial}{\partial V} \ln Z(\beta, \mu, V).$$

The functional integral is an object that you should know well from your course on statistical physics.

3 Operators and transfer matrix

Video: **Lecture03Video01.mp4**

Our approach to quantum field theory will be based on the discussion of functional integrals. These are a generalization of ordinary, multi-dimensional integrals to the limit of infinitely many degrees of freedom, i. e. infinite dimensional integrals. For bosons, the variables or fields all commute. (For fermions we will later use the anti-commuting Grassmann variables). One has learned that non-commuting operators play a crucial role in quantum mechanics. These non-commuting structures are not immediately visible in the bosonic functional integral which on first sight only contains commuting quantities. One may wonder how such integrals can describe the non-commutative properties of quantum mechanics. In the following we want to reveal the structural relation between the operator formalism, known from quantum mechanics, and the functional integral.

3.1 Transfer matrix for the Ising model

Boundary problem for Ising chain. Let us consider the one-dimensional Ising model

$$S = \sum_x \mathscr{L}(x) + \mathscr{L}_{\text{in}} + \mathscr{L}_{\text{f}},$$

with a next-neighbor interaction $\mathscr{L}(x) = -\beta s(x+\varepsilon)s(x)$ and initial and final boundary terms $\mathscr{L}_{\text{in}}$ and $\mathscr{L}_{\text{f}}$. (We combine interaction strength and inverse temperature into a single dimensionless parameter β.) We choose boundary conditions such that $s(x_{\text{in}}) = 1$ and $s(x_{\text{f}}) = 1$. This can be implemented by

$$e^{-\mathscr{L}_{\text{in}}} = \delta(s(x_{\text{in}}) - 1), \qquad e^{-\mathscr{L}_f} = \delta(s(x_{\text{f}}) - 1),$$

which in turn can be implemented by limits like

$$\mathscr{L}_{\rm in} = -\lim_{\kappa\to\infty} \kappa[s(x_{\rm in}) - 1].$$

The question arises now: What is the expectation value $\langle s(x)\rangle$ for x in the bulk, i. e. between the endpoints $x_{\rm in}$ and $x_{\rm f}$? The single configuration with minimal action has all spins aligned, $s(x) = 1$. There are, however, many more configurations where some of the spins take negative values. Even though the particular probability for one such configuration is smaller, this is outweighed by the number of configurations. Qualitatively one expects something like in figure 1. In the bulk, far

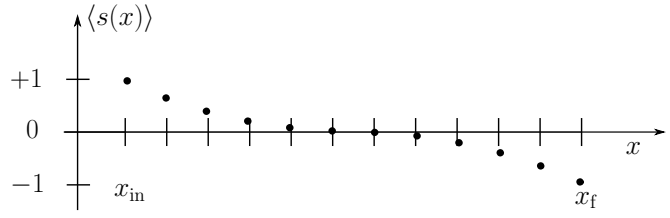

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Figure 1. Ising chain with spins at the endpoints fixed to $s(x_{\rm in}) = +1$ and $s(x_{\rm f}) = -1$. What is $\langle s(x)\rangle$ for x between the endpoints?

away from the boundaries, the average spin may vanish to a good approximation. We look for a formalism to compute this behaviour as a function of the parameter β.

Video: **Lecture03Video02.mp4**

Product form of probability distribution. We can write e^{-S} in product form

$$e^{-S} = e^{-\mathscr{L}_{\rm f}+\sum_x \mathscr{L}(x)+\mathscr{L}_{\rm in}} = \bar{f}_{\rm f}\left[\prod_x e^{-\mathscr{L}(x)}\right] f_{\rm in} = \bar{f}_{\rm f}\left[\prod_x \mathscr{K}(x)\right] f_{\rm in}$$

with boundary terms $\bar{f}_{\rm f} = e^{-\mathscr{L}_{\rm f}}$ and $f_{\rm in} = e^{-\mathscr{L}_{\rm in}}$. Here $\mathscr{K}(x)$ depends on the two spins $s(x)$ and $s(x+\varepsilon)$, while $f_{\rm in}$ depends on $s(x_{\rm in})$ and $\bar{f}_{\rm f}$ depends on $s(x_{\rm f})$.

Occupation number basis. Any function $f(s(x))$ that depends only on the spin $s(x)$ can be expanded in terms of two basis functions $h_\tau(s(x))$ where $\tau = 1, 2$,

$$f(s(x)) = q_1(x)\, h_1(s(x)) + q_2(x)\, h_2(s(x)).$$

We choose the occupation number basis with

$$h_1(s) = \frac{1+s}{2} = n, \qquad h_2(s) = \frac{1-s}{2} = (1-n).$$

This is easily seen by noting that the occupation number n has only the values 1 (for $s = 1$) and 0 (for $s = -1$), such that

$$n^2 = n.$$

Any polynomial in s can be written as $an + b$, such that any $f(s)$ can indeed be expressed in terms of the two basis functions.

We note some properties of the basis functions. The relation

$$h_\tau(s)h_\rho(s) = \delta_{\tau\rho}h_\rho(s)$$

is simply verified by $h_\tau^2(s) = h_\tau(s)$ and $h_1(s)h_2(s) = 0$). Other useful relations are

$$\sum_{s=\pm 1} h_\tau(s) = h_\tau(s=1) + h_\tau(s=-1) = 1,$$

$$\sum_{\tau} h_\tau(s) = h_1(s) + h_2(s) = 1,$$

and finally by combination

$$\sum_{s=\pm 1} h_\tau(s) h_\rho(s) = \delta_{\tau\rho}.$$

Video: **Lecture03Video03.mp4**

Transfer matrix. Let us now expand $\mathscr{K}(x)$ in terms of the basis functions $h_\tau(s(x+\varepsilon))$ and $h_\rho(s(x))$,

$$\mathscr{K}(x) = \hat{T}_{\tau\rho}(x)\, h_\tau(s(x+\varepsilon))\, h_\rho(s(x)).$$

We use here the Einstein summation convention which implies summation over the indices τ and ρ. The expansion coefficients $\hat{T}_{\tau\rho}(x)$ are the elements of the transfer matrix $\hat{T}$. This is a 2×2 matrix. Indeed using shorthands $\bar{n} = n(t+\varepsilon)$, $n = n(t)$ and similar for $\bar{h}_\tau$, h_τ, an arbitrary $\mathscr{K}(x)$ can be written as

$$\begin{aligned}\mathscr{K} &= a\bar{n}n + b\bar{n} + cn + d \\ &= \hat{T}_{11}\bar{h}_1 h_1 + \hat{T}_{12}\bar{h}_1 h_2 + \hat{T}_{21}\bar{h}_2 h_1 + \hat{T}_{22}\bar{h}_2 h_2.\end{aligned}$$

Matrix product for transfer matrix. Consider now the product of two neighbouring factors $\mathscr{K}(x+\varepsilon)$ and $\mathscr{K}(x)$, summed over the common spin $s(x+\varepsilon)$

$$\begin{aligned}\sum_{s(x+\varepsilon)} \mathscr{K}(x+\varepsilon)\mathscr{K}(x) &= \sum_{s(x+\varepsilon)} h_\tau(s(x+2\varepsilon))\hat{T}_{\tau\rho}(x+\varepsilon))h_\rho(s(x+\varepsilon))h_\alpha(s(x+\varepsilon))\hat{T}_{\alpha\beta}(x)h_\beta(s(x)) \\ &= \sum_{\rho}\sum_{s(x+\varepsilon)} h_\tau(s(x+2\varepsilon))\hat{T}_{\tau\rho}(x+\varepsilon)\hat{T}_{\rho\beta}(x)h_\rho(s(x+\varepsilon))h_\beta(s(x)) \\ &= \sum_{\rho} h_\tau(s(x+2\varepsilon))\hat{T}_{\tau\rho}(x+\varepsilon)\hat{T}_{\rho\beta}(x)h_\beta(s(x)) \\ &= h_\tau(s(x+2\varepsilon))\left[\hat{T}(x+\varepsilon)\hat{T}(x)\right]_{\tau\beta} h_\beta(s(x)).\end{aligned}$$

The second line uses $h_\tau h_\rho = \delta_{\tau\rho}h_\rho$ and the third line $\sum_s h_\rho = 1$. We observe that the matrix product of transfer matrices appears in this product. For the Ising model the factors $\mathscr{K}(x)$ are the same for all x (except for different spins being involved), and therefore $\hat{T}$ is independent of x. One simply finds

$$\sum_{s(x+\varepsilon)} \mathscr{K}(x+\varepsilon)\mathscr{K}(x) = h_\tau(s(x+2\varepsilon))\left[\hat{T}^2\right]_{\tau\rho} h_\rho(s(x)).$$

Doing one more similar step yields

$$\sum_{s(x+2\varepsilon)}\sum_{s(x+\varepsilon)} \mathscr{K}(x+2\varepsilon)\mathscr{K}(x+\varepsilon)\mathscr{K}(x) = h_\tau(s(x+3\varepsilon))\left[\hat{T}(x+2\varepsilon)\hat{T}(x+\varepsilon)\hat{T}(x)\right]_{\tau\rho} h_\rho(s(x)),$$

and so on.

Video: **Lecture03Video04.mp4**

Partition function as product of transfer matrices. One can write the partition function as

$$\begin{aligned} Z &= \left[\prod_{x=x_{\rm in}}^{x_{\rm f}} \sum_{s(x)}\right] \bar{f}_{\rm f}(s(x_{\rm f})) \left[\prod_{x=x_{\rm in}}^{(x_{\rm f}-\varepsilon)} \mathscr{K}(x)\right] f_{\rm in}(s(x_{\rm in})) \\ &= \sum_{s(x_{\rm f})} \sum_{s(x_{\rm in})} \bar{f}_{\rm f}(s(x_{\rm f}))\, h_\tau(s(x_{\rm f})) \left[\hat{T}(x_{\rm f}-\varepsilon)\cdots\hat{T}(x_{\rm in})\right]_{\tau\rho} h_\rho(s(x_{\rm in}))f_{\rm in}(s(x_{\rm in})) \\ &= \sum_{s(x_{\rm f})} \sum_{s(x_{\rm in})} \bar{q}_\beta(x_{\rm f})\, h_\beta(s(x_{\rm f}))\, h_\tau(s(x_{\rm f})) \left[\hat{T}\cdots\hat{T}\right]_{\tau\rho} h_\rho(s(x_{\rm in}))\, \tilde{q}_\alpha(s(x_{\rm in}))\, h_\alpha(s(x_{\rm in})). \end{aligned}$$

Here we have expanded $\bar{f}_{\rm f}$ and $f_{\rm in}$ in terms of the basis functions,

$$\begin{aligned} \bar{f}_{\rm f}(s(x_{\rm f})) =& \bar{q}_\beta(x_{\rm f})\, h_\beta(s(x_{\rm f})), \\ f_{\rm in}(s(x_{\rm in})) =& \tilde{q}_\alpha(x_{\rm in})\, h_\alpha(s(x_{\rm in})). \end{aligned}$$

Performing the sums over the initial and final spins leads to

$$Z = \bar{q}_\tau(x_{\rm f}) \left[\hat{T}(x_{\rm f}-\varepsilon)\cdots\hat{T}(x_{\rm in})\right]_{\tau\rho} \tilde{q}_\rho(x_{\rm in}).$$

This has the structure of an initial vector (or wave function) $\tilde{q}(x_{\rm in})$ multiplied by a matrix, and then contracted with a final vector (or conjugate wave function) $\bar{q}(x_{\rm f})$. We can use the bracket notation familiar from quantum mechanics,

$$Z = \langle \bar{q}(x_{\rm f})|\hat{T}(x_{\rm f}-\varepsilon)\cdots\hat{T}(x_{\rm in})|\tilde{q}(x_{\rm in})\rangle.$$

This product formulae resembles quantum mechanics if one associates the transfer matrix with the infinitesimal evolution operator $U(t)$

$$\psi(t+\varepsilon) = U(t)\psi(t),$$

where

$$U(t) = e^{i\varepsilon H(t)}.$$

With

$$\psi(t_{\rm f}) = U(t_{\rm f}-\varepsilon)\cdots U(t_{\rm in})\psi(t_{\rm in}),$$

one can write the transition amplitude in the form

$$\langle\phi(t_{\rm f})|\psi(t_{\rm f})\rangle = \langle\phi(t_{\rm f})U(t_{\rm f}-\varepsilon)\cdots U(t_{\rm in})|\psi(t_{\rm in})\rangle.$$

Formally, the map between quantum mechanics and the classical statistics of the Ising model is

Quantum mechanics	Classical statistics
U	$\hat{T}$
t	x
ψ	$\tilde{q}$
$\bar{\phi}$	$\bar{q}$

A main difference to quantum mechanics is that $\hat{T}$ does not preserve the norm of the wave function.

Video: **Lecture03Video05.mp4**

Computation of the transfer matrix. Let us compute the transfer matrix for the Ising model. We employ the defining relation of the transfer matrix by an expansion of the local factor in terms of basis functions,

$$e^{\beta\bar{s}s} = \hat{T}_{\tau\rho}\, h_\tau(\bar{s})\, h_\rho(s),$$

where we use the shorthand notation

$$\bar{s} = s(x+\varepsilon), \qquad s = s(x).$$

Using the decomposition

$$s = h_1 - h_2 = n - (1-n) = 2n - 1,$$

and

$$\beta\bar{s}s = \beta(\bar{h_1} - \bar{h_2})(h_1 - h_2) = \beta(\bar{h}_1 h_1 + \bar{h}_2 h_2 - \bar{h}_1 h_2 - \bar{h}_2 h_1),$$

one obtains by analyzing the four configurations of neighboring spins $(\bar{s}, s)$,

$$e^{\beta\bar{s}s} = e^{\beta}(\bar{h}_1 h_1 + \bar{h}_2 h_2) + e^{-\beta}(\bar{h}_1 h_2 + \bar{h}_2 h_1).$$

From this one can read off the transfer matrix

$$\hat{T} = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}.$$

In general the transfer matrix $\hat{T}$ is not a unitary matrix as for quantum mechanics. For the Ising model $\hat{T}(x)$ does not depend on x so that one obtains

$$Z = \bar{q}_\tau(x_{\mathrm{f}}) \left[\hat{T}^{P-1}\right]_{\tau\rho} \tilde{q}_\rho(x_{\mathrm{in}}).$$

Video: **Lecture03Video06.mp4**

Periodic Boundary Condition. Replace $\mathscr{L}_f + \mathscr{L}_{\mathrm{in}}$ by $-\beta s(x_{\mathrm{f}})s(x_{\mathrm{in}})$. This closes the circle by defining x_{f} and x_{in} as next neighbours. The partition function becomes

$$Z = \mathrm{Tr}\left\{\hat{T}^P\right\}.$$

Diagonalising $\hat{T}$ solves the Ising model in a simple way,

$$Z = {\lambda_+}^P + {\lambda_-}^P,$$

with $\lambda_\pm$ the two eigenvalues of the transfer matrix,

$$\lambda_+ = 2\cosh(\beta), \qquad \lambda_- = 2\sinh(\beta).$$

In the limit $P \to \infty$ only the largest eigenvalue λ_+ contributes.If we restore for β the product of coupling strength and inverse temperature, this is the exact solution for the canonical partition function for the Ising chain. The thermodynamics follows from there.

Video: **Lecture04Video01.mp4**

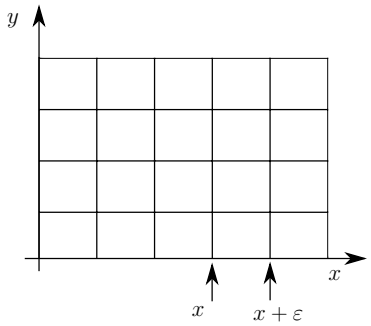


Figure 2. Illustration of the two dimensional Ising model.

Generalisations. The transfer matrix can be generalised to an arbitrary number of Ising spins $s_\gamma(x)$. For M spins, $\gamma = 1, \ldots, M$, the transfer matrix $\hat{T}$ is an $N \times N$ matrix, $N = 2^M$, $\tau = 1, \ldots, N$.

For example, if $M = 2$, $\hat{T}$ is a 4×4 matrix. The basis functions in the occupation number basis are taken as

$$h_1 = n_1 n_2, \qquad h_2 = (1 - n_1) n_2,$$
$$h_3 = n_1 (1 - n_2), \qquad h_4 = (1 - n_1)(1 - n_2).$$

This structure can be extended to arbitrary M. The basis functions obey the same rules as discussed for $M = 1$. In particular, γ may denote a second coordinate y such that,

$$s_\gamma(x) \to s(x, y).$$

Video: **Lecture04Video02.mp4**

Two-dimensional Ising model. In this way one can define formally the transfer matrix for the two-dimensional Ising model. The coordinate x denotes now lines in a two-dimensional plane, see fig. 2. More generally, in d dimensions, x denotes the position on a particular $d - 1$ dimensional hypersurface. The transfer matrix contains the information of what happens if one goes from one hypersurface to the next one.

3.2 Non-commutativity in classical statistics

Video: **Lecture04Video03.mp4**

Local observables and operators. A local observable $A(x)$ depends only on the local spin $s(x)$. We want to find an expression for its expectation value in terms of the transfer matrix. For this purpose we consider the expression

$$\sum_{s(x)} \mathscr{K}(x) A(x) \mathscr{K}(x - \varepsilon) = \sum_{s(x)} h_\tau(x + \varepsilon) \hat{T}_{\tau\rho}(x) h_\rho(x) A_\gamma(x) h_\gamma(x) h_\alpha(x) \hat{T}_{\alpha\beta}(x - \varepsilon) h_\beta(x - \varepsilon),$$

where we use the shorthand

$$h_\tau(x) = h_\tau(s(x)),$$

and the expansion

$$A(x) = A_\gamma(x)\, h_\gamma(s(x)).$$

We employ

$$A_\gamma(x)\sum_{s(x)} h_\rho(x)h_\gamma(x)h_\alpha(x) = \sum_\gamma A_\gamma(x)\delta_{\rho\gamma}\delta_{\gamma\alpha},$$

and introduce the diagonal operator

$$(\hat{A}(x))_{\rho\alpha} = \sum_\gamma A_\gamma(x)\delta_{\rho\gamma}\delta_{\gamma\alpha} = \begin{pmatrix} A_1(x) & 0 \\ 0 & A_2(x) \end{pmatrix}.$$

The last identity refers to the single spin Ising chain. The two observables A_1 and A_2 correspond to the values that the observable takes in the two local states of the Ising chain. The fact that the operator is diagonal reflects properties of the specific occupation number basis. For a general basis the operator is not diagonal.

Video: **Lecture04Video04.mp4**

In terms of this operator we can write

$$\sum_{s(x)} \mathscr{K}(x)A(x)\mathscr{K}(x-\varepsilon) = h_\tau(x+\varepsilon)\hat{T}_{\tau\rho}(x)\hat{A}_{\rho\alpha}(x)\hat{T}_{\alpha\beta}(x-\varepsilon)h_\beta(x-\varepsilon).$$

The expectation value of $A(x)$ obtains by an insertion of the operator $\hat{A}(x)$,

$$\begin{aligned}\langle A(x)\rangle &= \frac{1}{Z}\int Ds\, e^{-S}A(x) \\ &= \frac{1}{Z}\bar{q}_\tau(x_{\rm f})[\hat{T}(x_{\rm f}-\varepsilon)\cdots\hat{T}(x)\hat{A}(x)\hat{T}(x-\varepsilon)\cdots\hat{T}(x_{\rm in})]_{\tau\rho}\tilde{q}_\rho(x_{\rm in})\end{aligned}$$

The operators $\hat{T}(x)$ and $\hat{A}(x)$ do in general not commute,

$$[\hat{T}(x),\hat{A}(x)] \neq 0.$$

Non-commutativity is present in classical statistics if one asks questions related to hypersurfaces!

Video: **Lecture04Video05.mp4**

Let us concentrate on observables that are represented by operators $\hat{A}$ which are independent of x. As an example we take the local occupation number $n(x) = 2s(x) - 1$. The associated operator is

$$\hat{N} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

If we want to obtain the expectation value at x, we need to compute

$$\langle n(x)\rangle = \frac{1}{Z}\langle\bar{q_f}|\hat{T}(x_{\rm f}-\varepsilon)\cdots\hat{T}(x)\hat{N}\hat{T}(x-\varepsilon)\cdots\hat{T}(x_{\rm in})|\tilde{q}_{\rm in}\rangle,$$

where we employ a notation familiar from quantum mechanics,

$$\langle\bar{q}_{\rm f}|\hat{M}|\tilde{q}_{\rm in}\rangle = (\bar{q}_{\rm f}(x_{\rm f}))_\tau\hat{M}_{\tau\rho}(q_{\rm in}(x_{\rm in}))_\rho.$$

A normalisation with $Z = 1$ brings the expression even closer to quantum mechanics. We adopt it in the following.

Video: **Lecture04Video06.mp4**

We may next consider the operator

$$\hat{N}_+ = \hat{T}(x)^{-1}\,\hat{N}\,\hat{T}(x), \tag{3.1}$$

and compute

$$\langle \bar{q}_{\rm f}|\hat{T}(x_f-\varepsilon)\cdots\hat{T}(x)\hat{N}_+\hat{T}(x-\varepsilon)\cdots\hat{T}(x_{\rm in})|\tilde{q}_{\rm in}\rangle = \langle n(x+\varepsilon)\rangle.$$

When we use the same prescription (with x singled out as a reference point) the operator $\hat{N}$ corresponds to the observable $n(x)$, while $\hat{N}_+$ is associated to the observable $n(x+\varepsilon)$. The operator $\hat{N}_+$ is not diagonal and does not commute with $\hat{N}$,

$$[\hat{N}_+, \hat{N}] \neq 0.$$

We conclude that non-commuting operators do not only appear in quantum mechanics. The appearance of non-commuting structures is an issue of what questions are asked and which formalism is appropriate for the answer to these questions. One can actually device a Heisenberg picture for classical statistical systems in close analogy to quantum mechanics. The Heisenberg operators depend on x and do not commute for different x.

3.3 Classical Wave functions

Video: **Lecture04Video07.mp4**

We have seen how operators and non-commuting structures appear within classical probabilistic systems. The transfer matrix formalism is a type of Heisenberg picture for classical statistics. There is also a type of Schrödinger picture with wave functions as probability amplitudes. This will be discussed in the present lecture.

Video: **Lecture04Video08.mp4**

Local probabilities. We start from the "overall probability distribution" given for the Ising chain by

$$p[s] = \frac{1}{Z}e^{-S[s]}, \qquad Z = \int Ds\, e^{-S[s]}.$$

A *local* probability distribution at x, which involves only the spin $s(x)$, can be obtained by summing over all spins at $x' \neq x$,

$$p_l(s(x)) = \frac{1}{Z}\left[\prod_{x'\neq x}\sum_{s(x')=\pm 1}\right] e^{-S} \equiv p_l(x).$$

It is properly normalized,

$$\sum_{s(x)=\pm 1} p_l(s(x)) = 1.$$

The expectation value of the spin $s(x)$ can be computed from $p_l(s(x))$,

$$\langle s(x)\rangle = \sum_{s(x)=\pm 1} p_l(s(x))s(x).$$

If there would be a simple evolution law how to compute $p_l(x+\varepsilon)$ from $p_l(x)$, one could solve the boundary value problem iteratively, starting from the initial probability distribution $p_l(x_{\rm in})$. The evolution law would permit us to infer $p_l(x)$, and therefore to compute the expectation value of $s(x)$. Unfortunately, such a simple evolution law does not exist for the local probabilities. We will see next that it exists for local wave functions or probability amplitudes.

Video: **Lecture04Video09.mp4**

Wave Functions. Define for a given x the partial actions S_- and S_+ by

$$\begin{aligned} S_- =& \mathscr{L}_{\rm in} + \sum_{x'=x_{\rm in}}^{x-\varepsilon} \mathscr{L}(x'), \\ S_+ =& \sum_{x'=x}^{x_{\rm f}-\varepsilon} \mathscr{L}(x') + \mathscr{L}_{\rm f}, \\ S =& S_- + S_+. \end{aligned}$$

Here S_- depends only on the Ising spins $s(x')$ with $x' \leq x$, and S_+ depends on spins $s(x')$ with $x' \geq x$.

The wave function $f(x)$ is defined by

$$f(x) = \left[\prod_{x'=x_{\rm in}}^{x-\varepsilon} \sum_{s(x')=\pm 1} \right] e^{-S_-}.$$

Because we sum over all $s(x')$ with $x' < x$, and S_- depends only on those $s(x')$ and on $s(x)$, the wave function $f(x)$ depends only on the single spin $s(x)$. Similarly, we define the conjugate wave function

$$\bar{f}(x) = \left[\prod_{x'=x+\varepsilon}^{x_f} \sum_{s(x')=\pm 1} \right] e^{-S_+},$$

which also depends only on $s(x)$.

Video: **Lecture04Video10.mp4**

Wave functions and local probability distribution. The product

$$\bar{f}(x) f(x) = \left[\prod_{x' \neq x} \sum_{s(x')=\pm 1} \right] e^{-S} = Z\, p_l(x),$$

is closely related to the local probability distribution $p_l(x)$. One has

$$\sum_{s(x)=\pm 1} \bar{f}(x) f(x) = Z.$$

In the following we employ the possibility of an additive renormalisation $S \to S + C$ in order to normalise the partition function to $Z = 1$. This can be achieved by adding a constant to $\mathscr{L}(x)$, and similarly for the boundary terms $\mathscr{L}_{\rm in}$ and $\mathscr{L}_{\rm f}$. With $Z = 1$ the wave functions $\bar{f}$ and f are a type of probability amplitudes, similar as in quantum mechanics. We have, however, two distinct types of probability amplitudes, f and $\bar{f}$.

Video: **Lecture04Video11.mp4**

Quantum rule for expectations values of local observables. The expectation value of a local observable $A(x)$ can be written in terms of a bilinear in the wave functions.

$$\begin{aligned}\langle A(x)\rangle &= \sum_{s(x)=\pm 1} A(x)p_l(x) \\ &= \frac{1}{Z}\sum_{s(x)=\pm 1} \bar{f}(x)A(x)f(x).\end{aligned}$$

We expand again in the occupation number basis

$$\begin{aligned}f(x) &= \tilde{q}_\rho(x)h_\rho(x), \\ \bar{f}(x) &= \bar{q}_\tau(x)h_\tau(x), \\ A(x) &= A_\sigma(x)h_\sigma(x).\end{aligned}$$

Here $\tilde{q}_\rho(x)$ are the components of the wave function in the occupation number basis at x, and $\bar{q}_\tau(x)$ are the components of the conjugate wave function. This yields for the expectation values

$$\langle A(x)\rangle = \frac{1}{Z}\bar{q}_\tau(x)A_\sigma(x)\tilde{q}_\rho(x)\sum_{s(x)=\pm 1} h_\tau(x)h_\sigma(x)h_\rho(x).$$

Using again the product properties of the basis functions one finds the "quantum rule" for the expectation value as a bilinear in the wave functions,

$$\begin{aligned}\langle A(x)\rangle &= \frac{1}{Z}\langle\bar{q}(x)|\hat{A}(x)|\tilde{q}(x)\rangle \\ &= \frac{1}{Z}\sum_\sigma \bar{q}_\tau(x)A_\sigma(x)\delta_{\tau\sigma}\delta_{\sigma\rho}\tilde{q}_\rho(x).\end{aligned} \tag{3.2}$$

Knowledge of the wave function at x is therefore sufficient for the computation of $\langle A(x)\rangle$.

Video: **Lecture04Video12.mp4**

In particular, for $Z = 1$ the rule (3.2) is very close to quantum mechanics, except that $\tilde{q}$ and $\bar{q}$ are real wave functions and $\bar{q}$ is not related to $\tilde{q}$. As in quantum mechanics, it associates an operator to an observable, and employs the concept of probability amplitudes. We can not only express the expectation values of local observables as $n(x)$, represented by $\hat{N}(x)$, in this way. The relation (3.2) also holds for the observable $n(x+\varepsilon)$, represented by the operator $\hat{N}_+$ in equation (3.1). The rule (3.2) may be called the "quantum rule". In contrast to quantum mechanics it is not a new postulate. It follows from the basic probabilistic definition of expectation values in classical statistics by an appropriate organization of the probabilistic information.

Video: **Lecture04Video13.mp4**

Evolution equation for the wave function. In contrast to the local probability distribution, the x-dependence of the wave functions is a simple linear evolution law. This makes the wave function the appropriate object for the discussion of boundary value problems and beyond. From the definition of the wave function $f(x)$ one infers immediately

$$f(x+\varepsilon) = \sum_{s(x)=\pm 1} \mathscr{K}(x)f(x).$$

As it should be, $f(x+\varepsilon)$ depends on the spin $s(x+\varepsilon)$. The expansion in the occupation number basis yields

$$\begin{aligned} f(x+\varepsilon) &= \tilde{q}_\tau(x+\varepsilon)h_\tau(x+\varepsilon) \\ &= \sum_{s(x)=\pm 1} h_\tau(x+\varepsilon)\hat{T}_{\tau\rho}(x)h_\rho(x)\,\tilde{q}_\sigma(x)h_\sigma(x) \\ &= \hat{T}_{\tau\rho}(x)\tilde{q}_\rho(x)h_\tau(x+\varepsilon). \end{aligned}$$

The linear evolution operator for the wave function is the transfer matrix.

$$\tilde{q}_\tau(x+\varepsilon) = \hat{T}_{\tau\rho}(x)\tilde{q}_\rho(x),$$

or, in a vector matrix notation

$$\tilde{q}(x+\varepsilon) = \hat{T}(x)\tilde{q}(x).$$

Video: **Lecture04Video14.mp4**

By the same type of argument one obtains for the conjugate wave function (as a row vector)

$$\bar{q}(x) = \bar{q}(x+\varepsilon)\hat{T}(x),$$

or, written as a column vector,

$$\bar{q}(x) = \hat{T}^T(x)\bar{q}(x+\varepsilon),$$

and

$$\bar{q}(x+\varepsilon) = (\hat{T}^T(x))^{-1}\bar{q}(x).$$

In cases where $\hat{T}$ is orthogonal, $\hat{T}^{-1} = \hat{T}^T$, both $\bar{q}$ and $\tilde{q}$ obey the same evolution law. The evolution law is linear. The superposition law familiar from quantum mechanics follows. If $\tilde{q}_1(x)$ and $\tilde{q}_2(x)$ are two solutions of the evolution equation, this also holds for linear combinations $\alpha\tilde{q}_1(x)+\beta\tilde{q}_2(x)$.

Video: **Lecture04Video15.mp4**

Continuous evolution. For a sufficiently smooth wave function $\tilde{q}(x)$ one defines the derivative

$$\begin{aligned} \frac{\partial\tilde{q}}{\partial x} &= \frac{1}{2\varepsilon}(\tilde{q}(x+\varepsilon) - \tilde{q}(x-\varepsilon)) \\ &= \frac{1}{2\varepsilon}(\hat{T}(x) - \hat{T}^{-1}(x-\varepsilon))\tilde{q}(x). \end{aligned}$$

This yields the generalised Schrödinger equation

$$\begin{aligned} \partial_x\tilde{q} &= \frac{\partial}{\partial x}\tilde{q} = W\tilde{q}, \\ W(x) &= \frac{1}{2\varepsilon}\left[\hat{T}(x) - \hat{T}^{-1}(x-\varepsilon)\right]. \end{aligned}$$

For the same $\mathscr{L}$ at every x, both $\hat{T}$ and W are independent of x,

$$W = \frac{1}{2\varepsilon}\left[\hat{T} - \hat{T}^{-1}\right].$$

Video: **Lecture04Video16.mp4**

Step evolution operator. An additive renormalization of the action corresponds to a multiplicative renormalization of the transfer matrix. The step evolution operator is the transfer matrix normalized such that the absolute value of the largest eigenvalue equals unity. As the name indicates, the step evolution operator plays the same role as the discrete evolution operator in quantum mechanics. For the Ising model, the step evolution operator is given by

$$\hat{T} = \frac{1}{2\cosh(\beta)} \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}.$$

Equilibrium state. If only one eigenvalue of the step evolution operator equals unity in absolute magnitude, the eigenstate to this eigenvalue is the unique equilibrium state $\tilde{q}_*$. For the Ising model the equilibrium wave function is

$$\tilde{q}_* \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The equilibrium state is invariant under the evolution.

Video: **Lecture04Video17.mp4**

Boundary value problem. For given boundary conditions $\tilde{q}(x_{\rm in})$ and $\bar{q}(x_{\rm f})$ are fixed. One can use the evolution equation to compute both $\tilde{q}(x)$ and $\bar{q}(x)$. The value of a local observable $A(x)$, with associated operator $\hat{A}(x)$, follows from

$$\langle A(x)\rangle = \frac{1}{Z}\langle \bar{q}(x)|\hat{A}(x)|\tilde{q}(x)\rangle.$$

Choose for $\tilde{q}(x_{\rm in})$ a decomposition into eigenfunctions of the transfer matrix $\hat{T}$, e. g. with eigenvalues λ_+ and λ_-,

$$\tilde{q}(x_{\rm in}) = c_+(x_{\rm in})\tilde{q}_+ + c_-(x_{\rm in})\tilde{q}_-,$$

such that

$$\tilde{q}(x) = \tilde{q}(x_{\rm in} + N\varepsilon) = c_+(x_{\rm in})\,(\lambda_+)^N\,\tilde{q}_+ + c_-(x_{\rm in})\,(\lambda_-)^N\,\tilde{q}_-.$$

For $\lambda_+ = 1$, the corresponding eigenfunction is the equilibrium wave function,

$$\lambda_+^N \tilde{q}_+ = \tilde{q}_+.$$

For $\lambda_- < 1$ the contribution $\sim (\lambda_-)^N \tilde{q}_-$ vanishes for large N. This describes the approach to equilibrium. The correlation length is directly related to λ_-. Similar rules are valid for the conjugate wave function. For a finite distance from the boundary we can employ the two wave functions in order to compute the expectation value of $s(x)$ in dependence on given boundary conditions.

Video: **Lecture04Video18.mp4**

Generalisations and summary. The discussion for the Ising chain with a single spin at each site is easily generalised to M spins at each site, to multi-dimensional Ising models and to arbitrary overall probability distributions. The main purpose of this lecture was to give you a basic understanding how the functional integral and the operator formalism of quantum mechanics are related. When we discuss later the functional integral for relativistic quantum field theories no basic new concepts need to be introduced for the translation to the still more familiar operator formalism that is the starting point of most textbooks.

All these properties point to a close connection between quantum mechanics and classical statistics. Indeed, quantum mechanics can be understood as a sub-field of classical statistics. Quantum systems are realized as appropriate subsystems of "classical" probabilistic systems. This is not the topic of this lecture. If you are interested, you may read "The probabilistic world" [arXiv 2011.02867].

4 Quantum Fields and Functional Integral

Video: **Lecture05Video01.mp4**

In this lecture we will start from many body quantum mechanics and construct the functional integral for a quantum field theory. In the last lecture we have shown how the operator formalism emerges from a functional integral, in short: *functional integral* → *operators*. In this lecture we will proceed in the opposite direction. Starting from a formulation of many body quantum mechanics in terms of operators we will derive the equivalent functional integral, in short: *operators* → *functional integral*. The aim of the lecture is once more to show the equivalence of the functional integral and the operator formalism. Historically, this is the way how Feynman introduced the functional integral for quantum mechanics. This construction of the functional integral can be found in many textbooks on quantum field theory at a somewhat later stage. The present lecture should also help to establish this contact.

In the present lecture we introduce quantum fields, establishing in this way the basic concepts of quantum field theory in the operator formalism. We construct the functional integral for quantum fields. We take the non-relativistic example of phonons. This demonstrates that quantum field theory is not only needed for relativistic particle physics. Phonons are perhaps also easier to understand intuitively than photons. There is not much conceptual difference between phonons and photons. Phonons are excitations in a solid, photons are excitations of the vacuum. Photons are relativistic.

4.1 Phonons as quantum fields in one dimension

Video: **Lecture05Video02.mp4**

One-dimensional crystal. Consider a one-dimensional crystal of atoms with lattice sites $x_j = j\varepsilon$ and lattice distance ε. Denote the displacement from the equilibrium position at x_j by Q_j and the momentum of the atoms by P_j. The Hamiltonian for small displacements can be taken quadratic in Q_j, and we decompose $H = H_0 + H_{\rm nn}$ with

$$H_0 = \sum_j \left(\frac{P_j^2}{2M} + \frac{D}{2} {Q_j}^2 \right), \qquad H_{\rm nn} = -\frac{B}{2} \sum_j Q_{j+1} Q_j .$$

Here Q_j and P_j are quantum operators with the usual commutation relations

$$[Q_j, P_k] = i\delta_{jk}, \qquad [Q_j, Q_k] = 0, \qquad [P_i, P_j] = 0.$$

We use units where $\hbar = 1$.

The term H_0 alone describes decoupled harmonic oscillators at every lattice site j. The term H_{nn} couples the oscillators by a next neighbour interaction. Phonons are thus described by a coupled system of harmonic oscillators.

Quantum fields. The displacements are an example for a *quantum field*,

$$Q_j = Q(x).$$

Here x is a discrete variable labelling the lattice sites. In the continuum limit x will become a continuous position variable. The field $Q(x)$ is an *operator field*. For each x one has an operator

$Q(x)$. Often such operator fields are called "quantum fields". We use this expression here as well, but not exclusively for the operator fields in the operator formalism. We will also employ the notion of quantum fields in the equivalent functional integral formalism that does not employ operators for its formulation. Also the momentum field $P(x) = P_j$ is an operator field or quantum field. One may consider the pairs $\{Q_j, P_j\}$ as a common (two-component) quantum field.

Video: **Lecture05Video03.mp4**

Occupation number basis. At each site j we define annihilation and creation operators a_j and $a_j^\dagger$. The annihilation operators are

$$a_j = \frac{1}{\sqrt{2}} \left((DM)^{\frac{1}{4}} Q_j + i (DM)^{-\frac{1}{4}} P_j \right),$$

and the creation operators are given by

$$a_j^\dagger = \frac{1}{\sqrt{2}} \left((DM)^{\frac{1}{4}} Q_j - i (DM)^{-\frac{1}{4}} P_j \right).$$

The creation operators are the hermitian conjugates of the annihilation operators, $a_j^\dagger = (a_j)^\dagger$. The commutation relations are

$$[a_j, a_k^\dagger] = \delta_{jk}, \qquad [a_j, a_k] = 0, \qquad [a_j^\dagger, a_k^\dagger] = 0.$$

This can be verified by employing the commutation relations for Q and P. Both $a(x) = a_j$ and $a^\dagger(x) = a_j^\dagger$ are operator fields.

Inserting

$$Q(x) = Q_j = \frac{1}{\sqrt{2}} (DM)^{-\frac{1}{4}} \left(a_j + a_j^\dagger \right),$$

and similar for P_j, we can express the Hamiltonian in terms of a and $a^\dagger$,

$$H_0 = \omega_0 \sum_j \left(a_j^\dagger a_j + \frac{1}{2} \right) = \omega_0 \sum_j \left(\hat{n}_j + \frac{1}{2} \right),$$

with the frequency $\omega_0 = \sqrt{D/M}$. You recognise the standard treatment of harmonic oscillators in quantum mechanics. Occupation numbers at positions x_j are expressed in terms of the operator $\hat{n}_j = a_j^\dagger a_j$. They have the eigenvalues $n_j = (0, 1, 2, \ldots)$. At each site j there are a number n_j of "localised phonons". For $B = 0$ the system describes uncoupled harmonic oscillators, one at each lattice site.

We next discuss the effects of the next-neighbour interaction. It involves products of a_j, a_{j+1} etc., according to

$$\begin{aligned} H_{\rm nn} &= -\frac{B}{2} \sum_j Q_{j+1} Q_j \\ &= -\frac{B}{2} \frac{(DM)^{-\frac{1}{2}}}{2} \sum_j \left(a_{j+1} + a_{j+1}^\dagger \right) \left(a_j + a_j^\dagger \right). \end{aligned}$$

Video: **Lecture05Video04.mp4**

Momentum Space. It is possible to diagonalize H by a discrete Fourier transform. To this end, we write

$$a_j = \frac{1}{\sqrt{\mathscr{N}}} \sum_q e^{i\varepsilon q j} a_q, \qquad a_j^\dagger = \frac{1}{\sqrt{\mathscr{N}}} \sum_q e^{-i\varepsilon q j} a_q^\dagger.$$

Due to the finite lattice distance the sum is periodic in q,

$$\sum_q = \sum_{|q| \le \frac{\pi}{\varepsilon}},$$

and $\mathscr{N} = \sum_j$ is a normalization factor corresponding to the number of lattice sites. If we place the sites of the lattice on a torus with circumference L, the momentum sum is a discrete sum, with level distance given by $2\pi/L$. If you are not familiar with these formulae you may look up in some text book a chapter on discrete Fourier transforms. It is the most simple and basic case for a lattice in solids.

Video: **Lecture05Video05.mp4**

Hamiltonian. We next express the Hamiltonian in terms of the Fourier modes. Insertion of

$$\begin{aligned} Q_j &= \frac{1}{\sqrt{2\mathscr{N}}} (DM)^{-\frac{1}{4}} \sum_q (e^{i\varepsilon q j} a_q + e^{-i\varepsilon q j} a_q^\dagger) \\ &= \frac{1}{\sqrt{2\mathscr{N}}} (DM)^{-\frac{1}{4}} \sum_q e^{i\varepsilon q j} \left(a_q + {a_{-q}}^\dagger\right), \end{aligned}$$

yields

$$H_{\text{nn}} = -\frac{B}{4\mathscr{N}} (DM)^{-\frac{1}{2}} \sum_j \sum_q \sum_{q'} e^{i\varepsilon q' j} e^{i\varepsilon q (j+1)} \left(a_q + {a_{-q}}^\dagger\right) \left(a'_q + {a_{-q'}}^\dagger\right).$$

We use the following identity for discrete Fourier transforms,

$$\sum_j e^{i\varepsilon (q+q') j} = \mathscr{N} \delta_{q,-q'},$$

which corresponds to the familiar continuum expression

$$\int dx\, e^{i(q+q')x} = 2\pi \delta(q+q').$$

One obtains

$$\begin{aligned} H_{\text{nn}} &= -b \sum_q e^{i\varepsilon q} \left(a_q + {a_{-q}}^\dagger\right) \left(a_{-q} + a_q^\dagger\right) \\ &= -b \sum_q \cos(\varepsilon q) \left(a_q + a_q^\dagger\right) \left(a_{-q} + a_{-q}^\dagger\right), \end{aligned}$$

with $b = \frac{B}{4} (DM)^{-\frac{1}{2}}$. Similarly, one has

$$H_0 = \omega_0 \sum_q \left(a_q^\dagger a_q + \frac{1}{2} \right).$$

Video: **Lecture05Video06.mp4**

Momentum modes. At this stage, the Hamiltonian H involves separate q-blocks,

$$H = \sum_q H_q,$$

with

$$H_q = \omega_0 \left(a_q^\dagger a_q + \frac{1}{2} \right) - b \cos(\varepsilon q) \left(a_q + {a_{-q}}^\dagger \right) \left(a_{-q} + a_q^\dagger \right).$$

Each block involves q and $-q$. What remains is the diagonalization of the q-blocks, done by the *Bogoliubov transformation*,

$$a_q = \alpha(q) A_q + \beta(q) A_{-q}^\dagger, \qquad a_q^\dagger = \alpha(q) A_q^\dagger + \beta(q) A_{-q},$$

where the commutation relations

$$[a_q, a_q^\dagger] = 1, \qquad [A_q, A_q^\dagger] = 1,$$

require

$$\alpha(q)^2 - \beta(q)^2 = 1.$$

Video: **Lecture05Video07.mp4**

The coefficients $\alpha(q)$ are determined such that the Hamiltonian is diagonal,

$$H = \sum_q \omega_q \left(A_q^\dagger A_q + \frac{1}{2} \right).$$

The algebra is straightforward and one finds for the squared frequencies of the independent oscillation modes

$${\omega_q}^2 = \frac{D}{M} \left(1 - \frac{B}{D} \cos(\varepsilon q) \right).$$

In the momentum basis the phonons are described as uncoupled harmonic oscillators, one for every momentum q. They are a *free quantum field*, which means that they do not interact with themselves.

Video: **Lecture05Video08.mp4**

Quantum field theory So far we have just presented the most basic notion for a quantum description of solids. Conceptually, this is simply a quantum theory for many degrees of freedom. Phonons are a simple example for a quantum field theory. No additional concepts need to be introduced. The so called "second quantisation" is nothing else than quantum mechanics for many degrees of freedom. The continuum limit, for which x becomes a continuous variable, does not introduce any qualitative changes.

Many properties of quantum field theories, as the role of the vacuum and particles as excitations of the vacuum, can already be seen for phonons. The vacuum obeys, as usual $A_q|0\rangle = 0$. This is not the same as for $B = 0$, where one has $a_q|0\rangle = 0$. The vacuum state depends on B. It can be a complicated object. For phonons it remains possible to construct the vacuum state explicitly. For more complicated quantum field theories this is, in general, no longer possible. Phonons are considered as excitations of the vacuum. These excitations are called *quasiparticles* or simply *particles*. Their properties depend on the vacuum, e. g. the dispersion relation depends on B. This concept plays an important role for elementary particle physics. For example, the mass of the electron depends on the expectations value of the Higgs field in the vacuum state. An important insight may be phrased in the simple term: "The vacuum is not nothing."

Video: **Lecture05Video09.mp4**

Dispersion relation. The relation between frequency and momentum,

$$\omega(q) = \omega_q = \sqrt{\frac{D - B\cos(\varepsilon q)}{M}},$$

is called the dispersion relation. Consider the limit of small εq, where one can expand, $\cos(\varepsilon q) = 1 - \frac{1}{2}\varepsilon^2 q^2$, such that

$$\omega^2(q) = \frac{D - B}{M} + \frac{\varepsilon^2 B}{2M} q^2.$$

In our units frequency and energy are identical, such that the dispersion relation corresponds to the energy momentum relation of the *phonon-quasi-particles*.

For $D > B$ the occupation relation has a gap, one needs positive energy even for a phonon with $q = 0$. For many cases the interaction between atoms is of the form $(Q_j - Q_{j-1})^2$, involving only the distance between two neighbouring atoms. Then $D = B$, phonons are gapless and the dispersion relation becomes linear for small εq. The sound velocity is given here by

$$v = \left|\frac{d\omega}{dq}\right| = \frac{\varepsilon^2 B q}{2M\omega(q)}.$$

Video: **Lecture05Video10.mp4**

Generalisations. In three dimensions $d = 3$ one has $q \to \vec{q}$ and the dispersion relation becomes an equation for $\omega(\vec{q})$. For real solids it depends on the particular structure of the lattice and the form of the interactions.

Continuum limit. The continuum limit can be taken for situations where the expectation values of the relevant observables and corresponding wave functions are sufficiently smooth. This means that their variation with x is small on scales of the order ε. Typically, this concerns properties dominated by modes with low momenta q. The continuum limit corresponds to the limit $\varepsilon q \to 0$.

Video: **Lecture05Video11.mp4**

Photons. For photons the dispersion relation is (in units where the velocity of light is unity, $c = 1$),

$$\omega(\vec{q}) = |\vec{q}|.$$

There are two photon helicities, related to polarisation. Photons are conceptually very similar to phonons. We will discuss them in more detail later.

Quantum fields for photons. For photons, associated quantum fields are the electric field $\vec{E}(\vec{q})$ in momentum space or $\vec{E}(\vec{x})$ in position space, as well as the magnetic field $\vec{B}(\vec{q})$ or $\vec{B}(\vec{x})$, respectively. In other words, the electric field $\vec{E}$ and the magnetic field $\vec{B}$ are quantum operators! The corresponding operator fields consist of operators for each $\vec{x}$ or for each $\vec{q}$. There is conceptually no difference to phonons.

Bosonic atoms without interaction. For free, non-relativistic atoms, the dispersion relation is given by

$$\omega(\vec{q}) = \frac{\vec{q}^2}{2M}.$$

For the grand-canonical ensemble, one includes a chemical potential, multiplying the total particle number. This shifts effectively

$$\omega(\vec{q}) \to \epsilon(\vec{q}) = \frac{\vec{q}^2}{2M} - \mu.$$

We will not distinguish $\omega(\vec{q})$ and $\varepsilon(\vec{q})$ unless stated otherwise.

General free quantum field theories. Formulated in momentum space, free quantum field theories are described by separate harmonic oscillators for each momentum mode q. The detailed microscopic origin of the Hamiltonian does not matter. All properties are encoded in the particular form of H_q, as the dispersion relation. Phonons, photons or bosonic atoms have all the same status. This extends to excitations or quasiparticles in many domains of physics.

4.2 Functional integral for quantum fields

Video: **Lecture05Video12.mp4**

In this part we introduce the functional integral for quantum fields. We discuss both thermodynamic equilibrium and the time evolution for given initial conditions. The mathematical formalism is very similar for both cases. They are distinguished by an important factor of i multiplying the action. While this is crucial for the physical behaviour, the mathematical treatment for both cases is identical. We can construct the functional integral simultaneously for the equilibrium situation and for quantum dynamics.

Free quantum boson gas in thermal equilibrium. We start with quantum statistics for free fields. Quantum statistics is distiguished from the classical statistics discussed in the previous lecture by the operator nature of the quantum fields. We will, nevertheless, derive a functional integral formulation involving only commuting objects. This formulation involves one additional dimension of "euclidean time".

For the Hamiltonian

$$H = \sum_q \omega(q) \left(a_q^\dagger a_q + \frac{1}{2} \right),$$

the partition function in thermal equilibrium is given by the trace

$$Z = \mathrm{Tr}\, e^{-\beta H},$$

with $\beta = \frac{1}{k_B T} = \frac{1}{T}$. (We use units for the Boltzmannn constant $k_B = 1$). It decays into independent factors for every momentum mode,

$$Z = \prod_q \mathrm{Tr}\, e^{-\beta\omega_q \left(a_q^\dagger a_q + \frac{1}{2}\right)} = \prod_q Z_q.$$

One only has to compute the individual Z_q,

$$Z = \mathrm{Tr}\, e^{-\tilde{\beta}\left(a^\dagger a + \frac{1}{2}\right)},$$

with $\tilde{\beta} = \beta\omega_q$ (we omit the index q). As an example, for a free gas of bosonic atoms one has

$$\omega(q) = \frac{\vec{q}^2}{2M} - \mu,$$

with chemical potential μ. The logarithm of the partition function is simply a momentum sum of the individual logarithms. From the logarithm of $Z(\beta,\mu)$ one can derive all thermodynamics of the quantum boson gas. This will be done in lecture 6 including interactions.

Video: **Lecture05Video13.mp4**

In this lecture we will derive a functional integral representation of the partition function

$$Z = \mathrm{Tr}\, e^{-\beta H} = \int D\phi\, e^{-S[\phi]},$$

with Euclidean action

$$S = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \sum_q \phi^*(\tau,q)\left(\frac{\partial}{\partial\tau} + \omega(q)\right)\phi(\tau,q).$$

The complex fields $\phi(\tau,q)$ are periodic,

$$\phi(\tau+\beta,q) = \phi(\tau,q).$$

In consequence, the euclidean time τ parameterizes a torus with circumference β.

Video: **Lecture05Video14.mp4**

Partition function with boundary conditions. We will derive the functional integral below. In order to do this in parallel for the dynamical evolution in quantum field theory we introduce a formal boundary term in the expression

$$\tilde{Z} = \mathrm{Tr}\left\{b\, e^{-\tilde{\beta}\left(a^\dagger a + \frac{1}{2}\right)}\right\}.$$

For $b=1$ one has $\tilde{Z} = Z$ for thermodynamic equilibrium if $\tilde{\beta} = \beta\omega$ is real. A more general boundary term b has no direct physical meaning for the thermal equilibrium state of phonons, photons or atoms. It is used here as a technical device which permits us to discuss the functional integral for a larger class of operator problems. The boundary term b is a matrix in Hilbert space. For example, in the occupation number basis one has

$$\tilde{Z} = b_{nm}\left(e^{-\tilde{\beta}\left(a^\dagger a + \frac{1}{2}\right)}\right)_{mn}.$$

We may take the "boundary term" b as a product of wave functions,

$$b_{nm} = (\psi_{\mathrm{in}})_n (\phi_{\mathrm{f}})_m,$$

such that

$$\begin{aligned}\tilde{Z} &= (\phi_{\mathrm{f}})_m \left(e^{-\tilde{\beta}\left(a^\dagger a + \frac{1}{2}\right)}\right)_{mn} (\psi_{\mathrm{in}})_n \\ &= \left\langle \phi_{\mathrm{f}} \left| e^{-\tilde{\beta}\left(a^\dagger a + \frac{1}{2}\right)} \right| \psi_{\mathrm{in}} \right\rangle.\end{aligned}$$

Extension to complex formulation. The trace is well defined also for complex values of $\tilde{\beta}$. In particular, we may consider purely imaginary $\tilde{\beta}$,

$$\tilde{\beta} = i\omega\Delta t.$$

We can also choose a complex boundary term b and admit complex wave functions $\phi_{\rm f}$ and $\psi_{\rm in}$. We employ the notation of quantum mechanics with $\langle\phi_{\rm f}|$ involving complex conjugation, e. g. $\langle\phi_{\rm f}|_m = (\phi_{\rm f}^*)_m$. In general, $\tilde{Z}$ will now be a complex number.

Video: **Lecture05Video15.mp4**

Transition amplitude. With this setting $\tilde{Z}$ is the transition amplitude for the quantum mechanics of an harmonic oscillator,

$$\begin{aligned}\tilde{Z} &= \langle\phi_{\rm f}|e^{-i\Delta t\omega\left(a^\dagger a+\frac{1}{2}\right)}|\psi_{\rm in}\rangle \\ &= \langle\phi_{\rm f}|e^{-i\Delta t H}|\psi_{\rm in}\rangle.\end{aligned}$$

The operator $e^{-i\Delta t H}$ is the evolution operator in quantum mechanics between an initial time t_{in} and a final time $t_f = t_{in} + \Delta t$. We associate the boundary wave functions with

$$\psi_{\rm in} = \psi(t_{\rm in}), \qquad \phi_f = \phi(t_{\rm f}),$$

In quantum mechanics the evolution operator relates the wave function at t_f to the initial wave function at t_{in}

$$\psi(t_{\rm f}) = e^{-i(t_{\rm f}-t_{\rm in})H}\psi(t_{\rm in}).$$

We can therefore also interpret the quantity $\tilde{Z}$ as the transition amplitude between ψ and ϕ at the common time $t_{\rm f}$,

$$\tilde{Z} = \langle\phi(t_{\rm f})|\psi(t_{\rm f})\rangle, \qquad \Delta t = t_{\rm f} - t_{\rm in}.$$

The square $|\tilde{Z}|^2$ measures the probability that a given $\psi(t_{\rm in})$ coincides at $t_{\rm f}$ with $\phi(t_{\rm f})$.

We can generalise the single harmonic oscillator to a free quantum field theory. The Hamiltonian is a sum over Hamiltonians for every momentum mode q. Then $H = \omega\left(a^\dagger a + \frac{1}{2}\right)$ stands for H_q. With total Hamiltonian being the sum of all H_q, the expression $\tilde{Z}$ is the transition amplitude for a free quantum field theory. Adding interactions the transition amplitude is a key element for the S-matrix for scattering to be discussed in coming lectures.

Video: **Lecture05Video16.mp4**

Split into factors. The trace can be evaluated by splitting $\tilde{\beta}$ into small pieces, and therefore $e^{-\tilde{\beta}H}$ into many factors. For the transition amplitude this factorizes the evolution operator into many evolution operators for small time steps. For thermal equilibrium there is no such intuitive interpretation for small steps in euclidean time. Nevertheless, the method of splitting Δt or $\tilde{\beta}$ into small steps is the same.

We demonstrate this method for a single harmonic oscillator. The split of $\tilde{\beta}$ into small steps is done by writing $\tilde{\beta} = (2N+1)\delta$, where $|\delta| \ll 1$. For convenience we assume N to be even. The factorization yields

$$\exp\left\{-\tilde{\beta}\left[a^\dagger a + \frac{1}{2}\right]\right\} = \prod_{j=-N}^{N} \exp\left\{-\delta\left[a^\dagger a + \frac{1}{2}\right]\right\}. \tag{4.1}$$

The splitting is a formal method and the index j has nothing to do with lattice sites or other physical objects. For large N or small δ, the exponential simplifies. This would not be necessary for the present very simple case, but is very useful for more complicated Hamiltonians which involve pieces that do not commute with each other.

Video: **Lecture05Video17.mp4**

The split will be used to define a functional integral. Indeed, the expression (4.1) looks already like a product of transfer matrices. We can take $N \to \infty$ such that approximations for small δ become exact. Let us define the operators

$$\hat{x} = \frac{1}{\sqrt{2}} \left(a^\dagger + a \right), \qquad \hat{p} = \frac{i}{\sqrt{2}} \left(a^\dagger - a \right),$$

with commutation relation

$$[\hat{x}, \hat{p}] = i.$$

The operators $\hat{x}$ and $\hat{p}$ have similar properties as position and momentum operators. In our context they are abstract operators, since for photons or phonons already $a^\dagger a$ stands for $a_q^\dagger a_q$ or $A_q^\dagger A_q$ in momentum space. Thus $\hat{x}$ and $\hat{p}$ have nothing to do with position and momentum of phonons or photons.

In terms of the operators $\hat{x}$, $\hat{p}$ one has

$$\hat{H} = a^\dagger a + \frac{1}{2} = \frac{\hat{p}^2}{2} + V(\hat{x}), \qquad V(\hat{x}) = \frac{\hat{x}^2}{2}.$$

This yields the expression

$$\exp\left\{ -\tilde{\beta} \left[a^\dagger a + \frac{1}{2} \right] \right\} = \prod_{j=-N}^{N} \exp\left\{ -\delta \left[\frac{\hat{p}^2}{2} + V(\hat{x}) \right] \right\},$$

where

$$\tilde{H} = \frac{\hat{p}^2}{2} + V(\hat{x}).$$

For a general function $V(\hat{x})$ this is the Hamiltonian for one-dimensional quantum mechanics in a potential V, with a factor $1/M$ incorporated in δ. Many steps below are valid for general V. Our treatment covers the path integral for a quantum particle in a potential.

Video: **Lecture05Video18.mp4**

Eigenfunctions of $\hat{x}$ and $\hat{p}$. We define eigenfunctions of the operators $\hat{x}$ and $\hat{p}$,

$$|x\rangle \qquad \text{such that} \qquad \hat{x}|x\rangle = x|x\rangle,$$

and

$$|p\rangle \qquad \text{such that} \qquad \hat{p}|p\rangle = p|p\rangle.$$

Here x and p are continuous variables. We can choose a normalization such that

$$\langle x'|x\rangle = \delta(x' - x), \qquad \langle p'|p\rangle = 2\pi\delta(p' - p),$$

and

$$\int dx\,|x\rangle\langle x| = \mathbb{1}, \qquad \int \frac{dp}{2\pi}\,|p\rangle\langle p| = \mathbb{1}.$$

We insert complete systems of eigenfunctions between each of the factors,

$$\prod_{j=-N}^{N} e^{-\delta\tilde{H}} = \left[\prod_{j=-N}^{N+1} dx_j\right] |x_{N+1}\rangle\langle x_{N+1}|e^{-\delta\tilde{H}}|x_N\rangle\langle x_N|\cdots|x_{1-N}\rangle\langle x_{1-N}|e^{-\delta\tilde{H}}|x_{-N}\rangle\langle x_{-N}|.$$

Video: **Lecture05Video19.mp4**

Evaluation of factors. The factors $\langle x_{j+1}|e^{-\delta\tilde{H}}|x_j\rangle$ are complex numbers, no longer operators. For their computation it is convenient to insert a complete set of $\hat{p}$ -eigenstates,

$$\langle x_{j+1}|e^{-\delta\tilde{H}}|x_j\rangle = \int \frac{dp_j}{2\pi}\langle x_{j+1}|p_j\rangle\langle p_j|e^{-\delta\tilde{H}}|x_j\rangle.$$

We next use for $\delta \to 0$ the expansion

$$\exp\left\{-\delta\left[\tfrac{\hat{p}^2}{2} + V(\hat{x})\right]\right\} = \exp\left\{-\delta\tfrac{\hat{p}^2}{2}\right\}\exp\left\{-\delta V(\hat{x})\right\} + \mathcal{O}(\delta^2),$$

where the term $\sim \mathcal{O}(\delta^2)$ arises from the commutator of $\hat{x}$ and $\hat{p}$. Corrections $\sim \delta^2$ can be neglected for $\delta \to 0$ such that

$$\langle x_{j+1}|e^{-\delta\tilde{H}}|x_j\rangle = \int \frac{dp_j}{2\pi} e^{-\delta\frac{p_j^2}{2}} e^{-\delta V(x_j)}\langle x_{j+1}|p_j\rangle\langle p_j|x_j\rangle.$$

No operators appear anymore in this expression and we only need

$$\langle p_j|x_j\rangle = e^{-ip_jx_j}, \qquad \langle x_{j+1}|p_j\rangle\langle p_j|x_j\rangle = e^{ip_j(x_{j+1}-x_j)}.$$

This yields the expression

$$\langle x_{j+1}|e^{-\delta\tilde{H}}|x_j\rangle = \int \frac{dp_j}{2\pi}\exp\left\{ip_j(x_{j+1}-x_j) - \delta\left[\tfrac{p_j^2}{2} + V(x_j)\right]\right\}.$$

Functional integral. Insertion of these factors yields

$$e^{-\tilde{\beta}\tilde{H}} = \int dx_{-N}\int dx_{N+1}|x_{N+1}\rangle\; F\;\langle x_{-N}|,$$

with

$$F = \int D\phi'\exp\left\{\sum_{j=-N}^{N}\left[ip_j(x_{j+1}-x_j) - \delta\tfrac{p_j^2}{2} + \delta V(x_j)\right]\right\},$$

and functional measure

$$\int D\phi' = \left[\prod_{j=-N+1}^{N}\int_{-\infty}^{\infty} dx_j\right]\left[\prod_{j=-N}^{N}\int_{-\infty}^{\infty}\frac{dp_j}{2\pi}\right].$$

With boundary terms one obtains

$$\langle\phi_{\rm f}|e^{-\tilde{\beta}\tilde{H}}|\psi_{\rm in}\rangle = \int dx_{-N}\int dx_{N+1}\langle\phi_f|x_{N+1}\rangle\; F\;\langle x_{-N}|\psi_{\rm in}\rangle.$$

Video: **Lecture05Video20.mp4**

Summary. In conclusion, we have transformed the operator trace into a functional integral

$$\tilde{Z} = \mathrm{Tr}\left\{ b\, e^{-\tilde{\beta}(a^\dagger a + \frac{1}{2})} \right\} = \langle \phi_{\mathrm{f}} | e^{-\tilde{\beta}\tilde{H}} | \psi_{\mathrm{in}} \rangle = \int D\phi\, e^{-S}.$$

The action is given by

$$S = -\sum_{j=N}^{N} \left\{ i p_j (x_{j+1} - x_j) - \delta \left[\tfrac{p_j^2}{2} + V(x_j) \right] \right\},$$

and the integration measure reads

$$\int D\phi = \left[\prod_j \int dx_j \int \frac{dp_j}{2\pi} \right].$$

The boundary factor $\tilde{b}$ has the form

$$\tilde{b} = \int dx_{-N} \int dx_{N+1} \langle \phi_f | x_{N+1} \rangle \langle x_{-N} | \psi_{\mathrm{in}} \rangle.$$

From this expression Feynman's path integral obtains by performing the Gaussian integration over the variables p_j. What remains is an integral over all possible paths

$$\int Dx[t] = \prod_j \int dx_j.$$

This is not the direction we follow in this lecture. We rather develop a formulation with complex variables. This can then easily be extended to a field theory.

Video: **Lecture06Video01.mp4**

4.3 Thermodynamic equilibrium

In this section we discuss the thermal equilibrium state for a single quantum harmonic oscillator. This is a first example for the approach to quantum statistical equilibrium that can later be generalised to quantum field theories with interactions. For thermodynamic equilibrium, $Z = \mathrm{Tr}\, e^{-\tilde{\beta}\tilde{H}}$, one identifies x_{N+1} with x_{-N} and includes no integration over x_{N+1}. The variable j is periodic, reflecting in

$$x_{N+1} = x_{-N}, \qquad p_{N+1} = p_{-N}.$$

Formally, this can be achieved by choosing for $\tilde{b}$ a δ - function. For periodic boundary conditions one has

$$\tilde{Z} = \mathrm{Tr}\, e^{\tilde{\beta}\tilde{H}} = \int D\phi\, e^{-S},$$

with

$$S = -\sum_{j=N}^{N} \left\{ i p_j (x_{j+1} - x_j) - \delta \left[\tfrac{p_j^2}{2} + V(x_j) \right] \right\},$$

and

$$\int D\phi = \left[\prod_j \int dx_j \int \frac{dp_j}{2\pi} \right].$$

There are a total of $2N + 1$ factors, and δ is related to $\tilde{\beta}$ by $\delta = \frac{\tilde{\beta}}{2N+1}$.

Video: **Lecture06Video02.mp4**

Matsubara sum. Quantum statistics is described by the so called Matsubara formalism. We derive this formalism here for a single harmonic oscillator, with straightforward generalisations. We can diagonalize the action S by a type of Fourier transform

$$x_j = \sum_{n=-N}^{N} \exp\left(\frac{2\pi i n j}{2N+1}\right) \tilde{x}_n, \qquad \tilde{x}_{-n} = \tilde{x}_n^*,$$

$$p_j = \sum_{n=-N}^{N} \exp\left(\frac{2\pi i n (j+\frac{1}{2})}{2N+1}\right) \tilde{p}_n, \qquad \tilde{p}_{-n} = \tilde{p}_n^*,$$

such that

$$-\sum_{j=-N}^{N} [i p_j (x_{j+1} - x_j)] = \sum_{n=-N}^{N} [(2N+1)\sin\left(\frac{\pi n}{2N+1}\right) (\tilde{p}_n^* \tilde{x}_n - \tilde{p}_n \tilde{x}_n^*)].$$

Here we use the identity ($j = -N$ and $j = N+1$ identified)

$$\sum_{j=-N}^{N} \exp\left(\frac{2\pi i (m-n) j}{2N+1}\right) = (2N+1)\delta_{m,n}.$$

Similarly, with $V(x_j) = x_j^2/2$, one has

$$\frac{\delta}{2} \sum_{j=-N}^{N} (x_j^2 + p_j^2) = \frac{(2N+1)\delta}{2} \sum_{n=-N}^{N} (\tilde{x}_n^* \tilde{x}_n + \tilde{p}_n^* \tilde{p}_n) = \frac{\tilde{\beta}}{2} \sum_{n=-N}^{N} (\tilde{x}_n^* \tilde{x}_n + \tilde{p}_n^* \tilde{p}_n).$$

The action becomes a sum over independent pieces, labelled by n. The sum over n is the Matsubara sum.

Video: **Lecture06Video03.mp4**

Complex fields. We next introduce complex numbers ϕ_n by

$$\tilde{x}_n = \frac{1}{\sqrt{2}}(\phi_n + \phi_{-n}^*), \qquad \tilde{p}_n = -\frac{i}{\sqrt{2}}(\phi_n - \phi_{-n}^*),$$

With

$$\tilde{p}_n^* \tilde{x}_n - \tilde{x}_n^* \tilde{p}_n = i(\phi_n^* \phi_n - \phi_{-n}^* \phi_{-n}),$$

and

$$\tilde{x}_n^* \tilde{x}_n + \tilde{p}_n^* \tilde{p}_n = \phi_n^* \phi_n + \phi_{-n}^* \phi_{-n},$$

we finally obtain for the action

$$S = \sum_{n=-N}^{N} \left[2(2N+1) i \sin\left(\frac{\pi n}{(2N+1)}\right) + \tilde{\beta}\right] \phi_n^* \phi_n.$$

The modes ϕ_n are called Matsubara modes, and the sum over n is the Matsubara sum.

One can also translate the integration measure for the variables x_j and p_j to ϕ_n. With

$$\phi_n = \phi_{nR} + i\phi_{nI},$$

one has

$$\int D\phi = \prod_n \left(\int_{-\infty}^{\infty} d\phi_{nR} \int_{-\infty}^{\infty} d\phi_{nI}\right).$$

All variable transformations have been linear transformations and there is no non-trivial Jacobian. Recall that an overall constant factor of Z or additive constant in S is irrevelant.

Video: **Lecture06Video04.mp4**

Matsubara frequencies. At the end we take the limit $N \to \infty$. In this limit the neglected terms (from commutators of $\hat{x}$ and $\hat{p}$) vanish. This yields the central functional integral equation for thermodynamic equilibrium,

$$\mathrm{Tr}\{e^{-\beta H}\} = \int D\phi \; e^{-S}.$$

For $H = \omega(a^\dagger a + \frac{1}{2})$ one has

$$S = \sum_{n=-\infty}^{\infty} (2\pi i n + \beta\omega)\phi_n^* \phi_n.$$

(Recall that $\tilde{H} = a^\dagger a + \frac{1}{2}$ and $\tilde{\beta} = \beta\omega$.) The quantities

$$\tilde{\omega}_n = \frac{2\pi n}{\beta} = 2\pi n T$$

are called *Matsubara frequencies*.

Video: **Lecture06Video05.mp4**

Action for free quantum fields. This result extends directly to a free quantum field theory. The partition function Z factorises for the different momentum modes, $Z = \prod_q Z_q$, and correspondingly the action for all momentum modes is simply the sum of actions for individual momentum modes, $S = \sum_q S_q$. For a given momentum mode one has $\tilde{\beta} = \beta\omega_q$. Thus for

$$H = \sum_q \omega(q) \left[a_q^\dagger a_q + \frac{1}{2} \right],$$

one obtains

$$\begin{aligned} S &= \sum_n \sum_q \left[2\pi i n + \beta\omega(q) \right] \phi_n^*(q)\phi_n(q) \\ &= \sum_n \sum_q \beta \left[i\tilde{\omega}_n + \omega(q) \right] \phi_n^*(q)\phi_n(q). \end{aligned}$$

One often denotes the *dispersion relation* by $\omega(q)$ or by $\varepsilon(q)$. For non-relativistic particles the Matsubara frequencies

$$\tilde{\omega}_n = \frac{2\pi n}{\beta} = 2\pi n T$$

multiply a term quadratic in the Matsubara modes. At this point we have formulated the thermodynamics of phonons or atoms as a functional integral. It is gaussian and can easily be solved explicitely.

The solution of this functional integral is well known. It is the expression of the partition function in terms of mean occupation numbers, as derived in the course on theoretical statistical physics. It is a worthwhile exercise to reproduce this result by solving the functional integral. This involves suitable Matsubara sums. It is actually easier to compute derivatives as the mean energy.

Video: **Lecture06Video06.mp4**

Euclidean time. We can consider the Matsubara modes ϕ_n as the modes of a discrete Fourier transformation. Indeed, making a Fourier transformations of functions on a circle yields discrete modes. Consider a function $\phi(\tau)$, with τ parameterizing a circle with circumference β. Equivalently, we can take τ to be a periodic variable with period β

$$\tau + \beta \equiv \tau.$$

The Fourier expansion reads

$$\phi(\tau) = \sum_n \exp\left(\frac{2\pi i n\tau}{\beta}\right)\phi_n,$$

with integer n. With

$$\begin{aligned}\partial_\tau\phi(\tau) &= \sum_n \left(\frac{2\pi i n}{\beta}\right)\exp\left(\frac{2\pi i n\tau}{\beta}\right)\phi_n \\ &= \sum_n i\tilde{\omega}_n \exp\left(\frac{2\pi i n\tau}{\beta}\right)\phi_n,\end{aligned}$$

one has

$$\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau\,\{\phi^*(\tau)\partial_\tau\phi(\tau)\} = \sum_n i\tilde{\omega}_n\,\phi_n^*\phi_n.$$

Here we employ the identity for discrete Fourier transforms

$$\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \exp\left(\frac{2\pi i(n-m)\tau}{\beta}\right) = \beta\,\delta_{m,n}.$$

In this basis the action reads

$$S = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \sum_q \left[\phi^*(\tau,q)\,\partial_\tau\phi(\tau,q) + \omega(q)\,\phi^*(\tau,q)\phi(\tau,q)\right].$$

One calls τ the *Euclidean time*. The Fourier modes depend on an additional periodic variable - namely euclidean time.

We can also write formally the functional measure as $\int D\phi(\tau)$. This is the same as the integral over all Matsubara modes ϕ_n. Every periodic function can be specified by the coefficients of the discrete Fourier representation ϕ_n. Integration over all ϕ_n covers the space of all periodic functions. We can use this for a well defined functional integral $\int D\phi(\tau)$. For a finite range of n, $-N_M < n < N_M$, we have a finite dimensional integral. At the end we take the limit $N_M \to \infty$.

Video: **Lecture06Video07.mp4**

Local action and transfer matrix. This action is a *local action* in the sense of lectures 2 and 3. Discretizing τ on a lattice with distance ε, and with $\tau = j\varepsilon$, $j = -N\cdots N$ periodic, $\epsilon = \frac{\beta}{2N+1}$, the partial derivative is replaced by a lattice derivative

$$\partial_\tau\phi(\tau) = \frac{1}{\varepsilon}\left[\phi(\tau+\varepsilon) - \phi(\tau)\right],$$

One can write (with $\sum_\tau \equiv \sum_j$)

$$S = \sum_\tau \mathscr{L}(\tau),$$

with

$$\mathscr{L}(\tau) = \frac{1}{2}\sum_q \left\{\phi(\tau+\varepsilon)\phi^*(\tau) - \phi^*(\tau+\varepsilon)\phi(\tau) + \varepsilon\omega(q)\left[\phi^*(\tau+\varepsilon)\phi(\tau) + \phi(\tau+\varepsilon)\phi^*(\tau)\right]\right\}.$$

Here we omit the label q for the momentum modes. Note that $\mathscr{L}(\tau)$ is a complex function of complex variables $\phi(\tau)$ and $\phi(\tau+\varepsilon)$. With respect to τ the action involves *next neighbour interactions*, similar to the Ising model. We could go the inverse way and compute the transfer matrix. We know already the answer in the bosonic occupation number basis

$$\hat{T} = \exp\left[-\frac{\beta}{2N+1}\sum_q \omega(q)\left(a_q^\dagger a_q + \frac{1}{2}\right)\right],$$

with $2N+1$ the number of *time points*. This is compatible with

$$Z = \mathrm{Tr}\left\{\hat{T}^{2N+1}\right\}.$$

This closes the circle to our first approach. We could start with the functional integral, derive the transfer matrix, and define the partition function as a product of transfer matrices.

Video: **Lecture06Video08.mp4**

Quantum gas of bosonic atoms. For free bosonic atoms (without internal degrees of freedom) the dispersion relation is

$$\epsilon(q) = \frac{\vec{q}^2}{2M} - \mu,$$

with μ the chemical potential. We can make a Fourier-transform to three-dimensional position space,

$$S = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \int d^3x \{\phi^*(\tau,\vec{x})\partial_\tau\phi(\tau,x) + \frac{1}{2M}\vec{\nabla}\phi^*(\tau,\vec{x})\vec{\nabla}\phi(\tau,\vec{x}) - \mu\phi^*(x)\phi(x)\}$$

This is the action of a *field theory* in Euclidean time.

For a quantum field theory the action defines the weight factor in a functional integral. The extremum of the action yields the "classical field equation". This classical field equation is, however, a "microscopic" object. The field equations that are valid for a quantum field theory have to include the effects of fluctuations!

Video: **Lecture06Video09.mp4**

Interactions. So far we have discussed models that represent quantum fields without interactions. This is a very good approximation for phonons if the energy is not too high. Free quantum field theories can be represented in momentum space as uncoupled harmonic oscillators. For them the description is simple both in the functional integral formalism (gaussian integration) and in the operator formalism. The situation changes in the presence of interactions.

Consider a pointlike interaction between bosonic atoms.

$$H = H_0 + H_{int}$$

$$H_0 = \sum_q \omega(q)\left(a_q^\dagger a_q + \frac{1}{2}\right)$$

$$H_{int} = \frac{\lambda}{2} \sum_{q_1,q_2,q_3,q_4} a^\dagger_{q_4} a^\dagger_{q_3} a_{q_2} a_{q_1} \delta(q_1 + q_2 - q_3 - q_4).$$

Two atoms with momentum q_1 and q_2 are annihilated, two atoms with momenta q_3 and q_4 are created. Momentum conservation is guaranteed by the δ-function.

For the functional integral this adds to the action a piece

$$S_{int} = \tfrac{\lambda}{2} \int d\tau \int d^3x [(\phi^*(\tau,\vec{x})\phi(\tau,\vec{x}))^2 - 2\delta\mu\phi^*(\tau,\vec{x})\phi(\tau,\vec{x})]$$

with $\delta\mu \sim \lambda$ a *counterterm* that corrects μ. The additional interaction term is is the only modification needed for the functional integral! Euclidean time remains periodic with period β, and this is the only point where the value of the temperature enters. We will not perform here a derivation of the Matsubara formalism in the presence of interactions. Starting from the operator formalism one can divide β into small pieces and work with a basis of "coherent states". This cuts short the various transformations that we have performed for the free theory. We will simply take the functional integral in euclidean time as a starting point.

For an interacting gas of bosonic atoms the functional integral permits us to investigate phenomena as the Bose-Einstein condensation and the associated superfluidity in dependence on temperature and particle number density or chemical potential. For atoms at ultracold temperature this is a very interesting topic both for experiment and theory.

A systematic treatment of interactions beyond a perturbative expansion in small λ is rather hard in the operator formalism. For the functional integral formulation powerful methods are available. This is one of the main reasons why we concentrate on the functional integral.

Video: **Lecture06Video10.mp4**

Zero temperature limit. For $T \longrightarrow 0$ one has $\beta \longrightarrow \infty$. The circumference of the circle goes to infinity. Instead of discrete Matsubara modes one has continuous modes with frequency $\tilde{\omega} = q_0$ and therefore a continuous four-dimensional momentum integral. The momenta q_0 and $\vec{q}$ appear, however differently in the action. The same holds for the dependence of S on τ and $\vec{x}$. There is a first derivative with respect to τ, but a squared first derivative or second derivative with respect to $\vec{x}$. This difference will go away for relativistic particles. For bosonic atoms with a pointlike interaction one finds for the action in Fourier space for the $T \longrightarrow 0$ limit of the thermal equilibrium state

$$S = \int_q \phi^*(q) \left(i\tilde{\omega} + \frac{\vec{q}^2}{2M} - \mu + \lambda\delta\mu \right) \phi(q) + \frac{\lambda}{2} \int_{q_1} \int_{q_2} \int_{q_3} \int_{q_4} \phi^*(q_4)\phi^*(q_3)\phi(q_2)\phi(q_1)\delta(q_4 + q_3 - q_2 - q_1),$$

where we have chosen an appropriate *continuum normalization* of $\phi(q)$, with

$$\phi(q) \equiv \phi(\tilde{\omega}, \vec{q})$$

$$\int_q = \frac{1}{(2\pi)^4} \int d\tilde{\omega} d^3\vec{q}$$

$$\delta(q) = (2\pi)^4 \delta(\omega)\delta(q_1)\delta(q_2)\delta(q_3).$$

The δ function expresses conservation of the euclidean four momentum q. It reflects translation symmetry in space and euclidean time τ. The limit $T \longrightarrow 0$ can be associated in some sense with the vacuum, if one chooses μ such that the mean particle number vanishes.

Video: **Lecture06Video11.mp4**

Summary. At this stage we have established an important starting point for our lecture based on the functional integral. The functional integral can describe both classical statistical thermodynamic equilibrium and quantum statistical thermodynamic equilibrium. Different models or different microphysical laws are encoded in the particular form of the action. This form is often largely dictated by symmetry. The "fundamental laws" are formulated in terms of the action. It is often not necessary to know the precise form of the Hamiltonian in the operator formalism for quantum systems, or the precise form of the transfer matrix for classical probabilistic systems. This is an important advantage, since the operator formalism can become quite complicated for interacting many body systems.

The lecture is called "quantum field theory", but you may realise that the quantum aspects are actually not crucial. What counts are the presence of fluctuations. The origin of the fluctuations, be it quantum fluctuations or thermal fluctuations or both, is not important. A more adapted name for our lecture could be "probabilistic field theory". We stick to the traditional name of quantum field theory for historical reasons. It should also be clear that our treatment applies to arbitrary settings with fluctuations. Fluctuations may be market fluctuations in economy or fluctuations in the reproduction of species in biology. Whenever a system is described by a probability distribution there exists an associated action.

Video: **Lecture07Video01.mp4**

4.4 Real time evolution

The functional integral can also be employed for the time evolution of quantum systems. This is typically a problem with boundary conditions. An initial condition for the quantum state is given at some initial time t_{in} by the wave function $\psi(t_{\rm in})$. This wave function develops in time according to the unitary evolution in quantum mechanics and arrives at some final time t_f at

$$\psi(t_f) = U(t_f, t_{in})\psi(t_{in}).$$

For a time-independent Hamiltonian H the evolution operator obeys

$$U(t_f, t_{in}) = U(t_f - t_{in}) = e^{-i(t_f - t_{\rm in})H}.$$

We are interested in the transition amplitude to some different final wave function $\phi(t_f)$.

Video: **Lecture07Video02.mp4**

We want to derive the functional integral for the transition amplitude

$$\langle\phi(t_f)|\psi(t_f)\rangle = \langle\phi(t_f)|U(t_f - t_{\rm in})|\psi(t_{\rm in})\rangle = \langle\phi(t_f)|e^{-i(t_f - t_{\rm in})H}|\psi(t_{\rm in})\rangle.$$

Recalling our formulation of thermal equilibrium with boundary conditions and its extension to a complex formulation, the transition element can be obtained from thermal equilibrium by the replacement

$$\beta \to i(t_f - t_{\rm in}).$$

The split into infinitesimal pieces, Fourier-transforms etc can be done for complex β in the same way as before. For $\beta \to \infty$ ($T \to 0$), $t_f - t_{\rm in} \to \infty$ one finds

$$\langle \phi(t_f) | \psi(t_f) \rangle = B(t_f, t_{\rm in}) Z_t$$

$$Z_t = \int D\phi \, \exp(-S_t).$$

For the sake of clarity we denote by S_t the action for the dynamical time evolution, in contrast to S_{eq} for the thermal equilibrium. For obtaining S_t from S_{eq} we have to multiply the terms $\sim \beta$ by i before taking the limit $\beta \longrightarrow \infty$. The term $i\tilde{\omega}$ remains unchanged, while all other parts in the action get multiplied by i. This results in

$$S_t = \int_q \left[\phi^*(q) \left[i\tilde{\omega} + i \left(\frac{\vec{q}^2}{2M} - \mu + \lambda \delta\mu \right) \right] \phi(q) \right.$$
$$\left. + i\frac{\lambda}{2} \int_{q_1} \int_{q_2} \int_{q_3} \int_{q_4} \phi^*(q_4) \phi^*(q_3) \phi(q_2) \phi(q_1) \delta(q_3 + q_4 - q_1 - q_2) \right].$$

Video: **Lecture07Video03.mp4**

After a Fourier-transform in $\tilde{\omega}$ and $\vec{q}$ one finds, with time labelled now by t

$$S_t = \int_x \left[\phi^*(x) \partial_t \phi(x) + \frac{i}{2M} \left(\vec{\nabla} \phi^*(x) \right) \left(\vec{\nabla} \phi(x) \right) + \frac{i\lambda}{2} (\phi^*(x)\phi(x))^2 - (\mu - \lambda\delta\mu) \phi^*(x) \phi(x) \right] \quad (4.2)$$

where

$$x = (t, \vec{x}), \qquad \int_x = \int_{-\infty}^{\infty} dt \int d^3\vec{x}.$$

The transfer matrix for this functional integral is now

$$\hat{T}_t = \exp \left[-\frac{i(t_{\rm f} - t_{\rm in})}{(2N+1)} H \right],$$

instead of

$$\hat{T_{\rm eq}} = \exp \left[-\frac{\beta}{(2N+1)} H \right].$$

The matrix $\hat{T}_t$ is a unitary matrix if the Hamiltonian is hermitean, $H^\dagger = H$.

Video: **Lecture07Video04.mp4**

Local Physics. For observations and experiments done in some time involved around t the details of boundary conditions at t_f and $t_{\rm in}$ play no role for large $|t_f - t|$ and $|t - t_{\rm in}|$. Doing physics now is not much influenced by what happened precisely to the dinosaurs or what will happen in the year 10000. For many purposes the boundary term $B(t_f, t_{\rm in})$ is just an irrelevant multiplicative factor in Z which drops out from the expectation values of interest. One can then simply omit it and work directly with Z_t.

Video: **Lecture07Video05.mp4**

Minkowski action. We define the Minkowski action S_M by multiplying the euclidean action S with a factor i

$$S_M = iS, \qquad e^{-S} = e^{iS_M}.$$

This can be done both for S_t and S_{eq}. For S_t the Minkowski action reads

$$S_{M,t} = -\int_x \phi^* \left(-i\partial_t - \frac{\Delta}{2M}\right)\phi + \frac{\lambda}{2}(\phi^*(x)\phi(x))^2 + \dots.$$

Variation of $S_{M,t}$ or S_t with respect to ϕ^* yields for $\lambda = 0$ the Schrödinger equation for the wave function of a single free particle

$$\left(-i\partial_t - \frac{\Delta}{2M}\right)\phi = 0.$$

Video: **Lecture07Video06.mp4**

There is a reason for that, but the connection needs a few steps, concentrating on single particle states. Recall that the functional integral describes arbitrary particle numbers, such that one-particle states are only special cases. For $\lambda \neq 0$ the *classical field equation* $\frac{\delta S}{\delta \phi^*(x)} = 0$ is a non-linear equation, called Gross-Pitaevskii equation

$$i\partial_t \phi = -\frac{\Delta}{2M}\phi + \lambda(\phi^*\phi)\phi - (\mu - \lambda\delta\mu)\phi.$$

This is not a linear Schrödinger equation for a quantum wave function, but has a different interpretation. An equation of this type can describe the dynamics of a Bose-Einstein condensate.

Video: **Lecture08Video02.mp4**

Analytic continuation. Let us replace in the action S_t (4.2) the time coordinate

$$t = -i\tau$$

such that the integration becomes

$$\int_x = -i\int d\tau d^3\vec{x}.$$

For the time derivative term we have

$$\partial_t \phi = i\partial_\tau \phi.$$

This replacement is called "analytic continuation". The analytic continuation of S_t is the action S_{eq} for thermal equilibrium at zero temperature,

$$S_t \to S_{eq} = \int d\tau d^3x \left\{\phi^* \left(\partial_\tau - \frac{\Delta}{2M} - \mu\right)\phi + \frac{\lambda}{2}(\phi^*\phi)^2 + \lambda\delta\mu\phi^*\phi\right\}.$$

Analytic continuation can be done in both ways. The actions S_t and S_{eq} for two models, one for the time evolution, the other for the $T = 0$ limit of thermal equilibrium, are related by analytic continuation.

Note that S_M is not the analytic continuation of S, but rather related to S by a fixed definition. The sign of S_M is of historical origin. The Minkowski action $S_{M,t}$ is a real expression. In consequence, $e^{iS_{M,t}}$ is a phase. This is a profound change as compared to the situation for thermal

equilibrium, for which $e^{iS_{eq}} = e^{-S_{eq}}$ is a positive real quantity that can be associated to a probability distribution. The functional integral for the time evolution of quantum systems is described by an integration over phases. This is directly related to the unitary evolution in quantum mechanics. The transfer matrix $\hat{T}_t$ is a unitary matrix. No boundary information is lost, in contrast to the thermal equilibrium state, for which we have seen for the Ising chain how the memory of boundary information is lost in the bulk.

Video: **Lecture08Video03.mp4**

Fourier transform. For the Fourier transformation into frequency space we employ

$$\tilde{\omega}\tau = \omega_M t = -i\omega_M \tau.$$

This defines the Minkowski frequency

$$\omega_M = i\tilde{\omega} = q_0,$$

where q_0 is the zero-component of the four-momentum q_μ. Analytic continuation in time translates to analytic continuation in frequency or four-momentum between $\tilde{\omega}$ and ω_M.

The analytic continuation in momentum space is a very useful tool for the evaluation of correlation functions. One can first compute the correlation functions in "euclidean space", which corresponds to the $T \to 0$ limit of thermal equilibrium. This has the advantage that powerful methods can be used as, for example, numerical simulations. The correlation functions in momentum space depend on $\tilde{\omega}$. Subsequently, they can be continued analytically to Minkowski space, with replacement rules for the frequencyies

$$\tilde{\omega} \to -iq_0.$$

For the squared frequency one finds, using the Minkowski metric for raising and lowering indices, $\eta_{00} = -1$,

$$\tilde{\omega}^2 \to -(q_0)^2 = -(q^0)^2 = q^0 q^0 \eta_{00} = q^0 q_0.$$

For a relativistic theory one has

$$q_E^2 = \tilde{\omega}^2 + \vec{q}^2 \to q^0 q_0 + q^i q_i = q^\mu q_\mu = -(q^0)^2 + \vec{q}^2 = q_M^2,$$

and analytic continuation corresponds to

$$q_E^2 \to q_M^2.$$

For a vacuum state with translation and rotation symmetry the two-point correlation function can only depend on the invariant $q^2 = q^\mu q_\mu$. Only the meaning of $q^2 = q^\mu q_\mu$ differs between euclidean and Minkowski signature. For euclidean signature the zero-index is lowered by δ_{00}, while for Minkowski signature one employs η_{00}. Thus analytic continuation can also be formulated as an analytic continuation in the metric. For euclidean signature q^2 is invariant under $SO(4)$-rotations in four-dimensional euclidean space, while for Minkowski signature the Lorentz symmetry $SO(1,3)$ leaves q^2 invariant. Many properties can be understood by viewing momenta in the complex plane, for which analytic continuation can be formulated as a continuous rotation of q_0.

4.5 Expectation values of time ordered operators

Video: **Lecture08Video04.mp4**

So far we have established for the partition function a map between the operator formalism and the functional integral. This extends to the expectation values of observables. For the functional integral formulation expectation values are directly found by inserting the observable in the functional integral. An observable is a functional of the fields for which the functional integral is formulated. It is a c-number and no non-commuting structures are present at this level. The definition of the expectation value of an observable $A[\phi]$ holds independently of the particular form of the action,

$$\langle A\rangle = \frac{1}{Z}\int D\phi\, e^{-S[\phi]} A[\phi].$$

In particular, it is valid both for euclidean and Minkowski signature of the metric.

We have seen in sect.3.2 how operators can be associated to observables. They allow us to express expectation values in the functional integral by time-ordered products of Heisenberg operators. We will next establish the inverse direction and show how the expectation values of time ordered operators in the operator formalism translate to the functional integral expression. At the stage where we are this should no longer be a surprise. Nevertheless, we perform this step here, repeating partly the construction of the functional integral from the operator formalism. This provides for a link to many textbooks where the functional integral expression in introduced in this way.

Heisenberg picture in quantum mechanics. We briefly recapitulate the Heisenberg picture in quantum mechanics. While in the Schrödinger picture the wave function evolves and the operators are constant, in the Heisenberg picture the operators evolve instead. The central objects are $\hat{A}_H(t)$, the Heisenberg operators that depend on time. One can write them as

$$\hat{A}_H(t) = U^\dagger(t,t_{\rm in})\hat{A}_S U(t,t_{\rm in}),$$

where $\hat{A}_S$ is the operator in the Schrödinger picture. Consider for $t_2 \geq t_1$

$$\hat{A}_H(t_2)\hat{B}_H(t_1) = U^\dagger(t_2,t_{\rm in})\,\hat{A}_S\,U(t_2,t_{\rm in})U^\dagger(t_1,t_{\rm in})\,\hat{B}_S\,U(t_1,t_{\rm in}),$$

and use

$$U^\dagger(t_1,t_2) = U(t_2,t_1),$$

as well as

$$U(t_3,t_2)U(t_2,t_1) = U(t_3,t_1).$$

With

$$U(t_2,t_{\rm in})U^\dagger(t_1,t_{\rm in}) = U(t_2,t_1)U(t_1,t_{\rm in})U^\dagger(t_1,t_{\rm in}) = U(t_2,t_1),$$

one has

$$\hat{A}_H(t_2)\hat{B}_H(t_1) = U^\dagger(t_2,t_{\rm in})\,\hat{A}_S\,U(t_2,t_1)\,\hat{B}_S\,U(t_1,t_{\rm in}).$$

In the Heisenberg picture, one keeps the wave function fixed $|\psi\rangle = |\psi(t_{\rm in})\rangle$ and describes the time evolution by the time-dependence of the Heisenberg operators.

The transition amplitude for two time-ordered Heisenberg operators, where the larger t-argument stands on the left, is defined by

$$\langle\phi(t_{\rm in})|\hat{A}_H(t_2)\hat{B}_H(t_1)|\psi(t_{\rm in})\rangle = \langle A(t_2)B(t_1)\rangle_{\phi\psi}.$$

It reads in the Schrödinger picture

$$\begin{aligned}\langle A(t_2)B(t_1)\rangle_{\phi\psi} &= \langle\phi(t_{\rm in})|U^\dagger(t_2,t_{\rm in})\,\hat{A}_S\,U(t_2,t_1)\,\hat{B}_S\,U(t_1,t_{\rm in})|\psi(t_{\rm in})\rangle \\ &= \langle\phi(t_2)|\,\hat{A}_S\,U(t_2,t_1)\,\hat{B}_S\,|\psi(t_1)\rangle.\end{aligned}$$

We may insert a complete set of states

$$\int d\chi(t_1)|\chi(t_1)\rangle\langle\chi(t_1)| = \mathbb{1},$$

in order to obtain

$$\begin{aligned}\langle A(t_2)B(t_1)\rangle_{\varphi\psi} &= \int d\chi(t_1)\langle\varphi(t_2)|\,\hat{A}_S\,U(t_2,t_1)|\chi(t_1)\rangle\langle\chi(t_1)|\,\hat{B}_S\,|\psi(t_1)\rangle \\ &= \int d\chi(t_1)\langle\varphi(t_2)|\,\hat{A}_S\,|\chi(t_2)\rangle\langle\chi(t_1)|\,\hat{B}_S\,|\psi(t_1)\rangle\end{aligned}$$

This has an intuitive interpretation: The transition amplitudes are evaluated for B at time t_1 between $\psi(t_1)$ and arbitrary intermediate states $\chi(t_1)$. Then $\chi(t_1)$ propagates in time to $\chi(t_2)$, and one evaluates the transition amplitude at t_2 of A between $\chi(t_2)$ and $\phi(t_2)$. One finally sums over intermediate states.

It is our aim to derive a functional integral expression for this transition amplitude. We will do this first for a particular amplitude, namely the propagator. This will then be generalised to arbitrary chains of time-ordered operators.

Video: **Lecture08Video05.mp4**

4.6 Propagator

The propagator is a central quantity in quantum field theory. It contains the information how a one-particle wave function at t_1 has evolved at some later time t_2. We will express the propagator as a suitable transition amplitude for a product of annihilation and creation operators. In the functional integral formalism it will be given by a connected two-point function.

Since the propagator deals with the dynamics of a single particle we first define basis functions for localised single particle states. Particles are excitations of the vacuum. We therefore start at t_{in} with an initial vacuum state $|0\rangle$, evolve it to t_1, and apply a creation operator $a^\dagger(\vec{x})$. The result is a state for which at t_1 a single particle is located precisely at $\vec{x}$. We denote this one-particle state by

$$a^\dagger(\vec{x})U(t_1,t_{\rm in})|0\rangle = |(\vec{x},t_1);t_1\rangle.$$

Propagator as transition amplitude. For $t > t_1$ the particle will move. Correspondingly, the wave function changes in the Schrödinger picture according to the standard evolution in quantum mechanics,

$$|(\vec{x},t_1);t\rangle = U(t,t_1)|(\vec{x},t_1);t_1\rangle.$$

One has to distinguish the two different time arguments. For $|(\vec{x},t_1);t\rangle$ the time argument t_1 is a label (together with $\vec{x}$) specifying which state is meant. This state is the one for which at t_1 the

particle is located at $\vec{x}$. The time argument in the Schrödinger evolution of this wave function is given by t. For a given basis state t_1 is kept fixed, while the time evolution of the wave function in the Schrödinger picture is the evolution with varying t.

Let us define the transition amplitude of this one-particle state with a different one particle state $|(\vec{y}, t_2); t\rangle$ at a given time t. Its square is the probability to find a particle that was at time t_1 at $\vec{x}$ to be a particle that is at $\vec{y}$ at time t_2. This transition amplitude is the propagator,

$$G(\vec{y}, t_2; \vec{x}, t_1) = \langle(\vec{y}, t_2); t|(\vec{x}, t_1); t\rangle.$$

The propagator can be expressed by a product of Heisenberg operators. For this purpose we take $t = t_2$,

$$G(\vec{y}, t_2; \vec{x}, t_1) = \langle 0|U^\dagger(t_2, t_{\rm in})a(\vec{y})U(t_2, t_1)a^\dagger(\vec{x})U(t_1, t_{\rm in})|0\rangle.$$

In this expression we use that $\langle(\vec{y}, t_2); t|$ is the hermitean conjugate of $|(\vec{y}, t_2); t\rangle$ and we evolve$|(\vec{x}, t_1); t_1\rangle$ to $|(\vec{x}, t_1); t_2\rangle$. In the Heisenberg picture the propagator reads

$$G(\vec{y}, t_2; \vec{x}, t_1) = \langle 0|a_H(\vec{y}, t_2)a_H^\dagger(\vec{x}, t_1)|0\rangle.$$

This follows from the identity

$$U(t_2, t_1) = U(t_2, t_{in})U(t_{in}, t_1) = U(t_2, t_{in})U^\dagger(t_{in}, t_1)$$

and the definition of Heisenberg operators with reference point t_{in}. The transition amplitude G is called the propagator or Green's function. It is a central quantity in quantum field theory.

Video: **Lecture08Video06.mp4**

One particle wave function and Schrödinger equation. Before going on to derive the functional integral expression for the propagator we discuss next the Schrödinger equation for a one-particle state. This makes the connection to the standard formulation of quantum mechanics. Quantum mechanics obtains from quantum field theory by a restriction to states with a fixed particle number, typically a single particle or two particles. Since quantum field theory is quantum mechanics for many particles, it contains as a special case the quantum mechanical systems with a small fixed particle number. For a single particle we expect in our case the rather trivial quantum mechanics of a free particle, since we consider a translation invariant situation with a vanishing potential. In the functional integral formulation we could introduce a potential in the formulation of the action. For non-relativistic bosons one replaces the chemical potential by $\mu - V(\vec{x})$, thus breaking translation symmetry.

We first extract the Schrödinger wave function in the position basis. Using our basis of localised one-particle states a general one-particle wave function at time t is a superposition

$$|\psi_1(t)\rangle = \int_{\vec{x}} \varphi(\vec{x}, t)|(\vec{x}, t); t\rangle.$$

The position representation of the one-particle state or Schrödinger wave function is given by $\varphi(\vec{x}, t)$. As usual it can be extracted from $|\psi(t)\rangle$ by

$$\varphi(\vec{x}, t) = \langle(\vec{x}, t); t|\psi_1(t)\rangle$$

The proof is standard, using the orthogonality of basis functions

$$\begin{aligned}\langle(\vec{x},t);t|\psi_1(t)\rangle &= \int_{\vec{y}}\langle(\vec{x},t);t|\varphi(\vec{y},t)|(\vec{y},t);t\rangle \\ &= \int_{\vec{y}}\varphi(\vec{y},t)\langle(\vec{x},t);t|(\vec{y},t);t\rangle \\ &= \int_{\vec{y}}\varphi(\vec{y},t)\delta(\vec{x}-\vec{y}) \\ &= \varphi(\vec{x},t).\end{aligned}$$

From the position representation we can switch to the momentum representation $\phi(\vec{p},t)$ by a Fourier transform. For the momentum representation the evolution is trivial,

$$i\partial_t\phi(\vec{p},t) = \left[\frac{\vec{p}^2}{2M}-\mu\right]\phi(\vec{p},t).$$

This follows by applying to $|\psi_1\rangle$ an infinitesimal evolution operator

$$|\psi_1(t+dt)\rangle = U(t+dt,t)|\psi_1(t)\rangle = -i\hat{H}dt|\psi_1(t)\rangle,$$

and noting that $\hat{H}$ is diagonal in the momentum basis. The Schrödinger equation in position space obtains by a Fourier transform.

Video: **Lecture08Video07.mp4**

Huygens principle. You have learned before how to use a propagator for the evolution of wave functions, for example in electrodynamics. Our definition of the propagator plays exactly this role. We employ the time evolution of the position representation of the one particle wave function which can be found from the time evolution of $|(\vec{x},t_1);t\rangle$,

$$\begin{aligned}\varphi(\vec{y},t_2) &= \langle(\vec{y},t_2);t_2|\psi_1(t_2)\rangle \\ &= \langle(\vec{y},t_2);t_2|U(t_2,t_1)|\psi_1(t_1)\rangle \\ &= \int_{\vec{x}}\varphi(\vec{x},t_1)\langle(\vec{y},t_2);t_2|(\vec{x},t_1);t_2\rangle \\ &= \int_{\vec{x}}G(\vec{y},t_2;\vec{x},t_1)\varphi(\vec{x},t_1).\end{aligned}$$

The propagator G allows one to compute the one-particle wave function at t_2 from an *initial wave function* at t_1. This is Huygens' principle for the propagation of waves.

4.7 Functional integral for expectation values of time-ordered operators

We first derive the functional integral expression for the propagator. The corresponding technical steps are easily generalised to arbitrary chains of time-ordered operators.

Video: **Lecture08Video08.mp4**

Video: **Lecture08Video09.mp4**

Propagator from functional integral. For the derivation of a functional integral expression for the propagator we employ the functional integral expression for the evolution operator in the expression

$$G(\vec{y},t_2;\vec{x},t_1) = \langle 0|U^\dagger(t_2,t_{\rm in})a(\vec{y})U(t_2,t_1)a^\dagger(x)U(t_1,t_{\rm in})|0\rangle.$$

One often calls $|0\rangle = |0\rangle_{\rm in}$ the *initial vacuum* at $t_{\rm in}$, and $|0\rangle_{\rm f} = U(t_f,t_{\rm in})|0\rangle_{\rm in}$ the *final vacuum* at $t_{\rm f}$. For a time-translation invariant vacuum one has $|0\rangle_{\rm f} = |0\rangle_{\rm in}$. This implies

$$\langle 0|_{\rm f} = \langle 0|U^\dagger(t_{\rm f},t_{\rm in}).$$

Using

$$U^\dagger(t_f,t_{\rm in})U(t_f,t_2) = U^\dagger(t_2,t_{\rm in}),$$

we find

$$G(\vec{y},t_2;\vec{x},t_1) = \langle 0|_{\rm f}U(t_f,t_2)a(\vec{y})U(t_2,t_1)a^\dagger(\vec{x})U(t_1,t_{\rm in})|0\rangle.$$

This intuitive expression for the propagator involves evolution operators that can be expressed in terms of the functional integral.

Video: **Lecture09Video01.mp4**

We have derived before the functional integral expression for the evolution operator

$$U(t_2,t_1) = \int dx(t_2)\int dp(t_1)\,|x(t_2)\rangle\tilde{F}(t_2,t_1)\langle p(t_1)|,$$

with

$$\tilde{F}(t_2,t_1) = \int D\varphi(t_1<t'<t_2)\exp\left[-\int_{t_1}^{t_2}dt\mathscr{L}(t)\right].$$

Here $|x(t)\rangle$ and $|p(t)\rangle$ are eigenstates of the abstract operators $\hat{x}$ and $\hat{p}$ which are not related to positions in space and momenta,

$$\hat{x}|x(t)\rangle = x(t)|x(t)\rangle, \qquad \hat{p}|p(t)\rangle = p(t)|p(t)\rangle.$$

We employ now a mixed basis with x and p, which is reflected by the difference between $\tilde{F}$ and F as used previously. The integrals over $x(t_2)$ and $p(t_1)$ are not yet included in $\int D\phi(t_1<t'<t_2)$.

Video: **Lecture09Video02.mp4**

Our expression for the propagator contains factors

$$U(t_3,t_2)\,\hat{A}\,U(t_2,t_1) = \int dx(t_3)\;dp(t_2)\;dx(t_2)\;dp(t_1)$$
$$\times\,|x(t_3)\rangle\,\tilde{F}(t_3,t_2)\,\langle p(t_2)|\,\hat{A}\,|x(t_2)\rangle\,\tilde{F}(t_2,t_1)\,\langle p(t_1)|,$$

for which we need the matrix element

$$\langle p(t_2)|\,\hat{A}\,|x(t_2)\rangle = A(x(t_2),p(t_2)).$$

For $\hat{A}$ depending on $a^\dagger$ and a we first express it in terms of the operators $\hat{x}$ and $\hat{p}$, recalling the relations

$$a = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}), \qquad a^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p}).$$

We can then replace in the matrix element for $\hat{A}$

$$a \to \frac{1}{\sqrt{2}}\left[x(t_2) + ip(t_2)\right],$$
$$a^\dagger \to \frac{1}{\sqrt{2}}\left[x(t_2) - ip(t_2)\right],$$

provided that the ordering of operators is done such that $\hat{x}$-operators are on the right and $\hat{p}$-operators on the left.

With the matrix element replaced by a function $A(x_2, p_2)$, we can combine the remaining pieces to

$$U(t_3, t_2)\,\hat{A}\,U(t_2, t_1) = \int dx(t_3)dx(t_1)\,|x(t_3)\rangle \int D\varphi(t_1 < t' < t_3)$$
$$\times \exp\left\{-\int_{t_1}^{t_3} dt'\mathscr{L}(t')\right\} A(x(t_2), p(t_2))\,\langle x(t_1)|.$$

In summary we get the rule: The operator $\hat{A}$ at t_2 leads to the insertion of a function $A(t_2)$ into the functional integral.

Recall the inverse: an observable $A(t)$ in the functional integral results in the insertion of an operator $\hat{A}$ in the chain of transfer matrices.

Video: **Lecture09Video03.mp4**

Discrete formulation. We have been here a bit vague with the precise choice of integrations. In a precise discrete formulation one replaces

$$\langle x_{j+1}|e^{-i\Delta t\hat{H}}|x_j\rangle \quad \text{by} \quad \langle x_{j+1}|e^{-i\Delta t\hat{H}}\,\hat{A}|x_j\rangle$$

at the appropriate place in the chain.

Video: **Lecture09Video04.mp4**

Replacement rules We can now follow $A(x(t_2), p(t_2))$ through the chain of variable transformations

$$x_j \to \tilde{x}_n \to \frac{1}{\sqrt{2}}(\varphi_n + \varphi^*_{-n}) \to \frac{1}{\sqrt{2}}(\varphi(t) + \varphi^*(t)),$$

and similarly

$$p_j \to \tilde{p}_n \to -\frac{i}{\sqrt{2}}(\varphi_n - \varphi^*_{-n}) \to -\frac{i}{\sqrt{2}}(\varphi(t) - \varphi^*(t)),$$

resulting in the simple replacement rules

$$a \to \varphi(t), \qquad a^\dagger \to \varphi^*(t).$$

Video: **Lecture09Video05.mp4**

Propagator. These replacements yield for the propagator or correlation function

$$G(\vec{y},t_2,\vec{x},t_1) = Z^{-1}\int D\varphi\; e^{-S[\varphi]}\varphi(\vec{y},t_2)\varphi^*(\vec{x},t_1).$$

This is a simple and important result that permits us to derive the propagator directly from the functional integral. Once established, we need no longer the operator formalism for the description of the propagation of one-particle states. The above result is valid for vacua for which $\langle\phi(\vec{x},t)\rangle = 0$. We will generalise it to other vacua below.

Video: **Lecture09Video06.mp4**

Expectation values for complex functional integrals. For complex functional integrals in Minkowski space we define expectation values similar to classical statistical physics

$$\langle A\rangle = Z^{-1}\int D\varphi\; e^{-S[\varphi]}A[\varphi]$$

$$Z = \int D\varphi\; e^{-S[\varphi]}.$$

With this one can write the propagator as

$$G(\vec{y},t_2,\vec{x},t_1) = \langle\varphi(\vec{y},t_2)\varphi^*(\vec{x},t_1)\rangle,$$

which is also known as the two-point correlation function.

Video: **Lecture09Video07.mp4**

Normalisation factor Z. We have not paid much attention to the normalisation of the wave function, the additive normalisation of the action, and the formal boundary terms. All this yields constant factors for Z. These factors drop out in the expectation values of observables due to the factor Z^{-1}.

Video: **Lecture09Video08.mp4**

Fourier space. Since $A[\varphi]$ is a function (functional) of φ, variable transformations are straightforward. There are no longer complications with commutator relations as for a and $a^\dagger$. The Fourier transform of the correlation function reads

$$G(\vec{q},t_2;\vec{p},t_1) = \int_y\int_x e^{-i\vec{q}\cdot\vec{y}}\; e^{i\vec{p}\cdot\vec{x}}\; G(\vec{y},t_2;\vec{x},t_1).$$

Translation symmetry in space implies

$$G \sim \delta(\vec{q}-\vec{p}).$$

In case of translation invariance in time G depends only on the time difference $t_2 - t_1$.

Video: **Lecture09Video09.mp4**

Non-trivial field expectation values. So far we have assumed implicitly that the vacuum is trivial. In general $\langle\varphi(\vec{x},t)\rangle$ may be different from zero. A more general definition of the (connected) correlation function is given by

$$G(\vec{y},t_2;\vec{x},t_1) = \langle\delta\varphi(\vec{y},t_2)\delta\varphi(\vec{x},t_1)\rangle, \qquad \delta\varphi = \varphi - \langle\varphi\rangle.$$

This is the standard definition of the connected two-point function or the propagator in statistical physics. It reflects that particles are excitations of a given vacuum. We will also use this definition for the quantum field theory describing the dynamics of many body quantum systems. One can justify this expression in the operator formalism for simple models as phonons. For general theories with interactions it is quite cumbersome to give a solid derivation in the operator formalism. The problem starts with the fact that the vacuum state is often not known. Our functional integral formula is valid for all situations. The conceptual simplicity of the propagator underlines once more the important advantages of a functional integral formulation of quantum field theory. We can write this important formula also in the form

$$G(\vec{y},t_2;\vec{x},t_1) = \langle\varphi(\vec{y},t_2)\varphi(\vec{x},t_1)\rangle - \langle\varphi(\vec{y},t_2)\rangle\langle\varphi(\vec{x},t_1)\rangle.$$

Video: **Lecture09Video10.mp4**

At this point we are essentially done with the operator formalism, up to a few additions below. In the following we will base this lecture purely on the functional integral.

Video: **Lecture09Video11.mp4**

Definition of quantum field theory. A quantum field theory is defined by

(1) Choice of fields φ

(2) Action as functional of fields $S[\varphi]$

(3) Measure $\int D\varphi$

These three ingredients fix a given model or theory completely. For making contact to observation we also need observables $A[\varphi]$. Their expectation values are computed according to the general rule above.

Video: **Lecture09Video12.mp4**

Correlation function. In particular, a general expression for the correlation function is defined by

$$G_{\alpha\beta} = \langle\varphi_\alpha\varphi_\beta^*\rangle - \langle\varphi_\alpha\rangle\langle\varphi_\beta^*\rangle.$$

Here α,β are collective indices, e.g. $\alpha = (\vec{x},t)$ or $(\vec{p},t)$. An evaluation of these expectation values does not need a full knowledge of the vacuum. This is important, since the precise properties of the vacuum for interacting theories are not known.

We can consider φ_α as the components of a complex vector. The propagator is then a complex matrix, called the propagator matrix. Also one-particle wave functions are vectors similar to φ_α. Huygen's principle for the propagation of a wave function becomes a matrix equation.

Video: **Lecture09Video13.mp4**

Chains of operators. We finally generalise the functional integral expression for the propagator to arbitrary time-ordered chains of Heisenberg operators. Consider for $t_n > t_{n-1} > .....t_2 > t_1$ a chain of Heisenberg operators, with expectation value evaluated in the vacuum,

$$\tilde{G} = \langle 0|\hat{A}_H^{(n)}(t_n)\hat{A}_H^{(n-1)}(t_{n-1})\dots\hat{A}_H^{(2)}(t_2)\hat{A}_H^{(1)}(t_1)|0\rangle$$

The propagator is a special case

$$G = \langle 0|a_H(t_2)a_H^\dagger(t_1)|0\rangle.$$

In complete analogy to the discussion above one finds the functional integral expression

$$\tilde{G} = Z^{-1}\int D\varphi\, e^{-S}\bar{A} = \langle A\rangle$$

for the observable

$$\bar{A} = A(t_n)A(t_{n-1})\cdots A(t_2)A(t_1)$$

with

$$A(t_n) = A(\varphi^*(t_n), \varphi(t_n)).$$

The only difference to our treatment of the propagator is that we have different operators and typically more than two factors.

Video: **Lecture09Video14.mp4**

Time ordering. The product $A(t')A(t) = A(t)A(t')$ is commutative. The product $\hat{A}_H(t')\hat{A}_H(t)$ in general not. What happens to commutation relations?

We define the time ordering operator T by putting in a product of two Heisenberg operators the one with larger time argument to the left. e.g. for $t_2 > t_1$,

$$T\left(\hat{A}_H^{(2)}(t_2)\hat{A}_H^{(1)}(t_1)\right) = \hat{A}_H^{(2)}(t_2)\hat{A}_H^{(1)}(t_1)$$
$$T\left(\hat{A}_H^{(1)}(t_1)\hat{A}_H^{(2)}(t_2)\right) = \hat{A}_H^{(2)}(t_2)\hat{A}_H^{(1)}(t_1).$$

The time ordered operator product is commutative. There is therefore no contradiction. In the opposite direction, a given functional integral expression for the expectation values of observables with different time arguments gives a clear prescription in which order the Heisenberg operators appear for the vacuum expectation value in the operator formalism.

These remarks generalise to products with several factors. Defining observables A that may be products of observables with different time arguments we obtain the central identity

$$\langle 0|T\left(\hat{A}_H\right)|0\rangle = \langle A\rangle.$$

On the left one has an operator expression, and on the right functional integral expression.

Video: **Lecture09Video15.mp4**

Transition amplitude for multi-particle states. We may consider two particles at t_1 with momenta $\vec{p}_1$ and $\vec{p}_2$, and compute the transition amplitude to a two particle state at $t_2 > t_1$ with momenta $\vec{p}_3$ and $\vec{p}_4$. In analogy to our treatment of the propagator we first create the two particles from the vacuum at t_1, and annihilate two particles at t_2,

$$\begin{aligned}\tilde{G}_{2,2} &= \langle 0|a_H(\vec{p}_4,t_2)a_H(\vec{p}_3,t_2)a_H^\dagger(\vec{p}_2,t_1)a_H^\dagger(\vec{p}_1,t_1)|0\rangle \\ &= \langle \varphi(\vec{p}_4,t_2)\varphi(\vec{p}_3,t_2)\varphi^*(\vec{p}_2,t_1)\varphi^*(\vec{p}_1,t_1)\rangle.\end{aligned}$$

This is a four-point function. It is a basic element of scattering theory. We will see later that the scattering matrix for two incoming particles with momenta $\vec{p}_1, \vec{p}_2$, scattered into two outgoing particles with momenta $\vec{p}_3, \vec{p}_4$, obtains by squaring the transition amplitude, together with a suitable "phase space integration".

5 Relativistic scalar fields and O(N)-models

Video: **Lecture10Video01.mp4**

In the next chapter we discuss a first model with Lorentz symmetry. Lorentz symmetry is a key ingredient for elementary particle physics. We may focus on a simple model with a complex scalar. This is employed in order to understand how Lorentz symmetry is tightly connected with the existence of antiparticles or antimatter. We also will discuss the important concept of spontaneous symmetry breaking.

Examples for scalar fields. Neutral relativistic scalar fields are the neutral pion π^0 in QCD, or the inflaton or cosmon. In this case a scalar field is a real function $\chi(\vec{x},t)$. In principle, its expectation value can be measured, similar to the electric or magnetic field. Complex scalar fields are the charged pions $\pi^\pm$ and the kaons $K^\pm$, represented by a complex scalar field $\chi(\vec{x},t)$. An important complex field is the Higgs-doublet, represented by a two-component complex scalar field $\varphi_a(t)$ with $a = 1, 2$. In particle physics, its expectation value is responsible for the spontaneous breaking of the electroweak gauge symmetry, and the resulting masses of the W- and Z- bosons, quarks and charged leptons.

Video: **Lecture10Video02.mp4**

5.1 Lorentz invariant action.

Action. To formulate the action we first need the fields which are now fields in Minkowski space $\chi(x)$ where $x = (t, \vec{x})$. We consider local actions of the form

$$S = \int_x \mathscr{L}(x), \qquad \int_x = \int dt\, d^3x.$$

A typical form of the action is in an expansion in derivatives

$$\mathscr{L}(x) = \mathscr{L}_{\rm kin} + iV + \dots$$

The action will reflect the symmetries of the model. One important symmetry is Lorentz symmetry.

Video: **Lecture10Video03.mp4**

Kinetic term. The kinetic term $\mathscr{L}_{\rm kin}$ involves derivatives of fields. For non-relativistic free atoms we have found

$$\mathscr{L}_{\rm kin} = \chi^*(x)\partial_t\chi(x) + \frac{i}{2M}\partial_j\chi^*(x)\partial_j\chi(x), \quad \partial_j = \frac{\partial}{\partial x^j}.$$

The two space derivatives are needed for rotation symmetry. Lorentz-symmetry needs again two derivatives,

$$\mathscr{L}_{\rm kin} = i\partial^\mu\chi^*(x)\partial_\mu\chi(x),$$

with

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^j}\right) = (\partial_0, \partial_j),$$

and

$$\partial^\mu = \eta^{\mu\nu}\partial_\nu, \qquad \eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Derivatives of scalars $\partial_\mu\chi$ are covariant four-vectors. The scalar product of a covariant and a contravariant four-vector is invariant under Lorentz transformations. In momentum space the kinetic term involves the invariant squared momentum

$$\mathscr{L}_{\rm kin} = iq^2\chi^*(q)\chi(q)\ ,\ q^2 = q^\mu q_\mu = \eta^{\mu\nu}q_\mu q_\nu.$$

We conclude that relativistic theories of scalars involves two time derivatives. It is a direct consequence of Lorentz symmetry that the number of time- derivatives equals the number of space-derivatives. The kinetic term can be formulated for real fields in the same way.

Video: **Lecture10Video04.mp4**

From complex to real fields. Writing a complex field as two real fields

$$\chi = \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2)$$

one has

$$\mathscr{L}_{\rm kin} = \frac{i}{2}\sum_{a=1}^{N}\partial^\mu\chi_a(x)\partial_\mu\chi_a(x).$$

Here $N = 1$ for a real scalar, $N = 2$ for a complex scalar and $N = 4$ for the Higgs doublet.

Video: **Lecture10Video05.mp4**

Potential. The potential V involves no derivatives. It is a function of the fields and we write

$$V(x) = V(\chi(x)) = V(\chi).$$

Internal symmetries yield further restrictions. Charge conservation corresponds for complex field to the symmetry

$$\chi \to e^{i\alpha}\chi.$$

The potential can only depend on

$$\rho = \chi^*\chi = \frac{1}{2}(\chi_1^2 + \chi_2^2).$$

For the Higgs doublet, the symmetry is SU(2) such that

$$\rho = \chi^\dagger \chi = \frac{1}{2}\sum_{a=1}^{4} \chi_a^2.$$

Often one can expand

$$V(\rho) = \mu^2 \rho + \frac{1}{2}\lambda\rho^2 + \dots$$

For potentials depending only on ρ,

$$\mathscr{L}_{\rm kin} + V(\rho)$$

the action has $O(N)$ symmetry. Performing analytic continuation for a description of thermal equilibrium we recover the $O(N)$- symmetric models discussed in lecture 2.3. The classical statistical equilibrium and the quantum statistical equilibrium at vanishing temperature differ only by an additional dimension for the second, given by euclidean time. For the classical statistical setting the temperature enters as a parameter in the action, while for quantum statistics it appears in the periodic boundary condition.

Video: **Lecture10Video06.mp4**

5.2 Lorentz invariance and antiparticles.

We next want to show that antiparticles are a natural consequence of Lorentz symmetry.

Two fields with one time-derivative. In the following we concentrate on a single complex scalar field. We want to see how the Lorentz invariant action actually describes two degrees of freedom, namely a charged scalar particle and its antiparticle with opposite charge. Both charged pions π^- and π^+ are described by the same field.

In order to see this we recall that a differential equation with two derivatives is equivalent to two differential equations with one derivative. In other words, one complex field with two time-derivatives is equivalent to two complex fields with one time derivative. We will use this in order to rewrite the action in terms of two fields with only one time derivative. In this form we can make direct contact to the action for non-relativistic bosons that we have discussed previously.

Video: **Lecture10Video07.mp4**

Let us start with a free complex relativistic scalar field

$$\mathscr{L} = i(\partial^\mu \chi^* \partial_\mu \chi + M^2 \chi^* \chi).$$

The potential V describes only the mass M of the scalar particle. In momentum space, $\partial_t = \partial_0 = -\partial^0$, one has

$$\mathscr{L}_p = -i\partial_t \chi^* \partial_t \chi + i(\vec{p}^2 + M^2)\chi^* \chi,$$

and the partition function is

$$Z = \int D\chi \, e^{-\int dt \int_{\vec{p}} \mathscr{L}_p},$$

where $\int_{\vec{p}} = \int \frac{d^3p}{(2\pi)^3}$.

Video: **Lecture10Video08.mp4**

We treat every $\vec{p}$ mode separately. In order to switch to a formulation with two complex fields and only one time derivative we insert a unit factor

$$\int D\pi \, \exp\{-i(\partial_t\chi^* - \pi^*)(\partial_t\chi - \pi)\} = \text{const } " = " \, 1,$$

where $\pi(x)$ is a field. This factor yields only a constant which is independent of χ, as can be seen by a simple shift of the integration variable, $\pi' = \pi - \partial_t\chi$. Since multiplicative constants in Z do not matter, we can write the partition function equivalently as

$$Z = \int D\chi D\pi \, \exp\Big[-\int_t \{-i\partial_t\chi^*\partial_t\chi + i(\vec{p}^2 + M^2)\chi^*\chi + i\partial_t\chi^*\partial_t\chi - i\partial_t\chi^*\pi - i\pi^*\partial_t\chi + i\pi^*\pi\}\Big].$$

This eliminates the term with two derivatives. What remains are two complex fields χ and π with one time derivative,

$$Z = \int D\chi \, D\pi \, e^{-\int_t \mathscr{L}},$$

where, after doing a partial integration,

$$\mathscr{L} = i\chi^*\partial_t\pi - i\pi^*\partial_t\chi + i(\vec{p}^2 + M^2)\chi^*\chi + i\pi^*\pi.$$

Video: **Lecture10Video09.mp4**

At this stage we have the wanted number of fields and only first time derivatives. The time derivative term mixes the fields π and χ. We want to diagonalise this term by suitable variable transformations, such that the independent degrees of freedom are clearly visible. For this purpose we perform the variable transformation

$$\chi(t) = \frac{1}{\sqrt{2}}(\vec{p}^2 + M^2)^{-\frac{1}{4}}(\varphi_1(t) + \varphi_2(-t)),$$
$$\pi(t) = -\frac{i}{\sqrt{2}}(\vec{p}^2 + M^2)^{\frac{1}{4}}(\varphi_1(t) - \varphi_2(-t)).$$

This yields

$$(\vec{p}^2 + M^2)\chi^*(t)\chi(t) = \frac{1}{2}(\vec{p}^2 + M^2)^{\frac{1}{2}}\left[\varphi_1^*(t)\varphi_1(t) + \varphi_2^*(-t)\varphi_2(-t) + \varphi_1^*(t)\varphi_2(-t) + \varphi_2^*(t)\varphi_1(t)\right],$$

Similarly,

$$\pi^*(t)\pi(t) = \frac{1}{2}(\vec{p}^2 + M^2)^{\frac{1}{2}}\left[\varphi_1^*(t)\varphi_1(t) + \varphi_2^*(-t)\varphi_2(-t) - \varphi_1^*(t)\varphi_2(-t) - \varphi_2^*(-t)\varphi_1(t)\right],$$

Summing both expressions gives

$$i\left((\vec{p}^2 + M^2)\chi^*\chi + \pi^*\pi\right) = i(\vec{p}^2 + M^2)^{\frac{1}{2}}\left[\varphi_1^*(t)\varphi_1(t) + \varphi_2^*(-t)\varphi_2(-t)\right],$$

and the mixed term involving time derivatives is

$$\begin{aligned} i\left(\chi^*\partial_t\pi - \pi^*\partial_t\chi\right) &= \frac{1}{2}\left\{(\varphi_1^*(t)+\varphi_2^*(-t))\partial_t(\varphi_1(t)-\varphi_2(-t))\right. \\ &\qquad \left. +\,(\varphi_1^*(t)-\varphi_2^*(-t))\partial_t(\varphi_1(t)+\varphi_2(-t))\right\} \\ &= \varphi_1^*(t)\partial_t\varphi_1(t) - \varphi_2^*(-t)\partial_t\varphi_2(-t). \end{aligned}$$

Under the t-integral one can replace $-\varphi_2^*(-t)\partial_t\varphi_2(-t) \to \varphi_2^*(t)\partial_t\varphi_2(t)$.

Taking the terms together we find the action for two particles with dispersion relation $E = \omega_M = \sqrt{\vec{p}^2 + M^2}$,

$$S = \int dt \left\{\varphi_1^*\partial_t\varphi_1 + \varphi_2^*\partial_t\varphi_2 - i\sqrt{\vec{p}^2+M^2}(\varphi_1^*\varphi_1 + \varphi_2^*\varphi_2)\right\}$$

where $\varphi_i = \varphi_i(t)$. This has precisely the same form that we have encountered before for non-relativistic bosons as phonons. The only particularity is the form of the dispersion relation which reflects the relativistic relation between energy and momentum. The action is block-diagonal, and the two complex fields ϕ_1 and ϕ_2 describe two particles.

Video: **Lecture10Video10.mp4**

Antiparticles. The field χ with two time-derivatives describes a pair of fields φ_1, φ_2 with one time-derivative. One field is the antiparticle of the other. We want to show that the antiparticle has the opposite charge of the particle. For this purpose we couple the complex field χ to an "external" electromagnetic field. The different field equations for φ_1 and φ_2 will then reveal their opposite charges.

The coupling to the electromagnetic potential A_μ is dictated by the principle of gauge invariance. This requires to replace every derivative ∂_μ by a covariant derivative D_μ according to

$$\partial_\mu \to D_\mu\chi = (\partial_\mu - ieA_\mu)\chi.$$

We want to consider the particular field configuration $A_i = 0$ and constant electric potential A_0. In this case one only modifies the time derivative $\partial_t \to \partial_t - ieA_0$. Employing this modification also in the inserted unit factor one obtains for $\mathscr{L}$ an additional term

$$\Delta\mathscr{L} = eA_0\left[\chi^*(t)\pi(t) - \pi^*(t)\chi(t)\right].$$

We express this addition in terms of the fields φ_1 and φ_2

$$\begin{aligned} \Delta\mathscr{L} &= eA_0\left[-\frac{i}{2}(\varphi_1^*(t)+\varphi_2^*(-t))(\varphi_1(t)-\varphi_2(-t)) - \frac{i}{2}(\varphi_1^*(t)-\varphi_2^*(t))(\varphi_1(t)+\varphi_2(-t))\right] \\ &= -ieA_0(\varphi_1^*(t)\varphi_1(t) - \varphi_2^*(-t)\varphi_2(-t)). \end{aligned}$$

As a consequence, the time derivative part in the action becomes

$$S = \int dt\ \left\{\varphi_1^*(\partial_t - ieA_0)\varphi_1 + \varphi_2^*(\partial_t + ieA_0)\varphi_2 + \ldots\right\}$$

We conclude that φ_1 and φ_2 have opposite electric charge. An electric field, given by the gradient of A_0, will accelerate the two particles in opposite directions. The two fields show the characteristic properties of a pair of particle and antiparticle. They have the same mass, but opposite charge.

We have performed the insertion of unity and variable transformations merely in order to demonstrate the appearance of antiparticles in a simple way. In practise, one does not use this

variable transformation. The reason is that the "insertion of unity" for the introduction of the field π is not compatible with the Lorentz symmetry - time is singled out. Since we have not changed the functional integral, the Lorentz symmetry still governs the dynamics if $A_0 = 0$. The presence of this symmetry is hidden for the action formulated in terms of φ_1 and φ_2. Since Lorentz invariance is such an important symmetry for particle physics one wants to work with an action for which this symmetry is manifest.

In this lecture we turn again to the $O(N)$-symmetric scalar theories that we have already introduced in the lecture 2.3. These models are simple enough to serve as good examples, and rich enough to show many interesting physical properties characteristic for quantum field theories. They serve as a "working horse" for this lecture. With the formalism developed so far we can compute the propagator in the absence of interactions or in the limit of small interaction effects. We also discuss the setup for spontaneous symmetry breaking.

5.3 Unified Scalar field theories

Video: **Lecture11Video01.mp4**

Euclidean space. Scalars play an important role in quantum field theory. Prominent examples are the Higgs scalar for the standard model of particle physics, scalar mesons for the strong interactions, or the inflaton for cosmology. The corresponding Lorentz invariant quantum field theory is formulated in Minkowski space. Analytic continuation from Minkowski to Euclidean space yields

$$\eta^{\mu\nu}\partial_\mu\partial_\nu \to \delta^{\mu\nu}\partial_\mu\partial_\nu.$$

Another factor arises from $dt = -id\tau$. In Euclidean space the action therefore reads

$$S = \int_x \left\{ \frac{1}{2}\sum_a \partial^\mu\chi_a\partial_\mu\chi_a + V(\rho) \right\},$$

where now $\partial^\mu = \delta^{\mu\nu}\partial_\nu$ and $\int_x = \int\, dt \int d^3\vec{x}$. This is the four-dimensional $O(N)$-model introduced in lecture 2.3. The Euclidean action is also the one that appears for the $T \to 0$ limit of thermal equilibrium, while for $T > 0$ the τ-integration becomes periodic with period $1/T$.

In euclidean space, the Lorentz-symmetry $SO(1,3)$ gets replaced by the four dimensional rotations $SO(4)$. This symmetry is broken for $T > 0$ since space and time are no longer treated equally. One should distinguish two different symmetries: The internal symmetry $O(N)$ acts on the internal degrees of freedom, while the symmetry $SO(d)$ corresponds to the Lorentz symmetry and acts as a space-time transformation, changing coordinates or momenta.

Video: **Lecture11Video02.mp4**

Unified description of scalar theories. The euclidean $O(N)$-models in arbitrary dimension d, admit a classical statistical probability distribution, with real action,

$$p = \frac{1}{T}e^{-S}, \qquad Z = \int D\varphi e^{-S}.$$

They can be simulated on a computer.

We can classify important applications according to the dimension d of euclidean space and number N of real components of the scalar field:

$d = 1, 2, 3$	models of classical statistical systems in d-dimensions
$N = 3$	magnets, $\langle \chi_a(x) \rangle$ is magnetisation (order parameter)
$N = 1$	Ising type models
$N = 2,\ d = 2$	two-dimensional x-y model with Kosterlitz-Thouless phase transition
$d = 4$	relativistic scalar theories in thermal equilibrium at $T = 0$, or analytic continuation of quantum dynamics.

If the euclidean model is solved, the n-point functions can be analytically continued to Minkowski space, using

$$q_{0E} = q^0_E = -iq_{0M} = iq^0_M.$$

Video: **Lecture11Video03.mp4**

Correlation functions or n-point functions. The task is the computation of n-point functions

$$G^{(n)}_{ab\ldots f}(x_1 \ldots x_n) = \langle \chi_a(x_1)\chi_b(x_2)\cdots\chi_f(x_n)\rangle,$$

with space-time argument $x_i = x^\mu_i$. Alternatively in Fourier space the n-point functions are

$$G^{(n)}(p_1 \ldots p_n),$$

where $p_i = p^\mu_i$. As an example take the two point function or propagator

$$G_{ab}(p_1, p_2) = \langle \chi_a(p_1)\chi_b(-p_2)\rangle - \langle \chi_a(p_1)\rangle\langle \chi_b(-p_2)\rangle = G(p_1)\delta(p_1 - p_2)\delta_{ab}.$$

It can only depend on one momentum by virtue of d-dimensional translation symmetry. Invariance under $SO(d)$-rotations implies that G can only depend on

$$p^2 = p_\mu p_\nu \delta^{\mu\nu},$$

or, in other words, $G(p^\mu) = G(p^2)$. Analytic continuation does not change $G(p^2)$, one only has to switch to $p^2 = p_\mu p_\nu \eta^{\mu\nu}$ in momentum space.

5.4 Propagator for free field

Video: **Lecture11Video04.mp4**

We start from the action for a free field

$$S = \int_x \left\{ \frac{1}{2}\partial^\mu \chi_a \partial_\mu \chi_a + \frac{1}{2}M^2 \chi_a \chi_a \right\}.$$

This is a sum of independent pieces. Each particle with associated field can be treated separately. Consider for simplicity a single complex field

$$S = \int_x \left\{ \partial^\mu \chi^* \partial_\mu \chi + M^2 \chi^* \chi \right\},$$

and transform to Fourier space

$$S = \int_q (q^2 + M^2)\chi^*(q)\chi(q), \qquad \int_q = \int \frac{d^d q}{(2\pi)^d}.$$

The propagator in Fourier space is given by

$$G(p,q) = \langle \chi(p)\chi^*(q)\rangle - \langle \chi(p)\rangle\langle \chi^*(q)\rangle.$$

We want to compute this propagator. For this purpose we use a torus with discrete modes and take the volume to infinity at the end. For

$$S = \sum_q (q^2 + M^2)\chi^*(q)\chi(q)$$

the expectation value obeys

$$\langle \chi(p)\rangle = \frac{1}{Z}\int D\chi\; e^{-S}\;\chi(p) = 0.$$

This is a simple consequence of the invariance of S and $\int D\chi$ under the reflection $\chi \to -\chi$. Similarly, for $p \neq q$, one finds

$$\langle \chi(p)\chi^*(q)\rangle = \frac{1}{Z}\int D\chi e^{-S}\chi(p)\chi^*(q) = 0.$$

Video: **Lecture11Video05.mp4**

Only for equal momenta $p = q$ the two point function differs from zero,

$$\begin{aligned}\langle \chi(q)\chi^*(q)\rangle &= \frac{1}{Z}\int D\chi e^{-S}\chi(q)\chi^*(q)\\ &= \frac{\int d\chi(q) e^{-(q^2+M^2)\chi^*(q)\chi(q)}\chi^*(q)\chi(q)}{\int d\chi(q) e^{-(q^2+M^2)\chi^*(q)\chi(q)}}.\end{aligned}$$

For the second identity we use the fact that for all $q' \neq q$ the same factor appears in the numerator and denominator.

We first compute the Gaussian integral

$$\tilde{Z}(M^2) = \int d\chi(q) e^{-(q^2+M^2)\chi^*(q)\chi(q)},$$

and then take the derivative with respect to M^2,

$$\langle \chi(q)\chi^*(q)\rangle = -\frac{\partial}{\partial M^2}\ln(\tilde{Z}(M^2)).$$

The Gaussian integral has the solution

$$\begin{aligned}\tilde{Z}(M^2) &= \frac{\pi}{q^2+M^2},\\ -\ln(\tilde{Z}) &= \ln(q^2+M^2) - \ln(\pi),\\ -\frac{\partial}{\partial M^2}\ln(\tilde{Z}) &= \frac{1}{q^2+M^2}.\end{aligned}$$

We can summarise for the free propagator

$$G(q,p) = \frac{1}{q^2+M^2}\delta(q-p).$$

For the last identity we have performed the infinite volume limit for which the Kronecker delta $\delta_{p,q}$ becomes the distribution $\delta(p-q) = (2\pi)^{-d}\delta^d(p_\mu - q_\mu)$, which plays in our conventions the role of the unit matrix in momentum space.

Video: **Lecture11Video06.mp4**

Propagator in Minkowski space. The analytic continuation of the free euclidean propagator is straightforward in momentum space,

$$\begin{aligned} G(p,q) &= \frac{1}{(q^2+M^2)}\delta(p-q) \\ &= \frac{1}{-q_0^2+\vec{q}^2+M^2}\delta(p-q). \end{aligned}$$

This propagator has poles at

$$q_0 = \pm\sqrt{\vec{q}^2+M^2}.$$

These two poles correspond to a particle and its antiparticle.

Video: **Lecture11Video07.mp4**

The solutions of the free field equations are

$$\chi_+ = e^{-i\sqrt{\vec{q}^2+M^2}t}$$

and

$$\chi_- = e^{+i\sqrt{\vec{q}^2+M^2}t} = e^{-i\sqrt{\vec{q}^2+M^2}\tilde{t}}, \qquad \tilde{t} = -t.$$

Antiparticles appear formally as particles propagating "backwards in time". The oscillatory behaviour in time is also visible in the Fourier transform of the propagator.This contrasts with the behaviour in euclidean space. There the Fourier transform becomes a function of $r = |\vec{x}-\vec{y}|$. For $d = 3$ the result is a Yukawa potential $G(r)$ proportional $\exp(-Mr)/r$. The propagator vanishes rapidly for large separations $r \gg 1/M$.

Adding an interaction with strength λ, as specified by a potential

$$V = M^2\rho + \lambda\rho^2/2,$$

will modify the propagator through the effects of fluctuations. For small λ the leading effects are a shift of M^2 and a multiplicative constant for the terms in the action which are quadratic in χ. These effects can be absorbed by a multiplicative rescaling of fields and an additive "renormalisation" of M^2. Modifications of the momentum dependence of the propagator occur in the order λ^2 or higher. The free propagator remains often a very good approximation.

5.5 Magnetisation in classical statistics

Video: **Lecture11Video08.mp4**

In the next part we link our formalism to a first set of physical questions. We discuss magnetisation and the notion of spontaneous symmetry breaking. This is done in the view of a later treatment of the Higgs mechanism for the electroweak interactions in particle physics.

Action. We investigate the thermal equilibrium state for classical statistics of magnets. We employ microscopic fields $\sigma_a(x)$ which represent elementary magnets averaged over small volumes. The Hamiltonian with next neighbour interaction reads in this continuum description

$$H = \int_x \left\{ K\,\partial_i\sigma_a(x)\partial_i\sigma_\alpha(x) + c\,\sigma_a(x)\sigma_a(x) + d\,(\sigma_a(x)\sigma_a(x))^2 - B_a\sigma_a(x)\right\}.$$

We take $K > 0$, which tends to align magnets at neighbouring points. The homogeneous magnetic field B breaks the $O(N)$-symmetry. Typical isotropic magnets in three dimensions correspond to $N = 3$. The internal symmetry $O(3)$ reflects independent spin rotations that are decoupled from rotations in space. One can also consider asymmetric magnets with $N = 2$ (xy-models) or $N = 1$ (Ising-type models). Magnets in lower dimensions are also highly interesting, with $d = 2$ corresponding to physics dominated by layered structures as for materials leading to high temperature superconductivity. At this level there is no longer any difference between ferromagnets and antiferromagnets. The internal symmetry is the same.

The partition function in classical statistical thermal equilibrium obeys as usual

$$Z = \int D\sigma \; e^{-\beta H} = \int D\sigma e^{-S}$$

where the classical action is

$$S = \beta H.$$

Video: **Lecture11Video09.mp4**

Rescaled fields. By a rescaling of fields

$$\sigma_a(x) = \sqrt{\frac{1}{2\beta K}}\, \chi_a(x).$$

we can bring the action to the standard form for $O)N)$-models

$$S = \int_x \left\{ \frac{1}{2}\partial_i\chi_a(x)\partial_i\chi_a(x) + \frac{c}{2K}\chi_a(x)\chi_a(x) + \frac{d}{4\beta K^2}(\chi_a(x)\chi_a(x))^2 - \frac{B_a\sqrt{\beta}}{\sqrt{2K}}\chi_a(x) \right\},$$

or with other naming conventions for the couplings

$$S = \int_x \left\{ \frac{1}{2}\partial_i\chi_a(x)\partial_i\chi_a(x) + \frac{m^2}{2}\chi_a(x)\chi_a(x) + \frac{\lambda}{8}(\chi_a(x)\chi_a(x))^2 - J_a\chi_a(x) \right\}.$$

This relates the standard couplings m^2, λ and the source J to the microscopic model parameters. The parameter m^2 can be positive or negative. It is often called a "mass term", in analogy to the mass term for a relativistic particle.

Video: **Lecture11Video10.mp4**

Magnetisation. For $m^2 > 0$ the microscopic magnets have for $J = 0$ a preferred value $\chi_a = 0$. For $m^2 < 0$ the preferred value differs from zero for $J = 0$. The minimum of the potential

$$V_0(\rho) = m^2\rho + \frac{\lambda}{2}\rho^2, \qquad \rho = \frac{1}{2}\varphi_a\varphi_a,$$

obeys

$$\frac{\partial V_0}{\partial \rho} = m^2 + \lambda\rho = 0.$$

For $m^2 < 0$ it occurs at $\rho_0 = -\frac{m^2}{\lambda}$. A non-vanishing magnetic field J_a singles out a certain direction. The minimum of $V = m^2\rho + \frac{\lambda}{2}\rho^2 - J_a\varphi_a$ defines the microscopic magnetisation.

Video: **Lecture11Video11.mp4**

We want to compute the macroscopic magnetisation $\langle\chi(x)\rangle$ as a function of the magnetic field J_a. For this problem fluctuations play an important role. We concentrate on $m^2 < 0$ where things are most interesting. The factor e^{-S} is maximal if S is minimal. One may first look for the minimum of S and expand around it. This procedure is called the "saddle point approximation". The minimum of S is given by the microscopic magnetisation. Without loss of generality we choose $J = (J_1, 0, 0)$. The configuration with constant χ, $\chi_a(x) = \chi_{a,0}$ minimises the kinetic term. The minimum of the action is then given by the minimum of V. It occurs in the direction χ_1, for which the potential reduces to

$$V = \frac{1}{2}m^2\chi_1^2 + \frac{\lambda}{8}\chi_1^4 - J\chi_1.$$

The minimum of V is determined by the homogeneous field equation

$$\frac{\partial V}{\partial \chi} = m^2\chi_1 + \frac{\lambda}{2}\chi_1^3 - J = 0.$$

If we take $J > 0$ a positive $\chi_{1,0}$ is preferred, being the absolute minimum of V. The absolute minimum flips sign if we change the sign of J. At $J = 0$ one observes two degenerate minima. Such a behaviour is characteristic for a first order phase transition as a function of the magnetic field, as observed in ferromagnets or antiferromagnets.

In the limit of small $J > 0$ one has

$$\frac{\lambda}{2}\chi_{10}^2 = -m^2, \qquad \chi_{10} = \sqrt{-\frac{2m^2}{\lambda}}.$$

Video: **Lecture11Video12.mp4**

Fluctuations tend to wash out the microscopic magnetisation. If we want to know how strong is this effect, we have to compute the partition function $Z(J)$ as a function of the source J. Then the magnetisation $\tilde{M}$ in appropriate units is determined by

$$\frac{\partial \ln Z}{\partial J} = \left\langle \int_x \chi_1 \right\rangle = \Omega\langle\chi_1\rangle = \tilde{M},$$

where Ω the volume. We are interested here in small $J \to 0$.

To do thermodynamics we start from the free energy

$$F = -T\ln Z = -\frac{1}{\beta}\ln Z.$$

As well known in thermodynamics the magnetisation is determined by the minimum of the free energy.

Video: **Lecture11Video13.mp4**

Spontaneous symmetry breaking. Spontaneous symmetry breaking occurs if the magnetisation remains different from zero in the limit of vanishing magnetic field, $\tilde{M} \neq 0$ for $J \to 0$. The magnetisation $\tilde{M}_a$ is proportional to the expectation value

$$\varphi_a = \langle \chi_a \rangle.$$

For $J = 0$ the $O(N)$-symmetry is not violated. Any direction for φ_a in internal space is equivalent. Nevertheless, the state $\varphi_a = 0$, which corresponds to vanishing magnetisation, is not a minimum of the free energy, but rather a local maximum. The minimum occurs for $\rho_0 = (\varphi_a \varphi_a)/2$ different from zero, and the system has to choose "spontaneously" a direction of the magnetisation. Once this direction is chosen, the symmetry of the ground state is less than the symmetry of the action. This explains the name "spontaneous symmetry breaking". For the example of an $O(3)$-symmetry of the action the ground state only exhibits the symmetry $O(2)$ of rotations in the plane perpendicular to the vector $\vec{\varphi}$. In practice, the direction of $\vec{\varphi}$ is often determined by tiny amounts of symmetry breaking or a tiny effective source J. Nevertheless, a discussion of the simple situation $J = 0$ covers the relevant physics.

We will discuss this issue here in terms of the classical action. In view of the importance of fluctuation effects this may not seem to be a good idea at first sight. We will see later, however, that the main effect of the fluctuations is to replace the microscopic potential $V(\chi)$ by an "effective potential" $U(\varphi)$. Here φ_a are macroscopic fields. The symmetry of the "effective action" that includes the fluctuation effects is the same as for the microscopic or classical action S. Also the general form has often only small modifications, such that the dominant effect of the fluctuations is a change of parameters. The microscopic parameters m^2 and λ are replaced by macroscopic parameters of "renormalised couplings" m_R^2 and λ_R. Since we do not fix the parameters we can discuss many aspects in terms of the microscopic action S, keeping the later replacements in mind.

Video: **Lecture11Video14.mp4**

Goldstone bosons. One of the characteristic signs of spontaneous breaking of a global continuous symmetry (as $O(3)$ in our case) is the presence of massless "Goldstone bosons". They correspond to excitations perpendicular to $\chi_{a,0}$. For $J = 0$ the potential has the same height for arbitrary directions of $\chi_{a,0}$. A change of the direction will correspond to massless excitations, the Goldstone bosons.

"Massive" or "gapped" excitations correspond to a propagator G porportional $1/(q^2 + M^2)$, whereas for "massless" or "gapless" excitations one has $M = 0$ and therefor a propagator G which is proportional to $1/q^2$. For our translation invariant setting the propagator in momentum space is a matrix in internal space, $G_{ab}(q^2)$. In order to see the massless or massive excitations we have to diagonalise the propagator matrix.

Spontaneous symmetry breaking occurs for $m_R^2 < 0$, or in our "classical setting" for $m^2 < 0$. In this case it is useful to write the potential in the form

$$V = \frac{\lambda}{2}(\rho - \rho_0)^2, \qquad \rho_0 = -\frac{m^2}{\lambda}.$$

We concentrate for simplicity on a single complex field, $N = 2$,

$$\rho = \chi^* \chi = \frac{1}{2}(\chi_1^2 + \chi_2^2).$$

For the magnetisation in absence of a magnetic field, $J = 0$, we choose without loss of generality

$$\chi_{1,0} \neq 0, \qquad \chi_{2,0} = 0, \quad \rho_0 = \frac{1}{2}\chi_{10}^2.$$

We expand around $\chi_{1,0}$, with

$$\chi_1 = \chi_{10} + \delta\chi_1,$$
$$\frac{1}{2}\chi_1^2 = \rho_0 + \chi_{10}\delta\chi_1 + \frac{1}{2}\delta\chi_1^2,$$
$$\rho - \rho_0 = \chi_{10}\delta\chi_1 + \frac{1}{2}\delta\chi_1^2 + \frac{1}{2}\chi_2^2.$$

For the extraction of the propagator it is sufficient to keep only terms quadratic in the fields $\delta\chi_1$ and χ_2. A proof in terms of the effective potential will be given in later lectures. In quadratic approximation the potential reads

$$\frac{\lambda}{2}(\rho - \rho_0)^2 = \frac{\lambda}{2}\chi_{10}^2\delta\chi_1^2 = \lambda\rho_0\delta\chi_1^2.$$

In this approximation the potential does not depend on χ_2. The field χ_2 corresponds to a "flat direction of the potential" and will be associated with the Goldstone boson.

Video: **Lecture11Video15.mp4**

The kinetic term adds to the action in momentum space a piece $q^2(\delta\chi_1(q)\delta\chi_1(-q) + \chi_2(q)\chi_2(-q))$. In the quadratic approximation we therefore end with a free theory, for which we have already computed the propagator. We conclude that the excitation $\delta\chi_1$ behaves as massive field, with $M^2 = 2\lambda\rho_0$, and propagator

$$G = \frac{1}{q^2 + 2\lambda\rho_0}.$$

On the other hand, only the kinetic term contributes to the propagator of the excitation χ_2, which behaves as massless field with propagator

$$G = \frac{1}{q^2.}$$

This massless field is called a Goldstone boson.

Video: **Lecture11Video16.mp4**

We may add a small source J, which breaks the symmetry explicitly. This modifies the potential,

$$V = \frac{\lambda}{2}(\rho - \rho_0)^2 - J\chi_1$$
$$= \lambda\rho_0\delta\chi_1^2 - J\chi_{1,0} - J\delta\chi_1.$$

The action takes the form

$$S = S_0 + \Delta S,$$
$$S_0 = -\Omega J\chi_{1,0},$$
$$\Delta S = \int_x \frac{1}{2}\delta\chi_1(x)(-\Delta + 2\lambda\rho_0)\delta\chi_1(x) - J\ \delta\chi_1(x) + \frac{1}{2}\chi_2(x)(-\Delta)\chi_2(x).$$

Correspondingly, one obtains for the partition function in lowest order

$$Z_0 = e^{-S_0} = \exp(\Omega J\chi_{1,0}),$$
$$\ln Z_0 = \Omega J\chi_{1,0},$$
$$\tilde{M} = \frac{\partial \ln Z_0}{\partial J} = \Omega\chi_{1,0}.$$

Video: **Lecture11Video17.mp4**

Phase transitions and fluctuations. What remains is a computation of the fluctuation effects that relate the "microscopic parameters" m^2 and λ to the "macroscopic parameters" or "renormalised couplings" m_R^2 and λ_R. If m_R^2 turns out positive, the symmetry is not spontaneously broken and one speaks about the "symmetric phase". In contrast, for the range of (m^2, λ) for which m_R^2 is negative one has spontaneous symmetry breaking. One speaks about the "ordered phase" or "SSB phase". If both possibilities can be realised for suitable (m^2, λ), and m_R^2 depends continuously on these parameters, there must be a transition where $m_R^2 = 0$. This is a phase transition. There is a "critical surface" in the space of microscopic parameters for which the phase transition occurs. For two parameters this is a critical line, determined by the condition $m_R^2(m^2, \lambda) = 0$. Both m^2 and λ depend on the temperature T. For given functions $m^2(T)$ and $\lambda(T)$ one has $m_R^2(T)$. The critical temperature T_c for the phase transition is determined by the condition $m_R^2(T_c) = 0$.

Not all models admit a phase transition. For the example $d = 1$, $N > 3$, or for $d = 2$, $N > 2$, one can show that a true phase transition is not possible. For these models one finds $m_R^2 > 0$ for all possible values of m^2 and λ. This is the content of the Mermin-Wagner theorem. An interesting boundary case is $d = 2, N = 2$. In this case one encounters a "Kosterlitz-Thouless phase transition", which can be connected to vortices.

6 Non-relativistic bosons

Video: **Lecture12Video01.mp4**

6.1 Functional integral for spinless atoms

From relativistic to non-relativistic scalar fields. In this section we go from a relativistic quantum field theory back to non-relativistic physics but in a quantum field theoretic formalism. This non-relativistic QFT is in the few-body limit equivalent to quantum mechanics for a few particles but also has interesting applications to condensed matter physics (many body quantum theory) and it is interesting conceptually. We start from the action of a complex, relativistic scalar field in Minkowski space

$$S = \int dt d^3x \left\{ -\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \frac{\lambda}{2} (\phi^* \phi)^2 \right\}.$$

The quadratic part can be written in Fourier space with $(px = -p^0 x^0 + \vec{p}\vec{x})$,

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{ipx} \phi(p), \qquad \phi^*(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \phi^*(p),$$

as

$$\begin{aligned} S_2 &= -\int \frac{d^4p}{(2\pi)^4} \left\{ \phi^*(p) \left[-(p^0)^2 + \vec{p}^2 + m^2 \right] \phi(p) \right\} \\ &= -\int \frac{d^4p}{(2\pi)^4} \left\{ \phi^*(p) \left[-\left(p^0 - \sqrt{\vec{p}^2 + m^2} \right) \left(p^0 + \sqrt{\vec{p}^2 + m^2} \right) \right] \phi(p) \right\}. \end{aligned}$$

Video: **Lecture12Video02.mp4**

Two zero crossings. One observes that the so-called inverse propagator has two zero-crossings, one at $p^0 = \sqrt{\vec{p}^2 + m^2}$ and one at $p^0 = -\sqrt{\vec{p}^2 + m^2}$. At this points the quadratic part of the action become stationary in the sense

$$\frac{\delta}{\delta \phi^*(p)} S_2 = 0.$$

The zero-crossings also correspond to poles of the propagator. These so-called on-shell relations give the relation between frequency and momentum for propagating, particle-type excitations of the theory. In fact, $p^0 = \sqrt{\vec{p}^2 + m^2}$ gives the one for particles, $p^0 = -\sqrt{\vec{p}^2 + m^2}$ the one of anti-particles. In the non-relativistic theory, anti-particle excitations are absent. Intuitively, one assumes that the fields are close to fulfilling the dispersion relation for particles, $p^0 = \sqrt{\vec{p}^2 + m^2}$ which is for large m^2 rather far from the frequency of anti-particles. One can therefore replace in a first step

$$p^0 + \sqrt{\vec{p}^2 + m^2} \to 2\sqrt{\vec{p}^2 + m^2} \approx 2m.$$

Moreover, one can expand the dispersion relation for particles for $m^2 \gg \vec{p}^2$,

$$p^0 = \sqrt{\vec{p}^2 + m^2} = m + \frac{\vec{p}^2}{2m} + \dots$$

This leads us to a quadratic action of the form

$$S_2 = -\int \frac{d^d p}{(2\pi)^4} \left\{ \phi^*(p) \left(-p^0 + m + \frac{\vec{p}^2}{2m} \right) 2m\, \phi(p) \right\},$$

or for the full action in position space

$$S = \int dt d^3x \left\{ \phi^* \left(i\partial_t - m + \frac{\vec{\nabla}^2}{2m} \right) 2m\, \phi - \frac{\lambda}{2} (\phi^* \phi)^2 \right\}.$$

Video: **Lecture12Video03.mp4**

Rescaled fields and dispersion relation. It is now convenient to introduce rescaled fields by setting

$$\phi(t, \vec{x}) = \frac{1}{\sqrt{2m}} e^{-i(m - V_0)t} \varphi(t, \vec{x}).$$

The action becomes then

$$S = \int dt d^3x \left\{ \varphi^* \left(i\partial_t - V_0 + \frac{\vec{\nabla}^2}{2m} \right) \varphi - \frac{\lambda}{8m^2} (\varphi^* \varphi)^2 \right\}. \tag{6.1}$$

The dispersion relation is now with

$$\varphi(t, \vec{x}) = \int \frac{d\omega}{2\pi} \frac{d^3 p}{(2\pi)^3} e^{-i\omega t + i\vec{p}x} \varphi(\omega, \vec{p}),$$

given by

$$\omega = V_0 + \frac{\vec{p}^2}{2m}.$$

This corresponds to the energy of a non-relativistic particle where V_0 is an arbitrary normalization constant corresponding to the offset of an external potential. The action in equation (6.1) describes a non-relativistic field theory for a complex scalar field. As we will see, one can obtain quantum mechanics from there but it is also the starting point for a description of superfluidity.

Video: **Lecture12Video04.mp4**

Symmetries of non-relativistic theory. The non-relativistic action in equation (6.1) has a number of symmetries that are interesting to discuss. First we have translations in space and time as well as rotations in space as in the relativistic case. There is also a global U(1) internal symmetry,

$$\varphi(x) \to e^{i\alpha}\varphi(x), \qquad \varphi^*(x) \to e^{-i\alpha}\varphi^*(x).$$

By Noether's theorem this symmetry is related to particle number conservation (exercise).

Video: **Lecture12Video05.mp4**

Time-dependent U(1) symmetry. There is also an interesting extension of the global U(1) symmetry. One can in fact make it time-dependent according to

$$\varphi(x) \to e^{i(\alpha+\beta t)}\varphi(x), \qquad \varphi^*(x) \to e^{-i(\alpha+\beta t)}\varphi^*(x).$$

All terms in the action are invariant except for

$$\varphi^* i\partial_t\varphi \to \varphi^* e^{-i(\alpha+\beta t)}\; i\partial_t\; e^{i(\alpha+\beta t)}\varphi(x) = \varphi^*(i\partial_t - \beta)\varphi.$$

However, if we also change $V_0 \to V_0 - \beta$ we have for the combination

$$\varphi^*(i\partial_t - V_0)\varphi \to \varphi^*(i\partial_t - \beta - V_0 + \beta)\varphi = \varphi^*(i\partial_t - V_0)\varphi.$$

This shows that

$$\varphi(x) \to e^{i(\alpha+\beta t)}\varphi, \qquad \varphi^* \to e^{-i(\alpha+\beta t)}\varphi^*, \qquad V_0 \to V_0 - \beta,$$

is in fact another symmetry of the action in equationeq:nonrelativisticactionScalar. One can say here that $(i\partial_t - V_0)$ acts like a *covariant derivative*. This says that $(i\partial_t - V_0)\varphi$ transforms in the same (covariant) way as φ itself. The physical meaning of this transformation is a change in the absolute energy scale, which is possible in non-relativistic physics.

Video: **Lecture12Video06.mp4**

Galilei transformation. Note that the action in equation (6.1) is not invariant under Lorentz transformations any more. This is directly clear because derivatives with respect to time and space do not enter in an equal way. However, non-relativistic physics is invariant under another kind of space-time transformations, namely Galilei boosts,

$$t \to t,$$

$$\vec{x} \to \vec{x} + \vec{v}t.$$

One can go to another reference frame that moves relative to the original one with a constant velocity. How is this transformation realized in the non-relativistic field theory described by equation (6.1)? This is a little bit complicated and we directly give the transformation law,

$$\varphi(t,\vec{x}) \to \varphi'(t,\vec{x}) = e^{i\left(m\vec{v}\cdot\vec{x} - \frac{1}{2}m\vec{v}^2 t\right)}\varphi(t,\vec{x}-\vec{v}t).$$

Indeed one can confirm that

$$\left(i\partial_t + \frac{\vec{\nabla}^2}{2m}\right)\varphi(t,\vec{x}) \to e^{i\left(m\vec{v}\cdot\vec{x} - \frac{1}{2}m\vec{v}^2 t\right)}\left[\left(i\partial_t + \frac{\vec{\nabla}^2}{2m}\right)\varphi\right](t,\vec{x}-\vec{v}t),$$

so that the action (6.1) is invariant under Galilei transformations.

6.2 Spontaneous symmetry breaking, Bose-Einstein condensation and superfluidity

Video: **Lecture12Video07.mp4**

Effective potential. One can write the action in (6.1) also as

$$S = \int dt d^3x \left\{ \varphi^* \left(i\partial_t + \frac{\vec{\nabla}^2}{2m} \right) \varphi - V(\varphi^* \varphi) \right\}, \tag{6.2}$$

with microscopic potential as a function of $\rho = \varphi^* \varphi$,

$$V(\rho) = V_0 \rho + \frac{\lambda}{2} \rho^2 = -\mu\rho + \frac{\lambda}{2} \rho^2.$$

At non-vanishing density one has $V_0 = -\mu$, where μ is the chemical potential. For, $\mu > 0$ the minimum of the effective potential is at $\rho_0 > 0$. In a classical approximation where the effect of fluctuation is neglected, one has the equation of motion following from $\delta S = 0$.

Video: **Lecture12Video08.mp4**

Bose-Einstein condensate. If the solution $\varphi(x) = \phi_0$ is homogeneous (constant in space and time), it must correspond to a minimum of the effective potential. Without loss of generality we can assume $\phi_0 \in \mathbb{R}$ and

$$V'(\rho_0) = -\mu + \lambda\rho_0 = 0,$$

leads to

$$\phi_0 = \sqrt{\rho_0} = \sqrt{\frac{\mu}{\lambda}}.$$

Assuming that it survives the effect of quantum fluctuations, such a field expectation value breaks the global U(1) symmetry spontaneously, similar to magnetization. This phenomenon is known as Bose-Einstein condensation. One can see this as a macroscopic manifestation of quantum physics. The mode with vanishing momentum $\vec{p} = 0$ has a macroscopically large occupation number, which is possible for bosonic particles. On the other side, it arises here in a classical approximation to the quantum field theory described by the action in eq. (6.1). In this sense, a Bose-Einstein condensate can also be seen as a *classical* field, similar to the electro-magnetic field, for example.

Video: **Lecture12Video09.mp4**

Bogoliulov excitations. It is also interesting to study small perturbations around the homogeneous field value ϕ_0. Let us write

$$\varphi(x) = \phi_0 + \frac{1}{\sqrt{2}} \left[\phi_1(x) + i\, \phi_2(x) \right],$$

with real fields $\phi_1(x)$ and $\phi_2(x)$. The action in eq. (6.2) becomes (up to total derivatives)

$$S = \int dt\, d^3x \left\{ \phi_2 \partial_t \phi_1 + \frac{1}{2} \sum_{j=1}^{2} \phi_j \frac{\vec{\nabla}^2}{2m} \phi_j - V\left(\phi_0^2 + \sqrt{2}\phi_0\phi_1 + \tfrac{1}{2}\phi_1^2 + \tfrac{1}{2}\phi_2^2 \right) \right\}.$$

It is instructive to expand to quadratic order in the deviations from a homogeneous field ϕ_1 and ϕ_2. The quadratic part of the action reads

$$S_2 = \int dt\, d^3x \left\{ -\frac{1}{2}(\phi_1, \phi_2) \begin{pmatrix} -\frac{\vec{\nabla}^2}{2m} + 2\lambda\phi_0^2 & \partial_t \\ -\partial_t & -\frac{\vec{\nabla}^2}{2m} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right\}.$$

In momentum space, the matrix between the fields becomes

$$G^{-1}(\omega, \vec{p}) = \begin{pmatrix} \frac{\vec{p}^2}{2m} + 2\lambda\phi_0^2 & -i\omega \\ i\omega & \frac{\vec{p}^2}{2m} \end{pmatrix}.$$

In cases where the inverse propagator is a matrix, this holds also for the propagator. When the determinant of the inverse propagator has a zero-crossing, the propagator has a pole. This defines the dispersion relation for quasi-particle excitations,

$$\det G^{-1}(\omega, \vec{p}) = 0.$$

Here this leads to

$$-\omega^2 + \left(\frac{\vec{p}^2}{2m} + 2\lambda\phi_0^2 \right) \frac{\vec{p}^2}{2m} = 0,$$

or

$$\omega = \sqrt{\left(\frac{\vec{p}^2}{2m} + 2\lambda\phi_0^2 \right) \frac{\vec{p}^2}{2m}}. \tag{6.3}$$

This is known as Bogoliubov dispersion relation.

Video: **Lecture12Video10.mp4**

Linear and quadratic regimes. For small momenta, such that

$$\frac{\vec{p}^2}{2m} \ll 2\lambda\phi_0^2,$$

one finds

$$\omega \approx \sqrt{\frac{\lambda\phi_0^2}{m}} |\vec{p}|. \tag{6.4}$$

In contrast, for

$$\frac{\vec{p}^2}{2m} \gg 2\lambda\phi_0^2,$$

one recovers the usual dispersion relation for non-relativistic particles

$$\omega \approx \frac{\vec{p}^2}{2m}. \tag{6.5}$$

The low-momentum region describes phonons (quasi-particles of sound excitations), while the large-momentum region describes normal particles.

Video: **Lecture12Video11.mp4**

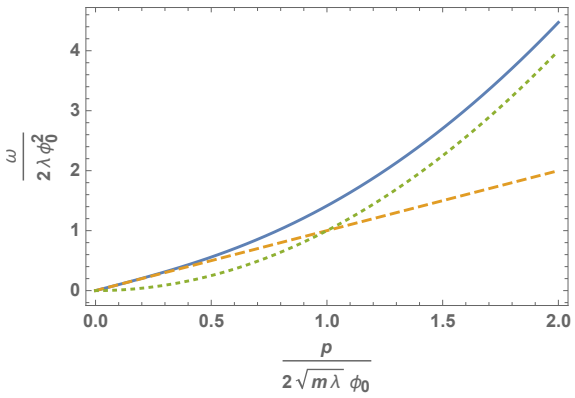


Figure 3. Bogoliubov dispersion relation as in eq. (6.3) (solid line). Also shown is the low momentum approximation (6.4) (dashed line) and the large-momentum approximation (6.5) (dotted line).

Superfluidity. The fact that the dispersion relation is linear for small momenta is also responsible for another interesting phenomenon, namely superfluidity, a fluid motion without friction. To understand this consider an interacting Bose-Einstein condensate flowing past some body of through a capillary. If the energy and momentum of the fluid are $E = E_0$ and $\vec{P} = 0$ in the fluid rest frame, they are

$$E' = E + \vec{P}\vec{v} + \frac{1}{2}M\vec{v}^2 = E_0 + \frac{1}{2}M\vec{v}^2, \qquad \vec{P}' = \vec{P} + M\vec{v} = M\vec{v},$$

in the rest frame of the body or capillary. We used here first the general transformation of energy E and momentum $\vec{P}$ under Galilei boost transformations and then the particular values for the homogeneous fluid state.

Imagine now that we can create an excitation or quasi-particle in the fluid with energy $\epsilon(\vec{p})$ and momentum $\vec{p}$. In the fluid rest frame we have now $E = E_0 + \epsilon(\vec{p})$ and $\vec{P} = \vec{p}$. The energy and momentum in the rest frame of the capillary are then

$$E' = E_0 + \epsilon(\vec{p}) + \vec{p}\cdot\vec{v} + \frac{1}{2}M\vec{v}^2, \qquad \vec{P}' = \vec{p} + M\vec{v}.$$

Comparison to the corresponding relation for the homogeneous state shows that the energy and momentum associated to the excitation are in the rest frame of the capiliary $\epsilon(\vec{p}) + \vec{p}\cdot\vec{v}$ and $\vec{p}$, respectively.

Video: **Lecture12Video12.mp4**

Landau's criterion for superfluidity. Now the point is that at small temperature, excitations will only be created in the fluid in appreciable numbers when it is energetically favorable, i.e. for

$$\epsilon(\vec{p}) + \vec{p}\cdot\vec{v} < 0,$$

such that the energy of the fluid is lowered. If this relation is not fulfilled for any momentum $\vec{p}$, no excitations that could transport momentum out of a local fluid cell will be created. This means that there is no viscosity and the flow is superfluid. It follows that for friction to become possible, the fluid needs to have a fluid velocity larger than

$$v_c = \min_{\vec{p}} \frac{\epsilon(\vec{p})}{|\vec{p}|},$$

known as critical velocity. For the Bogoliubov dispersion relation (6.3) the critical velocity equals the velocity of sound.

Video: **Lecture12Video13.mp4**

Summary. We have seen that relativistic quantum field theories can have a non-relativistic limit where Lorentz symmetry is replaced by Galilei symmetry. In the few-body limit this leads to the same predictions as quantum mechanics but the field theoretic formalism can have advantages, for example in the context of condensed matter theory. As an example we have discussed Bose-Einstein condensates where the low energy excitations are collective excitations of many particles in the form of sound waves or phonons. We have discussed here in particular the non-relativistic limit of a complex relativistic scalar field and have dropped the anti-particle excitations. One can also consider real relativistic scalar field theories which have a non-relativistic limit in terms of a complex scalar field, see for example [arXiv:2005.11359].

7 Scattering

In this section we will discuss a rather useful concept in quantum field theory – the S-matrix. It describes situations where the incoming state is a perturbation of a symmetric (homogeneous and isotropic) vacuum state in terms of particle excitations and the outgoing state similarly. We are interested in calculating the transition amplitude, and subsequently transition probability, between such few-particle states. An important example is the scattering of two particles with a certain center-of-mass energy. This is an experimental situation in many high energy laboratories, for example at CERN. The final states consists again of a few particles (although "few" might be rather many if the collision energy is high). Another interesting example is a single incoming particle, or resonance, that can be unstable and decay into other particles. For example $\pi^+ \to \mu^+ + \nu_\mu$. As we will discuss later on in more detail, particles as excitations of quantum fields are actually closely connected with symmetries of space-time, in particular translations in space and time as well as Lorentz transformations including rotations. (In the non-relativistic limit, Lorentz transformations are replaced by Galilei transformations). The standard application of the S-matrix concept assumes therefore that the vacuum state has these symmetries. The S-matrix is closely connected to the functional integral. Technically, this connection is somewhat simpler to establish for non-relativistic quantum field theories. This will be discussed in the following. The relativistic case will be discussed in full glory in the second part of the lecture course.

7.1 Scattering of non-relativistic bosons

Video: **Lecture13Video01.mp4**

Mode function expansion. Let us recall that one can expand fields in the operator picture as follows

$$\varphi(t,\vec{x}) = \int_{\vec{p}} v_{\vec{p}}(t,\vec{x})\, a_{\vec{p}}, \qquad \varphi^\dagger(t,\vec{x}) = \int_{\vec{p}} v^*_{\vec{p}}(t,\vec{x})\, a^\dagger_{\vec{p}},$$

with $\int_{\vec{p}} = \int \frac{d^3p}{(2\pi)^3}$, annihilation operators $a_{\vec{p}}$, creation operators $a^\dagger_{\vec{p}}$, and the *mode functions*

$$v_{\vec{p}}(t,\vec{x}) = e^{-i\omega_{\vec{p}}t + i\vec{p}\vec{x}}.$$

The dispersion relation in the non-relativistic limit is

$$\omega_{\vec{p}} = \frac{\vec{p}^2}{2m} + V_0.$$

Note that in contrast to the relativistic case, the expansion of $\varphi(t,\vec{x})$ contains no creation operator and the one of $\varphi^*(t,\vec{x})$ no annihilation operator. This is a consequence of the absence of anti-particles.

Video: **Lecture13Video02.mp4**

Scalar product. For the following discussion, it is useful to introduce a scalar product between two functions of space and time $f(t,\vec{x})$ and $g(t,\vec{x})$,

$$(f,g)_t = \int d^3x \left\{ f^*(t,\vec{x}) g(t,\vec{x}) \right\}.$$

The integer goes over the spatial coordinates at fixed time t. Note that if f and g were solutions of the non-relativistic, single-particle Schrödinger equation, the above scalar product were actually independent of time t as a consequence of unitarity in non-relativistic quantum mechanics.

Video: **Lecture13Video03.mp4**

Normalization of mode functions. The mode functions are normalized with respect to this scalar product as

$$(v_{\vec{p}}, v_{\vec{p}'})_t = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}').$$

One can write

$$\begin{aligned} a_{\vec{p}} =& (v_{\vec{p}}, \varphi)_t = \int d^3x e^{i\omega_{\vec{p}} t - i\vec{p}\vec{x}} \varphi(t,\vec{x}), \\ a^\dagger_{\vec{p}} =& (v^*_{\vec{p}}, \varphi^*)_t = \int d^3x e^{-i\omega_{\vec{p}} t + i\vec{p}\vec{x}} \varphi^*(t,\vec{x}). \end{aligned}$$

Video: **Lecture13Video04.mp4**

Time dependence of creation annihilation and creation operators. The right hand side depends on time t and it is instructive to take the time derivative,

$$\begin{aligned} \partial_t a_{\vec{p}(t)} &= \int d^3x \; e^{i\omega_{\vec{p}} t - i\vec{p}\vec{x}} [\partial_t + i\omega_{\vec{p}}] \varphi(t,\vec{x}) \\ &= \int d^3x \; e^{i\omega_{\vec{p}} t - i\vec{p}\vec{x}} \left[\partial_t + i \left(\frac{\vec{p}^2}{2m} + V_0 \right) \right] \varphi(t,\vec{x}) \\ &= \int d^3x \; e^{i\omega_{\vec{p}} t - i\vec{p}\vec{x}} \left[\partial_t + i \left(-\frac{\overleftarrow{\vec{\nabla}}^2}{2m} + V_0 \right) \right] \varphi(t,\vec{x}). \end{aligned}$$

We used here first the dispersion relation and expressed them $\vec{p}^2$ as a derivative acting on the mode function (it acts to the left). In a final step one can use partial integration to make the derivative operator act to the right,

$$\partial_t a_{\vec{p}}(t) = i \int d^3x \; e^{i\omega_{\vec{p}} t - i\vec{p}\vec{x}} \left[-i\partial_t - \frac{\vec{\nabla}^2}{2m} + V_0 \right] \varphi(t,\vec{x}).$$

This expression confirms that $a_{\vec{p}}$ were time-independent if $\varphi(t,\vec{x})$ were a solution of the one-particle Schrödinger equation. More general, it is a time-dependent, however. In a similar way one finds (exercise)

$$\partial_t a^\dagger_{\vec{p}}(t) = -i \int d^3x\, e^{-i\omega_{\vec{p}} t + i\vec{p}\vec{x}} \left[i\partial_t - \frac{\vec{\nabla}^2}{2m} + V_0 \right] \varphi^*(t,\vec{x}).$$

Video: **Lecture13Video05.mp4**

Incoming states. To construct the S-matrix, we first need incoming and out-going states. Incoming states can be constructed by the creation operator

$$a^\dagger_{\vec{p}}(-\infty) = \lim_{t\to-\infty} a^\dagger_{\vec{p}}(t).$$

For example, an incoming two-particle state would be

$$|\vec{p}_1,\vec{p}_2;\text{in}\rangle = a^\dagger_{\vec{p}_1}(-\infty) a^\dagger_{\vec{p}_2}(-\infty)|0\rangle.$$

Video: **Lecture13Video06.mp4**

Bosonic exchange symmetry. We note as an aside point that these state automatically obey bosonic exchange symmetry

$$|\vec{p}_1,\vec{p}_2;\text{in}\rangle = |\vec{p}_2,\vec{p}_1;\text{in}\rangle,$$

as a consequence of

$$a^\dagger_{\vec{p}_1}(-\infty) a^\dagger_{\vec{p}_2}(-\infty) = a^\dagger_{\vec{p}_2}(-\infty) a^\dagger_{\vec{p}_1}(-\infty).$$

Video: **Lecture13Video07.mp4**

Fock space. We note also general states of few particles can be constructed as

$$|\psi;\text{in}\rangle = C_0|0\rangle + \int_{\vec{p}} C_1(\vec{p})\, |\vec{p};\text{in}\rangle + \int_{\vec{p_1},\vec{p_2}} C_2(\vec{p_1},\vec{p_2})|\vec{p_1},\vec{p_2};\text{in}\rangle + \dots$$

This is a superposition of vacuum (0 particles), 1-particle states, 2-particle states and so on. The space of such states is known as *Fock space*. In the following we will sometimes use an abstract index α to label all the states in Fock space, i. e. $|\alpha;\text{in}\rangle$ is a general incoming state. These states are complete in the sense such that

$$\sum_\alpha |\alpha;\text{in}\rangle\langle\alpha;\text{in}| = \mathbb{1},$$

and normalized such that $\langle\alpha;\text{in}|\beta;\text{in}\rangle = \delta_{\alpha\beta}$.

Video: **Lecture13Video08.mp4**

Outgoing states. In a similar way to incoming states one can construct outgoing states with the operators

$$a^\dagger_{\vec{p}}(\infty) = \lim_{t\to\infty} a^\dagger_{\vec{p}}(t).$$

For example

$$|\vec{p}_1,\vec{p}_2;\text{out}\rangle = a^\dagger_{\vec{p}_1}(\infty) a^\dagger_{\vec{p}_2}(\infty)|0\rangle.$$

7.2 The S-matrix

Video: **Lecture13Video09.mp4**

S-matrix. The S-matrix denotes now simply the transition amplitude between incoming and out-going general states $|\alpha;\text{in}\rangle$ and $|\beta;\text{out}\rangle$,

$$S_{\beta\alpha} = \langle\beta;\text{out}|\alpha;\text{in}\rangle.$$

Because α labels all states in Fock space, the S-matrix is a rather general and powerful object. It contains the vacuum-to-vacuum transition amplitude as well as transition amplitudes between all particle-like excited states.

Video: **Lecture13Video10.mp4**

Unitarity of the S-matrix. Let us first prove that the scattering matrix is unitary,

$$\begin{aligned}(S^\dagger S)_{\alpha\beta} &= \sum_\gamma (S^\dagger)_{\alpha\gamma} S_{\gamma\beta} \\ &= \sum_j \langle\gamma;\text{out}|\alpha;\text{in}\rangle^* \langle\gamma;\text{out}|\beta;\text{in}\rangle \\ &= \sum_j \langle\alpha;\text{in}|\gamma;\text{out}\rangle\langle\gamma;\text{out}|\beta;\text{in}\rangle \\ &= \langle\alpha;\text{in}|\beta;\text{in}\rangle \\ &= \delta_{\alpha\beta}.\end{aligned}$$

We have used here the completeness of the out states

$$\sum_j |\gamma;\text{out}\rangle\langle\gamma;\text{out}| = \mathbb{1}.$$

Video: **Lecture13Video11.mp4**

Conservation laws. The S-matrix respects a number of conservation laws such as for energy and momentum. There can also be conservation laws for particle numbers, in particular also in the non-relativistic domain. One distinguishes between elastic collisions where particle numbers do not change, e.g. $2 \to 2$, and inelastic collisions, such as $2 \to 4$. In a non-relativistic theory, such inelastic processes can occur for bound states, for example two H_2 - molecules can scatter into their constituents

$$H_2 + H_2 \to 4H.$$

Video: **Lecture13Video12.mp4**

Connection between outgoing and incoming states. What is the connection between incoming and outgoing states? Let us write

$$\begin{aligned} a_{\vec{p}}(\infty) - a_{\vec{p}}(-\infty) &= \int_{-\infty}^{\infty} \partial_t a_{\vec{p}}(t) \\ &= i \int_{-\infty}^{\infty} dt \int d^3x \; e^{i\omega_{\vec{p}} t - i\vec{p}\vec{x}} \left[-i\partial_t - \frac{\vec{\nabla}^2}{2m} + V_0 \right] \varphi(t,\vec{x}). \end{aligned}$$

Annihilation operators at asymptotically large incoming and outgoing times differ by an integral over space-time of the Schrödinger operator acting on the field. In momentum space with ($px = -p^0 x^0 + \vec{p}\vec{x} = -p^0 t + \vec{p}\vec{x}$),

$$\varphi(t,\vec{x}) = \int \frac{dp^0}{2\omega} \frac{d^3\vec{p}}{(2\pi)^3} e^{ipx} \varphi(p),$$

this would read

$$a_{\vec{p}}(\infty) - a_{\vec{p}}(-\infty) = i \left[-p^0 + \frac{\vec{p}^2}{2m} + V_0 \right] \varphi(p).$$

In a similar way one finds

$$\begin{aligned} a^\dagger_{\vec{p}}(\infty) - a^\dagger_{\vec{p}}(-\infty) &= -i \int_{-\infty}^{\infty} dt \int d^3x \; e^{-i\omega_{\vec{p}} t + i\vec{p}\vec{x}} \left[-i\partial_t - \frac{\vec{\nabla}^2}{2m} + V_0 \right] \varphi^*(t,\vec{x}) \\ &= -i \left[-p^0 + \frac{\vec{p}^2}{2m} + V_0 \right] \varphi^*(p). \end{aligned}$$

Video: **Lecture13Video13.mp4**

Relation between S-matrix elements and correlation functions. For concreteness, let us consider $2 \to 2$ scattering with incoming state

$$|\vec{p_1}, \vec{p_2}; \text{in}\rangle = a^\dagger_{\vec{p}_1}(-\infty) a^\dagger_{\vec{p}_2}(-\infty)|0\rangle,$$

and out-going state

$$|\vec{q_1}, \vec{q_2}; \text{out}\rangle = a^\dagger_{\vec{q}_1}(\infty) a^\dagger_{\vec{q}_2}(\infty)|0\rangle.$$

The S-matrix element can be written as

$$\begin{aligned} S_{\vec{q}_1\vec{q}_2,\vec{p}_1\vec{p}_2} &= \langle \vec{q}_1, \vec{q}_2; \text{out} | \vec{p}_1, \vec{p}_2; \text{in} \rangle \\ &= \langle 0 | T\{ a_{\vec{q}_1}(\infty) \; a_{\vec{q}_2}(\infty) \; a^\dagger_{\vec{p}_1}(-\infty) \; a^\dagger_{\vec{p}_2}(-\infty) \} | 0 \rangle. \end{aligned}$$

We have inserted a time-ordering symbol but the operators are time-ordered already anyway.

Video: **Lecture13Video14.mp4**

Now, one can use

$$a_{\vec{q}_1}(\infty) = a_{\vec{q}_1}(-\infty) + i \left[-q_1^0 + \frac{\vec{q}_1^2}{2m} + V_0 \right] \psi(q_1).$$

However, $a_{\vec{q}_1}(-\infty)$ is moved to the right by time ordering and leads to a vanishing contribution because of

$$a_{\vec{q}_1}(-\infty)|0\rangle = 0.$$

So, effectively under time ordering, one can replace

$$a_{\vec{q}_1}(\infty) \to i\left[-q_1^0 + \frac{\vec{q}_1^2}{2m} + V_0\right]\varphi(q_1).$$

By a similar argument, one can replace creation operators for $t \to -\infty$ like

$$a^\dagger_{\vec{p}_1}(-\infty) \to i\left[-p_1^0 + \frac{\vec{p}_1^2}{2m} + V_0\right]\varphi^*(p_1).$$

The above argument is not fully correct. There is one contribution from the operators $a_{\vec{q}}(-\infty)$ we have forgotten here. In fact, the replacements $a_{\vec{q}_1}(\infty) \to a_{\vec{q}_1}(-\infty)$ and $a_{\vec{q}_2}(\infty) \to a_{\vec{q}_2}(-\infty)$ give

$$\langle 0|a_{\vec{q}_1}(-\infty)\; a_{\vec{q}_2}(-\infty)\; a^\dagger_{\vec{p}_1}(-\infty)\; a_{\vec{p}_2}(-\infty)|0\rangle.$$

We need to commute the annihilation operators to the right using the commutation relation

$$\left[a_{\vec{q}}(-\infty), a^\dagger_{\vec{p}}(-\infty)\right] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}).$$

This gives rise to a contribution to the S-matrix element

$$(2\pi)^6\left[\delta^{(3)}(\vec{p}_1-\vec{q}_1)\;\delta^{(3)}(\vec{p}_2-\vec{q}_2) + \delta^{(3)}(\vec{p}_1-\vec{q}_2)\;\delta^{(3)}(\vec{p}_2-\vec{q}_1)\right].$$

But this is just the "transition" amplitude for the case that no scattering has occurred! There is always this trivial part of the S-matrix and in fact one can write

$$S_{\alpha\beta} = \delta_{\alpha\beta} + \text{contributions from interactions.}$$

Let us keep this in mind and concentrate on the contribution from interactions in the following.

Video: **Lecture13Video15.mp4**

Interacting part. We obtain thus for the S-matrix element

$$\begin{aligned}&\langle \vec{q}_1, \vec{q}_2; \text{out}|\vec{p}_1, \vec{p}_2; \text{in}\rangle \\ &= i^4\left[-q_1^0 + \frac{\vec{q}_1^2}{2m} + V_0\right]\left[-q_2^0 + \frac{\vec{q}_2^2}{2m} + V_0\right]\left[-p_1^0 + \frac{\vec{p}_1^2}{2m} + V_0\right]\left[-p_2^0 + \frac{\vec{p}_2^2}{2m} + V_0\right] \\ &\quad\times \langle 0|T\{\varphi(q_1)\varphi(q_2)\varphi^*(p_1)\varphi^*(p_2)\}|0\rangle.\end{aligned}$$

This shows how S-matrix elements are connected to time ordered correlation functions. This relation is known as the *Lehmann-Symanzik-Zimmermann (LSZ) reduction formula*, here applied to non-relativistic quantum field theory.

Video: **Lecture13Video16.mp4**

Relativistic scalar theories. Let us mention here that for a relativistic theory the LSZ formula is quite similar but one needs to replace

$$\left[-q^0 + \frac{\vec{q}^2}{2m} + V_0\right] \to \left[-(q^0)^2 + \vec{q}^2 + m^2\right],$$

and for particles $\varphi(q) \to \phi(q)$, $\varphi^*(q) \to \phi^*(q)$, while for anti-particles $\varphi(q) \to \phi^*(-q)$, $\varphi^*(q) \to \phi(-q)$.

Video: **Lecture13Video17.mp4**

Correlation functions from functional integrals. The time-ordered correlation functions can be written as functional integrals,

$$\langle 0|T\{\varphi(q_1)\varphi(q_2)\varphi^*(p_1)\varphi^*(p_2)\}|0\rangle = \frac{\int D\varphi\ \varphi(q_1)\varphi(q_2)\varphi^*(p_1)\varphi^*(p_2)\ e^{iS[\varphi]}}{\int D\varphi\ e^{iS[\varphi]}}.$$

We can now calculate S-matrix elements from functional integrals!

7.3 Perturbation theory for interacting scalar fields

Video: **Lecture14Video01.mp4**

Partition function. Let us now consider a non-relativistic theory with the action

$$S[\varphi] = \int dt d^3x \left\{ \varphi^* \left(i\partial_t + \frac{\nabla^2}{2m} - V_0 \right) \varphi - \frac{\lambda}{2}(\varphi^*\varphi)^2 \right\}.$$

Compared to equation (6.1) we have rescaled the interaction parameter, $\frac{\lambda}{4m^2} \to \lambda$. We introduce now the partition function in the presence of source terms J as

$$Z[J] = \int D\varphi\ \exp\left[iS[\varphi] + i\int_x \{J^*(x)\varphi(x) + J(x)\varphi^*(x)\} \right],$$

with $x = (t, \vec{x})$ and $\int_x = \int dt \int d^3x$.

Video: **Lecture14Video02.mp4**

Source term. The source term can also be written in momentum space,

$$\int_x \{J^*(x)\varphi(x) + J(x)\varphi^*(x)\} = \int_p \{J^*(p)\varphi(p) + J(p)\varphi^*(p)\},$$

where

$$\varphi(x) = \int_p e^{ipx}\varphi(p), \qquad \varphi^*(x) = \int_p e^{-ipx}\varphi^*(p),$$

with

$$\int_p = \int \frac{dp^0}{2\pi}\frac{d^3\vec{p}}{(2\pi)^3},$$

and similar for the source J. Because the source term has the same form in position and momentum space, we will sometimes simple write it as

$$\int \{J^*\varphi + \varphi^* J\}.$$

Video: **Lecture14Video03.mp4**

Correlation functions from functional derivatives. One can generate correlation functions from functional derivatives of $Z[J]$, for example

$$\begin{aligned}\langle\varphi(x)\varphi^*(y)\rangle &= \langle 0|T\left\{\varphi(x)\varphi^*(y)\right\}|0\rangle \\ &= \frac{\int D\varphi\ \varphi(x)\ \varphi^*(y)\ e^{iS[\varphi]}}{\int D\varphi\ e^{iS[\varphi]}} \\ &= \left(\frac{(-i)^2}{Z[J]}\frac{\delta^2}{\delta J^*(x)\delta J(y)}Z[J]\right)_{J=0}.\end{aligned}$$

Video: **Lecture14Video04.mp4**

Functional derivatives in momentum space. One can also take functional derivatives directly in momentum space, for example

$$\frac{\delta}{\delta J^*(P)}\ \exp\left[i\int\left\{J^*\varphi+\varphi^*J\right\}\right] = \frac{i}{(2\pi)^4}\ \varphi(p)\ \exp\left[i\int\left\{J^*\varphi+\varphi^*J\right\}\right].$$

In this sense one can write

$$\langle\varphi(p)\ \varphi^*(q)\rangle = \left(\frac{(-i)^2}{Z[J]}(2\pi)^8\frac{\delta^2}{\delta J^*(p)\delta J(q)}Z[J]\right)_{J=0}.$$

Video: **Lecture14Video05.mp4**

Perturbation theory for partition function. Let us write the partition function formally as

$$Z[J] = \int D\varphi\ \exp\left[-i\frac{\lambda}{2}\int_x\left(-i\frac{\delta}{\delta J(x)}\right)^2\left(-i\frac{\delta}{\delta J^*(x)}\right)^2\right]\ \exp\left[iS_2[\varphi]+i\int\left\{J^*\varphi+\varphi^*J\right\}\right],$$

where the quadratic action is

$$S_2[\varphi] = \int_x\varphi^*\left(i\partial_t+\frac{\vec{\nabla}^2}{2m}-V_0\right)\varphi.$$

Note that when acting on the source term in the exponent, every functional derivative $-i\frac{\delta}{\delta J(x)}$ results in a field $\varphi^*(x)$ and so on. In this way, the quartic interaction term has been separated and written in terms of derivatives with respect to the source field.

Video: **Lecture14Video06.mp4**

Separate interaction term. We can now pull it out of the functional integral and write

$$Z[J] = \exp\left[-i\frac{\lambda}{2}\int_x\left(-i\frac{\delta}{\delta J(x)}\right)^2\left(-i\frac{\delta}{\delta J^*(x)}\right)^2\right]Z_2[J],$$

with the partition function for the quadratic theory

$$Z_2[J] = \int D\varphi\ e^{iS_2[\varphi]+i\int\{J^*\varphi+\varphi^*J\}}.$$

The latter is rather easy to evaluate this in momentum space.

Video: **Lecture14Video07.mp4**

Quadratic part. One can write

$$\begin{aligned} S_2 + \int \{J^*\varphi + \varphi^* J\} &= \int_p \left\{ -\varphi^* \left(-p^0 + \tfrac{\vec{p}^2}{2m} + V_0 \right) \varphi + J^*\varphi + \varphi^* J \right\} \\ &= \int_p \left\{ - \left[\varphi^* - J^* \left(-p^0 + \tfrac{\vec{p}^2}{2m} + V_0 \right)^{-1} \right] \left(-p^0 + \tfrac{\vec{p}^2}{2m} + V_0 \right) \right. \\ &\qquad \left. \times \left[\varphi - \left(-p^0 + \tfrac{\vec{p}^2}{2m} + V_0 \right)^{-1} J \right] \right\} \\ &\quad + \int_p \left\{ J^*(p) \left(-p^0 + \tfrac{\vec{p}^2}{2m} + V_0 \right)^{-1} J(p) \right\}. \end{aligned}$$

Note that the last term is independent of the field φ and can be pulled out of the functional integral.

Video: **Lecture14Video08.mp4**

Evaluate Gaussian integral. The functional integral over φ is of Gaussian form. One can shift the integration variable

$$\left[\varphi - \left(-p^0 + \tfrac{\vec{p}^2}{2m} + V_0 \right)^{-1} J \right] \to \varphi,$$

and perform the functional integration in $Z_2[\varphi]$. It yields then only an irrelevant constant and as a result one finds

$$Z_2[J] = \exp \left[i \int_p J^*(p) \left(-p^0 + \tfrac{\vec{p}^2}{2m} + V_0 \right)^{-1} J(p) \right].$$

Video: **Lecture14Video09.mp4**

Relating functional derivatives in position and momentum space. In the following it will be useful to write also the interaction term in momentum space. One may use

$$\begin{aligned} \frac{\delta}{\delta J(x)} &= \int d^4p \frac{\delta J(p)}{\delta J(x)} \frac{\delta}{\delta J(p)} = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} (2\pi)^4 \frac{\delta}{\delta J(p)} \\ &= \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \delta_{J(p)} = \int_p e^{-ipx} \delta_{J(p)}. \end{aligned}$$

Here we defined the abbreviation

$$\delta_{J(p)} = (2\pi)^4 \frac{\delta}{\delta J(p)}.$$

In a similar way

$$\frac{\delta}{\delta J^*(x)} = \int_p e^{ipx} \delta_{J^*(p)}.$$

We used also

$$\int_x e^{ipx} = (2\pi)^4 \delta^{(4)}(p).$$

Video: **Lecture14Video10.mp4**

Perturbation series. One finds for the partition function

$$\begin{aligned} Z[J] &= \exp\left[-i\frac{\lambda}{2}\int_x \left(\tfrac{\delta}{\delta J(x)}\right)^2\left(\tfrac{\delta}{\delta J^*(x)}\right)^2\right] Z_2[J] \\ &= \exp\left[-i\frac{\lambda}{2}\int_{k_1\ldots k_4}\left\{(2\pi)^4\delta^4(k_1+k_2-k_3-k_4)\delta_{J(k_1)}\delta_{J(k_2)}\delta_{J^*(k_3)}\delta_{J^*(k_4)}\right\}\right] \\ &\quad\times\exp\left[i\int_p J^*(p)\left(-p^0+\tfrac{\vec{p}^2}{2m}+V_0\right)^{-1}J(p)\right]. \end{aligned} \tag{7.1}$$

One can now expand the exponential to obtain a formal perturbation series in λ.

Video: **Lecture14Video11.mp4**

S-matrix element. Let us now come back to the S-matrix element for $2\to 2$ scattering

$$\begin{aligned} &\langle\vec{q}_1,\vec{q}_2;\text{out}|\vec{p}_1,\vec{p}_2;\text{in}\rangle \\ &= i^4\left[-q_1^0+\tfrac{\vec{q}_1^2}{2m}+V_0\right]\left[-q_2^0+\tfrac{\vec{q}_2^2}{2m}+V_0\right]\left[-p_1^0+\tfrac{\vec{p}_1^2}{2m}+V_0\right]\left[-p_2^0+\tfrac{\vec{p}_2^2}{2m}+V_0\right] \\ &\quad\times\langle\varphi(q_1)\varphi(q_2)\varphi^*(p_1)\varphi^*(p_2)\rangle \\ &= i^4\left[-q_1^0+\tfrac{\vec{q}_1^2}{2m}+V_0\right]\left[-q_2^0+\tfrac{\vec{q}_2^2}{2m}+V_0\right]\left[-p_1^0+\tfrac{\vec{p}_1^2}{2m}+V_0\right]\left[-p_2^0+\tfrac{\vec{p}_2^2}{2m}+V_0\right] \\ &\quad\times\left(\frac{1}{Z[J]}\delta_{J^*(q_1)}\delta_{J^*(q_2)}\delta_{J(p_1)}\delta_{J(p_2)}Z[J]\right)_{J=0}. \end{aligned}$$

If we now insert the perturbation expansion for Z[J], we can concentrate on the contribution at order $\lambda^1=\lambda$, because at order $\lambda^0=1$ we have only the trivial S-matrix element for no scattering that we already discussed.

Video: **Lecture14Video12.mp4**

Order λ. At order λ we have different derivatives acting on $Z_2[J]$,

- $\delta_{J(p_1)}$ for incoming particles with momentum $\vec{p}_1$
- $\delta_{J^*(q_1)}$ for outgoing particle with momentum $\vec{q}_1$
- $\delta_{J(k)}$ and $\delta_{J^*(k)}$ for the interaction term.

Video: **Lecture14Video13.mp4**

Propagator. At the end, all these derivatives are evaluated at $J=J^*=0$. Therefore, there must always be derivatives δ_J and δ_J^* acting together on one integral appearing in $Z_2[J]$. Note that

$$\delta_{J(p_1)}\delta_{J^*(q_1)}\left[i\int_p J^*(p)\left(-p^0+\tfrac{\vec{p}^2}{2m}+V_0\right)^{-1}J(p)\right] = i\left(-p_1^0+\tfrac{\vec{p}_1^2}{2m}+V_0\right)^{-1}(2\pi)^4\delta^{(4)}(p_1-q_1).$$

Video: **Lecture14Video14.mp4**

Momentum conservation. If two derivatives representing external particles would hit the same integral in $Z_2[J]$, one would have no scattering because $\vec{p}_1 = \vec{q}_1$ and as a result of momentum conservation then also $\vec{p}_2 = \vec{q}_2$. This is no real scattering. Only if a derivative representing an incoming or outgoing particle is combined with a derivative from the interaction term, this is avoided.

Video: **Lecture14Video15.mp4**

Resulting contribution to S-matrix. By doing the algebra one finds at order λ the term for scattering

$$\langle \vec{q}_1, \vec{q}_2; \text{out} | \vec{p}_1, \vec{p}_2; \text{in} \rangle = -i \frac{\lambda}{2} 4 \, (2\pi)^4 \delta^{(4)}(q_1 + q_2 - p_1 - p_2).$$

The factor $4 = 2 \times 2$ comes from different ways to combine functional derivatives with sources.

Video: **Lecture14Video16.mp4**

Momentum conservation. The overall Dirac function makes sure that the incoming four-momentum equals the out-going four-momentum,

$$p^{\text{in}} = p_1 + p_2 = q_1 + q_2 = p^{\text{out}}.$$

Video: **Lecture14Video17.mp4**

Transition amplitude. Quite generally, one can define for the non-trivial part of an S-matrix

$$\langle \beta; \text{out} | \alpha; \text{in} \rangle = (2\pi)^4 \delta^{(4)}(p^{\text{out}} - p^{\text{in}}) \, i \, \mathcal{T}_{\beta\alpha}.$$

Together with the trivial part from "no scattering", one can write

$$S_{\beta\alpha} = \delta_{\beta\alpha} + (2\pi)^4 \delta^{(4)}(p^{\text{out}} - p^{\text{in}}) \, i \, \mathcal{T}_{\beta\alpha}.$$

By comparison of expressions we find for the $2 \to 2$ scattering of non-relativistic bosons at lowest order in λ simply

$$\mathcal{T} = -2\lambda,$$

independant of momenta. More generally, the transition amplitude $\mathcal{T}$ is expected to depend on the momenta of incoming and outgoing particles.

Video: **Lecture14Video18.mp4**

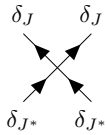
Diagrammatic representation. To keep the overview over a calculation it is sometimes useful to introduce a graphical representation. For the perturbation series discussed above we may represent incoming particles by

$$= i \left[-p_1^0 + \frac{\vec{p}_1^{\,2}}{2m} + V_0 \right] \delta_{J(p_1)} = i \left[-p_1^0 + \frac{\vec{p}_1^{\,2}}{2m} + V_0 \right] \frac{\delta}{\delta J(p_1)},$$

and similarly outgoing particles by

$$= i\left[-q_1^0 + \frac{\vec{q}_1^{\,2}}{2m} + V_0\right]\delta_{J^*(q_1)} = i\left[-q_1^0 + \frac{\vec{q}_1^{\,2}}{2m} + V_0\right]\frac{\delta}{\delta J^*(q_1)}.$$

These functional derivatives are acting on the partition function $Z[J]$. The partition function in (7.1) can be written in a perturbative series with the interaction term represented by



$$= -i\frac{\lambda}{2}\int_{k_1\ldots k_4}\left\{(2\pi)^4\delta^4(k_1+k_2-k_3-k_4)\delta_{J(k_1)}\delta_{J(k_2)}\delta_{J^*(k_3)}\delta_{J^*(k_4)}\right\}$$

which itself acts on the quadratic partition function $Z_2[J]$. The latter is an exponential of the propagator term with sources, which gets graphically represented by

$$= i\int_p J^*(p)\left(-p^0 + \frac{\vec{p}^{\,2}}{2m} + V_0\right)^{-1} J(p).$$

One can the let the functional derivatives act on the sources and at the end evaluate everything at $J = 0$. While the diagrammatic representation is useful, it is only an auxiliary tool to organize the algebra. With a bit of experience one can work well with it.

7.4 From the S-matrix to a cross-section

Video: **Lecture15Video01.mp4**

Transition propability. Let us start from an S-matrix element in the form

$$\langle\beta;\text{out}|\alpha;\text{in}\rangle = (2\pi)^4\delta^{(4)}(p^{\text{out}} - p^{\text{in}})\, i\,\mathcal{T}$$

with transition amplitude $\mathcal{T}$ which may depend on the momenta itself. (For $2 \to 2$ scattering of non-relativistic bosons, and at lowest order in λ, we found simply $\mathcal{T} = -2\lambda$.) Let us now discuss how one can relate S-matrix elements to actual scattering cross-sections that can be measured in an experiment. We start by writing the transition probability from a state α to a state β as

$$P = \frac{|\langle\beta;\text{out}|\alpha;\text{in}\rangle|^2}{\langle\beta;\text{out}|\beta;\text{out}\rangle\langle\alpha;\text{in}|\alpha;\text{in}\rangle}.$$

Video: **Lecture15Video02.mp4**

Transition rate. The numerator contains a factor

$$\left[(2\pi)^4\delta^{(4)}(p^{\text{out}} - p^{\text{in}})\right]^2 = (2\pi)^4\delta^{(4)}(p^{\text{out}} - p^{\text{in}})(2\pi)^4\delta^{(4)}(0).$$

This looks ill defined but becomes meaningful in a finite volume V and for finite time interval ΔT. In fact

$$(2\pi)^4\delta^4(0) = \int d^4x\, e^{i0x} = V\Delta T.$$

For the transition rate $\dot{P} = \frac{P}{\Delta T}$ we can therefore write

$$\dot{P} = \frac{V(2\pi)^4\delta^{(4)}(p^{\rm out} - p^{\rm in})|\mathcal{T}|^2}{\langle\beta;{\rm out}|\beta;{\rm out}\rangle\langle\alpha;{\rm in}|\alpha;{\rm in}\rangle}.$$

Video: **Lecture15Video03.mp4**

Normalization of incoming and outgoing states. Moreover, for incoming and outgoing two-particle states, their normalization is obtained from

$$\begin{aligned}\langle\vec{p}_1,\vec{p}_2;{\rm in}|\vec{q}_1,\vec{q}_2;{\rm in}\rangle &= \lim_{\vec{q}_j\to\vec{p}_j}\langle\vec{p}_1,\vec{p}_2;{\rm in}|\vec{p}_1,\vec{p}_2;{\rm in}\rangle \\ &= \lim_{\vec{q}_j\to\vec{p}_j}\Big[(2\pi)^6\Big(\delta^{(3)}(\vec{p}_1-\vec{q}_1)\delta^{(3)}(\vec{p}_2-\vec{q}_2)+\delta^{(3)}(\vec{p}_1-\vec{q}_2)\delta^{(3)}(\vec{p}_2-\vec{q}_1)\Big)\Big] \\ &= \Big[(2\pi)^3\delta^{(3)}(0)\Big]^2 \\ &= V^2.\end{aligned}$$

Video: **Lecture15Video04.mp4**

Counting of momentum states. In a finite volume $V = L^3$, and with periodic boundary conditions, the final momenta are of the form

$$\vec{p} = \frac{2\pi}{L}(m,n,l),$$

with some integer numbers m, n, l. We can count final states according to

$$\sum_{m,n,l} = \sum_{m,n,l}\Delta m\Delta n\Delta l = L^3\sum_{m,n,l}\frac{\Delta p_1\Delta p_2\Delta p_3}{(2\pi)^3}.$$

In the continuum limit this becomes

$$V\int\frac{d^3p}{(2\pi)^3}.$$

The differential transition rate has one factor $Vd^3p/(2\pi)^3$ for each final state particle.

Video: **Lecture15Video05.mp4**

Differential transition rate. For $2\to 2$ scattering,

$$d\dot{P} = (2\pi)^4\delta^{(4)}(p^{\rm out}-p^{\rm in})|\mathcal{T}|^2\frac{1}{V}\frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}.$$

This can be integrated to give the transition rate into a certain region of momentum states.

Video: **Lecture15Video06.mp4**

Flux of incoming particles. We can go from the transition probability to a cross-section by dividing through the flux of incoming particles

$$\mathcal{F} = \frac{1}{V} v = \frac{2|\vec{p}_1|}{mV}.$$

Here we have a density of one particle per volume V and the relative velocity of the two particles is $v = 2|\vec{p}_1|/m$, in the center-of-mass frame where $|\vec{p}_1| = |\vec{p}_2|$, for identical particles with equal mass m.

Video: **Lecture15Video07.mp4**

Differential cross-section. This cancels the last factor V and we find for the differential cross-section

$$d\sigma = \frac{|\mathcal{T}|^2 m}{2|\vec{p}_1|} (2\pi)^4 \delta^{(4)}(p^{\text{out}} - p^{\text{in}}) \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3}.$$

Video: **Lecture15Video08.mp4**

Phase space integrals. In the center-of-mass frame one has also $\vec{p}^{\,\text{in}} = \vec{p}_1 + \vec{p}_2 = 0$ and accordingly

$$\delta^{(4)}(p^{\text{out}} - p^{\text{in}}) = \delta(E^{\text{out}} - E^{\text{in}})\, \delta^{(3)}(\vec{q}_1 + \vec{q}_2).$$

The three-dimensional part can be used to perform the integral over $\vec{q}_2$. In doing these integrals over final state momenta, a bit of care is needed because the two final state particles are indistinguishable. An outgoing state $|\vec{q}_1, \vec{q}_2; \text{out}\rangle$ equals the state $|\vec{q}_2, \vec{q}_1; \text{out}\rangle$. Therefore, in order to count only really different final states, one must divide by a factor 2 if one simply integrates d^3q_1 and d^3q_2 independently. Keeping this in mind, we find for the differential cross-section after doing the integral over $\vec{q}_2$,

$$d\sigma = \frac{|\mathcal{T}|^2 m}{2|\vec{p}_1|(2\pi)^2} \delta(E^{\text{out}} - E^{\text{in}}) d^3q_1.$$

Video: **Lecture15Video09.mp4**

Magnitude and solid angle. We can now use

$$d^3\vec{q}_1 = |\vec{q}_1|^2 d|\vec{q}_1|\, d\Omega_{q_1}$$

where $d\Omega_{q_1}$ is the differential solid angle element. Moreover

$$E^{\text{out}} = \frac{\vec{q}_1^{\,2}}{2m} + \frac{\vec{q}_2^{\,2}}{2m} + 2V_0 = \frac{\vec{q}_1^{\,2}}{m} + 2V_0,$$

and

$$\frac{dE^{\text{out}}}{d|\vec{q}_1|} = 2\frac{|\vec{q}_1|}{m}.$$

With this, and using the familiar relation $\delta(f(x)) = \delta(x - x_0)/|f'(x_0)|$, one can perform the integral over the magnitude $|\vec{q}_1|$ using the Dirac function $\delta(E^{\text{out}} - E^{\text{in}})$. This yields $|\vec{q}_1| = |\vec{p}_1|$ and

$$d\sigma = \frac{|\mathcal{T}|^2 m^2}{16\pi^2}\, d\Omega_{q_1}.$$

Video: **Lecture15Video10.mp4**

Total cross-section. For the simple case where $\mathcal{T}$ is independent of the solid angle ω_{q_1}, we can calculate the total cross-section. Here we must now take into account that only half of the solid angle 4π corresponds to physically independent configurations. The total cross-sections is therefore

$$\sigma = \frac{|\mathcal{T}|^2 m^2}{8\pi}.$$

In a final step we use $\mathcal{T} = -2\lambda$ to lowest order in λ (equivalent to the Born approximation in quantum mechanics) and find here the cross-section

$$\sigma = \frac{\lambda^2 m^2}{2\pi}.$$

Video: **Lecture15Video11.mp4**

Dimensions. Let us check the dimensions. The action

$$S = \int dt\, d^3x \left\{ \varphi^* \left(i\partial_t + \frac{\vec{\nabla}^2}{2m} - V_0 \right) \varphi - \frac{\lambda}{2} (\varphi^* \varphi)^2 \right\}$$

must be dimensionless. The field φ must have dimension

$$[\varphi] = \text{length}^{-\frac{3}{2}}.$$

The interaction strength λ must accordingly have dimension

$$[\lambda] = \frac{\text{length}^3}{\text{time}}.$$

Because

$$\left[\frac{\vec{\nabla}^2}{2m}\right] = \frac{1}{\text{time}},$$

one has $[m] = \frac{\text{time}}{\text{length}^2}$ and therefore $[\lambda m] = \text{length}$. It follows that indeed

$$[\sigma] = \text{length}^2$$

as appropriate for a cross-section.

8 Fermions

Video: **Lecture15Video12.mp4**

So far we have discussed bosonic fields and bosonic particles as their excitations. Let us now turn to fermions. Fermions as quantum particles differ in two central aspects from bosons. First, they satisfy fermionic statistics. Wave functions for several particles are anti-symmetric under the exchange of particles and occupation numbers of modes can only be 0 or 1. Second, fermionic particles have half integer spin, i. e. 1/2, 3/2, and so on, in contrast to bosonic particles which have integer spin 0, 1, 2 and so on. Both these aspects lead to interesting new developments. Half-integer spin in the context of relativistic theories leads to a new and deeper understanding of space-time symmetries and fermionic statistics leads to a new kind of functional integral based on anti-commuting numbers. The latter appears already for functional integral representations of non-relativistic quantum fields. We will start with this second-aspect and then turn to aspects of space-time symmetry for relativistic theories later on.

8.1 Non-relativistic fermions

Video: **Lecture15Video13.mp4**

Pauli spinor fields. In non-relativistic quantum mechanics, particles with spin 1/2 are described by a variant of Schrödinger's equation with two-component fields. The fields are so-called Pauli spinors with components describing spin-up and spin-down parts with respect to some axis. One can write this as

$$\Psi(t,\vec{x}) = \begin{pmatrix} \psi_\uparrow(t,\vec{x}) \\ \psi_\downarrow(t,\vec{x}) \end{pmatrix}$$

We also use the notation $\psi_a(t,\vec{x})$ where $a = 1,2$ and

$$\psi_1(t,\vec{x}) = \psi_\uparrow(t,\vec{x}), \qquad \psi_2(t,\vec{x}) = \psi_\downarrow(t,\vec{x}).$$

Video: **Lecture15Video14.mp4**

Pauli equation. The Pauli equation is a generalisation of Schrödinger's equation (neglecting spin-orbit coupling),

$$\left[\left(-i\partial_t - \frac{\vec{\nabla}^2}{2m} + V_0\right)\mathbb{1} + \mu_B\,\vec{\sigma}\cdot\vec{B}\right]\Psi(t,\vec{x}) = 0,$$

or equivalently

$$\left[\left(-i\partial_t - \frac{\vec{\nabla}^2}{2m} + V_0\right)\delta_{ab} + \mu_B\,\vec{\sigma}_{ab}\cdot\vec{B}\right]\psi_b(t,\vec{x}) = 0.$$

Here we use the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\vec{B} = (B_1, B_2, B_3)$ is the magnetic field, while μ_B is the magneton that quantifies the magnetic moment.

Video: **Lecture15Video15.mp4**

Attempt for an action. Based on this, one would expect that the quadratic part of an action for a non-relativistic field describing spin-1/2 particles is of the form

$$S_2 \stackrel{?}{=} \int dt d^3x \left\{-\Psi^\dagger\left[\left(-i\partial_t - \frac{\vec{\nabla}^2}{2m} + V_0\right)\mathbb{1} + \mu_B\,\vec{\sigma}\cdot\vec{B}\right]\Psi\right\}$$

However, we also need to take care of fermionic (anti-symmetric) exchange symmetry, such that for fermionic states

$$|\vec{p}_1,\vec{p}_2;\mathrm{in}\rangle = -|\vec{p}_2,\vec{p}_1;\mathrm{in}\rangle.$$

To this aspect we turn next.

8.2 Grassmann fields

Video: **Lecture15Video16.mp4**

Grassmann variables. So-called *Grassmann variables* are generators θ_i of an algebra, and they are anti-commuting such that

$$\theta_i\theta_j + \theta_j\theta_i = 0.$$

An immediate consequence is that ${\theta_j}^2 = 0$.

Basis. If there is a finite set of generators $\theta_1, \theta_2, \ldots, \theta_n$, one can write general elements of the Grassmann algebra as a linear superposition (with coefficients that are ordinary complex (or real) numbers) of the following basis elements

$$\begin{aligned}
&1,\\
&\theta_1, \theta_2, \ldots, \theta_n,\\
&\theta_1\theta_2, \theta_1\theta_3, \ldots, \theta_2\theta_3, \theta_2\theta_4, \ldots, \theta_{n-1}\theta_n,\\
&\ldots\\
&\theta_1\theta_2\theta_3\cdots\theta_n.
\end{aligned}$$

There are 2^n such basis elements, because each Grassmann variable θ_j can be either present or absent.

Video: **Lecture15Video17.mp4**

Grade of monomial. To a monomial $\theta_{j_1}\cdots\theta_{j_q}$ one can associate a *grade* q which counts the number of generators in the monomial. For A_p and A_q being two such monomials one has

$$A_pA_q = (-1)^{p\cdot q}A_qA_p.$$

In particular, the monomials of even grade

$$\begin{aligned}
&1,\\
&\theta_1\theta_2, \theta_1\theta_3, \ldots, \theta_2\theta_3, \ldots, \theta_{n-1}\theta_n,\\
&\ldots
\end{aligned}$$

commute with other monomials, be the latter of even or odd grade.

Video: **Lecture15Video18.mp4**

Grassmann parity. One can define a Grassmann parity transformation P that acts on all generators according to

$$P(\theta_j) = -\theta_j, \qquad P^2 = \mathbb{1}.$$

Even monomials are even, odd monomials are odd under this transformation. The parity even part of the algebra, spanned by the monomials of even grade, constitutes a sub-algebra. Because its elements commute with other elements of the algebra they behave "bosonic", while elements of the Grassmann algebra that are odd with respect to P behave "fermionic".

Video: **Lecture16Video01.mp4**

Functions of Grassmann variables. Because of $\theta^2 = 0$, functions of a Grassmann variable θ are always linear,

$$f(\theta) = f_0 + \theta f_1.$$

Note that f_0 and f_1 could depend on other Grassmann variables but not θ.

Video: **Lecture16Video02.mp4**

Differentiation for Grassmann variables. To define differentiation of $f(\theta)$ with respect to θ we first bring it to the form

$$f(\theta) = f_0 + \theta f_1$$

and set then

$$\frac{\partial}{\partial\theta} f(\theta) = f_1.$$

Note that similar to $\theta^2 = 0$ one has also $\left(\frac{\partial}{\partial\theta}\right)^2 = 0$. One may verify that the chain rule applies. Take $\sigma(\theta)$ to be an odd element and $x(\theta)$ an even element of the Grassmann algebra. One has then

$$\frac{\partial}{\partial\theta} f(\sigma(\theta), x(\theta)) = \frac{\partial\sigma}{\partial\theta}\frac{\partial f}{\partial\sigma} + \frac{\partial x}{\partial\theta}\frac{\partial f}{\partial x}.$$

The derivative we use here is a left derivative.

Consider for example

$$f = f_0 + \theta_1\theta_2.$$

One has then

$$\frac{\partial}{\partial\theta_1} f = \theta_2, \qquad \frac{\partial}{\partial\theta_2} f = -\theta_1,$$
$$\frac{\partial}{\partial\theta_2}\frac{\partial}{\partial\theta_1} f = 1, \qquad \frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2} f = -1.$$

One could also define a right derivative such that

$$f\frac{\overleftarrow{\partial}}{\partial\theta_1} = -\theta_2, \qquad f\frac{\overleftarrow{\partial}}{\partial\theta_2} = \theta_1.$$

Video: **Lecture16Video03.mp4**

Integration for Grassmann variables. To define integration for Grassmann variables one takes orientation from two properties of integrals from $-\infty$ to ∞ for ordinary numbers. One such property is linearity,

$$\int_{-\infty}^{\infty} dx\; c\, f(x) = c\int_{-\infty}^{\infty} dx\; f(x).$$

The other is invariance under shifts of the integration variable,

$$\int_{-\infty}^{\infty} dx\; f(x+a) = \int_{-\infty}^{\infty} dx\; f(x).$$

For a function of a Grassmann variable

$$f(\theta) = f_0 + \theta f_1$$

One sets therefore

$$\int d\theta\ f(\theta) = f_1.$$

In other words, we have defined

$$\int d\theta = 0, \qquad \int d\theta\ \theta = 1.$$

This is indeed linear and makes sure that

$$\int d\theta\ f(\theta+\sigma) = \int d\theta\,\{(f_0+\sigma f_1) + f_1\ \theta\} = \int d\theta\ f(\theta) = f_1.$$

Note that one has formally

$$\int d\theta\ f(\theta) = \frac{\partial}{\partial\theta} f(\theta).$$

Video: **Lecture16Video04.mp4**

Several variables. For functions of several variables one has

$$\int d\theta_1 \int d\theta_2 f(\theta_1,\theta_2) = \frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2} f(\theta_1,\theta_2).$$

It is easy to see that derivatives with respect to Grassmann variables anti-commute

$$\frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_k} = -\frac{\partial}{\partial\theta_k}\frac{\partial}{\partial\theta_j},$$

and accordingly also the differentials anti-commute

$$d\theta_j d\theta_k = -d\theta_k d\theta_j.$$

Video: **Lecture16Video05.mp4**

Functions of several Grassmann variables. A function that depends on a set of Grassmann variables $\theta_1,\ldots,\theta_n$ can be written as

$$f(\theta) = f_0 + \theta_j f_1^j + \frac{1}{2}\theta_{j_1}\theta_{j_2} f_2^{j_1\ j_2} + \ldots + \frac{1}{n!}\theta_{j_1}\cdots\theta_{j_n} f_n^{j_1\cdots j_n}.$$

We use here Einsteins summation convention with indices j_k being summed over. The coefficients $f_k^{j_1\cdots j_k}$ are completely anti-symmetric with respect to the interchange of any part of indices. In particular, the last coefficient can only be of the form

$$f_n^{j_1\cdots j_n} = \tilde{f}_n \varepsilon_{j_1\cdots j_n},$$

where $\varepsilon_{j_1\cdots j_n}$ is the completely anti-symmetric Levi-Civita symbol in n dimensions with $\varepsilon_{12\ldots n} = 1$.

Video: **Lecture16Video06.mp4**

Differentiation and integration. Let us now discuss what happens if we differentiate or integrate $f(\theta)$. One has

$$\frac{\partial}{\partial\theta_k}f(\theta) = f_1^k + \theta_{j_2}f_2^{kj_2} + \ldots + \frac{1}{(n-1)!}\theta_{j_2}\cdots\theta_{j_n}f_n^{kj_2\cdots j_n},$$

and similar for higher order derivatives. In particular

$$\frac{\partial}{\partial\theta_n}\cdots\frac{\partial}{\partial\theta_1}f(\theta) = f_n^{12\ldots n} = \tilde{f}_n.$$

This defines also the integral with respect to all n variables,

$$\int d\theta_n\cdots d\theta_1 f(\theta) = f_n^{12\ldots} = \tilde{f}_n$$
$$= \int d^n\theta f(\theta) = \int D\theta f(\theta).$$

Video: **Lecture16Video07.mp4**

Linear change of Grassmann variables. Let us consider a linear change of the Grassmann variables in the form (summation over k is implied)

$$\theta_j = J_{jk}\theta'_k,$$

where J_{jk} is a matrix of commuting variables. We can write

$$f(\theta) = f_0 + \ldots + \frac{1}{n!}\left(J_{i_1j_1}\theta'_{j_1}\right)\cdots\left(J_{i_nj_n}\theta'_{jn}\right)\varepsilon_{i_1\ldots i_n}\tilde{f}_n.$$

Now one can use the identity

$$\varepsilon_{i_1\ldots i_n}J_{i_1j_1}\cdots J_{i_nj_n} = \det(J)\,\varepsilon_{j_1\ldots j_n}.$$

This can actually be seen as the definition of the determinant. One can therefore write

$$f(\theta) = f_0 + \ldots + \frac{1}{n!}\theta'_{j_1}\cdots\theta'_{j_n}\varepsilon_{j_1\ldots j_n}\det(J)\tilde{f}_n.$$

The integral with respect to θ' is

$$\int d^n\theta' f(\theta) = \det(J)\tilde{f}_n.$$

In summary, one has

$$\int d^n\theta f(\theta) = \frac{1}{\det(J)}\int d^n\theta' f(\theta).$$

Video: **Lecture16Video08.mp4**

Linear change of ordinary variables. One should compare this to the corresponding relation for conventional integrals with $x_j = J_{jk}x'_k$. In that case one has

$$\int d^n x f(x) = \det(J)\int d^n x' f(x').$$

Note that the determinant appears in the denominator for Grassmann variables while it appears in the numerator for conventional integrals.

Video: **Lecture16Video09.mp4**

Gaussian integrals of Grassmann variables. Consider a Gaussian integral of two Grassmann variables

$$\int d\theta d\xi\, e^{-\theta\xi b} = \int d\theta d\xi\, (1-\theta\xi b) = \int d\theta d\xi\, (1+\xi\theta b) = b.$$

For a Gaussian integral over conventional complex variables one has instead

$$\int d(\mathrm{Re}\, x)\; d(\mathrm{Im}\, x)\, e^{-x^* x b} = \frac{\pi}{b}.$$

Again, integrals over Grassmann and ordinary variables behave in some sense "inverse".

Video: **Lecture16Video10.mp4**

Higher dimensional Gaussian integrals. For higher dimensional Gaussian integrals over Grassmann numbers we write

$$\int d^n\theta d^n\xi e^{-\theta_j a_{jk}\xi_k} = \int d\theta_n d\xi_n \cdots d\theta_1 d\xi_1 e^{-\theta_j a_{jk}\xi_k}.$$

One can now employ two unitary matrices with unit determinat to perform a change of variables

$$\theta_j = \theta'_l U_{lj}, \qquad \xi_k = V_{km}\xi'_m,$$

such that

$$U_{lj}a_{jk}V_{km} = \tilde{a}_l\delta_{lm},$$

is diagonal. This is always possible. The Gaussian integral becomes

$$d^n\theta d^n\xi\, e^{-\theta_j a_{jk}\xi_k} = \det(U)^{-1}\det(V)^{-1}\int d^n\theta'\, d^n\xi'\, e^{-\theta'_l\xi'_l\tilde{a}_l} = \prod_{l=1}^{n}\tilde{a}_l = \det(a_{jk}).$$

Again this is in contrast to a similar integral over commuting variables where the determinant would appear in the denominator.

Video: **Lecture16Video11.mp4**

Gaussian integrals with sources. Finally let us consider a Gaussian integral with source forms,

$$\int d^n\bar{\psi}d^n\psi\;\exp\left[-\bar{\psi}M\psi + \bar{\eta}\psi + \bar{\psi}\eta\right] = Z(\bar{\eta},\eta).$$

We integrate here over independent Grassmann variables $\psi = (\psi_1,\ldots,\psi_n)$ and $\bar{\psi} = (\bar{\psi}_1,\ldots,\bar{\psi}_n)$ and we use the abbreviation

$$\bar{\psi}M\psi = \bar{\psi}_j M_{jk}\psi_k.$$

The source forms are also Grassmann variables $\eta = (\eta_1,\ldots,\eta_n)$ and $\bar{\eta} = (\bar{\eta}_1,\ldots,\bar{\eta}_n)$ with

$$\bar{\eta}\psi = \bar{\eta}_j\psi_j, \qquad \bar{\psi}\eta = \bar{\psi}_j\eta_j.$$

As usual, we can write

$$Z(\bar{\eta},\eta) = \int d^n\bar{\psi}d^n\psi\;\exp\left[-(\bar{\psi}-\eta M^{-1})M(\psi - M^{-1}\eta) + \bar{\eta}M^{-1}\eta\right].$$

A shift of integration variables does not change the result and thus we find

$$Z(\bar{\eta},\eta) = \det(M)\exp\left[\bar{\eta}M^{-1}\eta\right].$$

In this sense, Gaussian integrals over Grassmann variables can be manipulated similarly as Gaussian integrals over commuting variables. Note again that $\det(M)$ appears in the numerator while it would appear in the denominator of bosonic variables.

Video: **Lecture16Video12.mp4**

Functional integral over Grassmann fields. We can now take the limit $n \to \infty$ and write

$$\int d^n\bar{\psi}d^n\psi \to \int D\bar{\psi}D\psi, \qquad Z(\bar{\eta},\eta) \to Z[\bar{\eta},\eta],$$

with

$$Z[\bar{\eta},\eta] = \int D\bar{\psi}D\psi \;\exp[-\bar{\psi}M\psi + \bar{\eta}\psi + \bar{\psi}\eta] = \det(M)\exp\left[\bar{\eta}M^{-1}\eta\right].$$

In this way we obtain a formalism that can be used for fermionic or Grassmann fields.

Video: **Lecture16Video13.mp4**

Action for free non-relativistic scalars. We can now write down an action for non-relativistic fermions with spin 1/2. It looks similar to what we have conjectured before,

$$S_2 = \int dtd^3x\left\{-\bar{\psi}\left[\left(-i\partial_t - \tfrac{\vec{\nabla}^2}{2m} + V_0\right)\mathbb{1} + \mu_B\vec{\sigma}\cdot\vec{B}\right]\psi\right\},$$

but the two-component fields $\psi = (\psi_1,\psi_2)$ and $\bar{\psi} = (\bar{\psi}_1,\bar{\psi}_2)$ are in fact *Grassmann fields*. Such fields anti-commute, for example $\psi_1(x)\psi_2(y) = -\psi_2(y)\psi_1(x)$. One should see the field at different space-time positions x to be independent Grassmann numbers. Also, ψ_1 and $\bar{\psi}_1$ are independent as Grassmann fields. In particular $\psi_1(x)^2 = 0$ but $\bar{\psi}_1(x)\psi_1(x) \neq 0$.

Video: **Lecture16Video14.mp4**

Partition function. A partition function with sources for the above free theory could be written down as

$$Z_2[\bar{\eta},\eta] = \int D\bar{\psi}D\psi \;\exp\left[iS_2[\bar{\psi},\bar{\psi}] + i\int_x\left\{\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)\right\}\right]$$

Correlation functions can be obtained from functional derivatives of $Z[\bar{\eta},\eta]$ with respect to the source field $\bar{\eta}(x)$ and $\eta(x)$. Some care is needed to take minus signs into account that may arise from possible commutation of Grassmann numbers. For the quadratic theory one can easily complete the square, perform the functional integral and write the partition function formally as

$$Z_2[\bar{\eta},\eta] = \exp\left[i\int_{x,y}\bar{\eta}(x)\left[\left(-i\partial_t - \tfrac{\vec{\nabla}^2}{2m} + V_0\right)\mathbb{1} + \mu_B\,\vec{\sigma}\cdot\vec{B}\right]^{-1}(x,y)\,\eta(y)\right].$$

Video: **Lecture16Video15.mp4**

Greens function. The inverse of the operator

$$\left(-i\partial_t - \frac{\vec{\nabla}^2}{2m} + V_0\right) \mathbb{1} + \mu_B \, \vec{\sigma} \cdot \vec{B}$$

is a matrix valued Greens function. For a magnetic field that is constant in space and time, for example pointing in z-direction, one can easily invert this operator in Fourier space,

$$\Upsilon(x-y) = \int \frac{d^4p}{(2\pi)^4} \left[\left(-p^0 + \frac{\vec{p}^2}{2m} + V_0\right) \mathbb{1} + \mu_B \, \vec{\sigma} \cdot \vec{B}\right]^{-1} e^{ip(x-y)}.$$

In the following we will set $\vec{B} = 0$ for simplicity such that

$$\Upsilon(x-y) = \mathbb{1} \int_p \frac{1}{-p^0 + \frac{\vec{p}^2}{2m} + V_0 - i\epsilon} \, e^{ip(x-y)}. \tag{8.1}$$

The term $i\epsilon$ makes sure that we take the right Greens function with time ordering. For a non-relativistic theory at zero temperature and density, this equals the retarded Greens function.

8.3 Yukawa theory

Video: **Lecture17Video01.mp4**

Yukawa theory. Let us now investigate a theory for a non-relativistic fermion with spin 1/2 and a real, relativistic scalar boson

$$S = \int dt d^3x \left\{ -\bar{\psi} \left(-i\partial_t - \frac{\vec{\nabla}^2}{2m} + V_0 - i\epsilon\right) \psi - \tfrac{1}{2}\phi \left(\partial_t^2 - \vec{\nabla}^2 + M^2 - i\epsilon\right) \phi - g\phi\bar{\psi}\psi \right\}.$$

Partition function for Yukawa theory. We will discuss this theory in terms of the partition function

$$Z[\bar{\eta}, \eta, J] = \int D\bar{\psi} D\psi D\phi \; e^{iS[\bar{\psi},\psi,\phi] + i\int_x \{\bar{\eta}\psi + \bar{\psi}\eta + J\phi\}}.$$

As usual, by taking functional derivatives with respect to the source fields, one can obtain various correlation functions. Our strategy will be to perform a perturbation expansion in the cubic term $\sim g$.

Video: **Lecture17Video02.mp4**

Quadratic action. Let us first concentrate on the quadratic theory and the corresponding partition function derived from the action

$$S_2 = \int dt d^3x \left\{ -\bar{\psi} \left(-i\partial_t - \frac{\vec{\nabla}^2}{2m} + V_0 - i\epsilon\right) \psi - \frac{1}{2}\phi(\partial_t^2 - \vec{\nabla}^2 + M^2 - i\epsilon)\phi \right\}.$$

By doing the Gaussian integration one finds

$$\begin{aligned} Z_2[\bar{\eta}, \eta, J] &= \int D\bar{\psi} D\psi D\phi \; \exp\left[iS_2 + i\int_x \{\bar{\eta}\psi + \bar{\psi}\eta + J\phi\}\right] \\ &= \exp\left[i \int d^4x d^4y \left\{\bar{\eta}(x)\Upsilon(x-y)\eta(y) + \frac{1}{2}J(x)\Delta(x-y)J(y)\right\}\right] \end{aligned}$$

where $\Upsilon(x-y)$ is the Greens function for fermions in eq. (8.1). For the scalar bosons, the Green function is

$$\Delta(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{-(p^0)^2 + \vec{p}^2 + M^2 - i\epsilon} e^{ip(x-y)}.$$

Video: **Lecture17Video03.mp4**

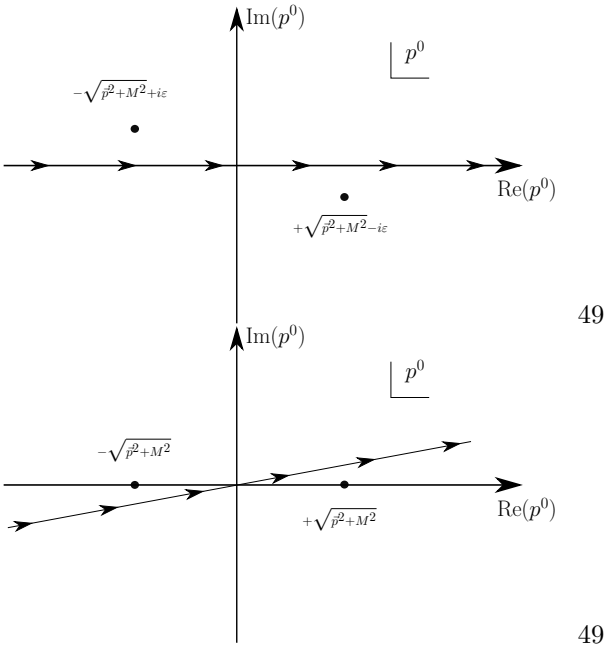


Figure 4. Illustration of the contour integral for the time-ordered Feynman propagator. In the left panel the poles are shifted slightly into the complex plane, in the right panel the integration contour is slightly shifted. Both prescriptions lead to equivalent results.

Time-ordered boundary conditions or $i\epsilon$ prescription. Again, the $i\epsilon$ term makes sure that the Greens function corresponds to the time-ordered or Feynman boundary conditions. One can also obtain this from a careful consideration of analytic continuation from Euclidean space to real time or Minkowski space. Note that the $i\epsilon$ term has in the functional integral the form

$$e^{iS} = e^{i[\ldots + i\epsilon \int_x \phi^2(x)]} = e^{-\epsilon \int_x \phi(x)^2 + i\ldots}.$$

This is the same suppression term that also appears in the Euclidean functional integral. It makes sure that functional integrals are converging and that the theory approaches the ground state on long time scales.

In the complex plane, positions of poles are shifted slightly away from the real axis. This is illustrated in the left panel of figure 4. In fact this is equivalent to keeping the poles at $p^0 = \pm\sqrt{\vec{p}^2 + M^2}$ but moving slightly in the integration contour. This is illustrated in the right panel of figure 4.

Video: **Lecture17Video04.mp4**

Time ordered or Feynman propagator in position space. Let us use either of these prescriptions to calculate the scalar field propagator in position space

$$\Delta(x-y) = \int \frac{dp^0}{2\pi} \frac{d^3p}{(2\pi)^3} \frac{e^{-ip^0(x^0-y^0)+i\vec{p}(\vec{x}-\vec{y})}}{\left(-p^0 + \sqrt{\vec{p}^2+M^2} - i\epsilon\right)\left(p^0 + \sqrt{\vec{p}^2+M^2} - i\epsilon\right)}.$$

The strategy will be to close the integration contour at $|p^0| \to \infty$ and to use the residue theorem. First, for $x^0 - y^0 > 0$, we can close the contour in the lower half of the complex p^0-plane because

$e^{-ip^0(x^0-y^0)} \to 0$ there. There is then only the residue at $p^0 = \sqrt{\vec{p}^2 + M^2}$ inside the integration contour (the $i\epsilon$ has already been dropped there). The residue theorem gives for the p^0 integral

$$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{i}{2\sqrt{\vec{p}^2+M^2}}\, e^{-i\sqrt{\vec{p}^2+M^2}(x^0-y^0)}\, e^{i\vec{p}(\vec{x}-\vec{y})} \qquad (\text{for } \; x^0-y^0 > 0).$$

In contrast, for $x^0 - y^0 < 0$ we need to close the p^0-integral in the upper half of the complex p^0 plane. The residue theorem given then

$$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{i}{2\sqrt{\vec{p}^2+M^2}}\, e^{i\sqrt{\vec{p}^2+M^2}(x^0-y^0)}\, e^{i\vec{p}(\vec{x}-\vec{y})} \qquad (\text{for } \; x^0-y^0 < 0).$$

These results can be combined to

$$\begin{aligned}\Delta(x-y) &= \int \frac{d^3p}{(2\pi)^3} \frac{i}{2\sqrt{\vec{p}^2+M^2}} e^{-i\sqrt{\vec{p}^2+M^2}|x^0-y^0|+i\vec{p}(\vec{x}-\vec{y})} \\ &= i\theta(x^0-y^0) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\sqrt{\vec{p}^2+M^2}} e^{-i\sqrt{\vec{p}^2+M^2}(x^0-y^0)+i\vec{p}(\vec{x}-\vec{y})} \\ &\quad + i\theta(y^0-x^0) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\sqrt{\vec{p}^2+M^2}} e^{i\sqrt{\vec{p}^2+M^2}(x^0-y^0)+i\vec{p}(\vec{x}-\vec{y})}.\end{aligned}$$

One can understand the first term as being due to particle-type excitations, while the second is due to anti-particle-type excitations. The above Greens function is known as time ordered or Feynmann propagator.

Video: **Lecture17Video05.mp4**

Propagator for non-relativistic fermions. For the non-relativistic fermion, the propagator integral over p^0 has just a single pole at $p^0 = \frac{\vec{p}^2}{2m} + V_0 - i\epsilon$,

$$\Upsilon(x-y) = \mathbb{1} \int \frac{dp^0}{2\pi} \frac{d^3p}{(2\pi)^3} \frac{1}{-p^0 + \frac{\vec{p}^2}{2m} + V_0 - i\epsilon} e^{-ip^0(x^0-y^0)+i\vec{p}(\vec{x}-\vec{y})}.$$

When $x^0 - y^0 > 0$ the contour can be closed below the real p^0-axis, leading to

$$\Upsilon(x-y) = i\, \mathbb{1} \int \frac{d^3p}{(2\pi)^3}\, e^{-i\left(\frac{\vec{p}^2}{2m}+V_0\right)(x^0-y^0)+i\vec{p}(\vec{x}-\vec{y})} \qquad (x^0-y^0 > 0).$$

In contrast, for $x^0 - y^0 < 0$, the contour can be closed above and there is no contribution at all. In summary

$$\Upsilon(x-y) = i\, \theta(x^0-y^0)\, \mathbb{1} \int \frac{d^3p}{(2\pi)^3}\, e^{-i\left(\frac{\vec{p}^2}{2m}+V_0\right)(x^0-y^0)+i\vec{p}(\vec{x}-\vec{y})}.$$

As a consequence of the absence of anti-particle-type excitations, the time-ordered and retarded propagators agree here.

Video: **Lecture17Video06.mp4**

Propagator and correlation functions. Let us also note the relation between propagators and correlation functions. For the free (quadratic) theory one has in the fermionic sector

$$\begin{aligned}\langle \psi_a(x)\bar{\psi}_b(y)\rangle &= \left(\frac{1}{Z_2}\frac{\delta}{\delta\bar{\eta}_a(x)}\frac{\delta}{\delta\eta_b(y)}Z_2[\bar{\eta},\eta,J]\right)_{\bar{\eta}=\eta=J=0}\\ &= -i\Upsilon_{ab}(x-y),\end{aligned}$$

Note that some care is needed with interchanges of Grassmann variables to obtain this expression. Similarly for the bosonic scalar field

$$\begin{aligned}\langle \phi(x)\phi(y)\rangle &= \left(\frac{1}{Z_2}\frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)}Z_2[\bar{\eta},\eta,J]\right)_{\bar{\eta}=\eta=J=0}\\ &= -i\Delta(x-y).\end{aligned}$$

Video: **Lecture17Video07.mp4**

Wick's theorem. More generally one finds for the free theory

$$\begin{aligned}\langle \phi(x_1)\dots\phi(x_n)\rangle &= \left(\frac{1}{Z_2}\left(-i\frac{\delta}{\delta J(x_1)}\right)\cdots\left(-i\frac{\delta}{\delta J(x_n)}\right)Z_2[\bar{\eta},\eta,J]\right)_{\bar{\eta}=\eta=J=0}\\ &= \sum_{\text{pairings}}\left[-i\Delta(x_{j_1}-x_{j_2})\right]\cdots\left[-i\Delta(x_{j_{n-1}}-x_{j_n})\right].\end{aligned}$$

The sum in the last line goes over all possible ways to distribute $x_1,\dots,x_n$ into pairs (x_{j_1},x_{j_2}), (x_{j_3},x_{j_4}), …, $(x_{j_{n-1}},x_{j_n})$. This result is known as *Wick's theorem*. It follows directly from the combinatorics of functional derivatives acting on Z_2.

For example,

$$\begin{aligned}\langle \phi(x_1)\,\phi(x_2)\,\phi(x_3)\phi(x_4)\rangle =&[-i\Delta(x_1-x_2)][-i\Delta(x_3-x_4)]\\ &+[-i\Delta(x_1-x_3)][-i\Delta(x_2-x_4)]\\ &+[-i\Delta(x_1-x_4)][-i\Delta(x_2-x_3)].\end{aligned}$$

Video: **Lecture17Video08.mp4**

In a similar way correlation functions involving $\bar{\psi}$ and ψ can be written as sums over the possible ways to pair ψ and $\bar{\psi}$. For example

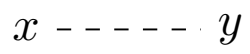
$$\begin{aligned}\left\langle \psi_{a_1}(x_1)\psi_{a_2}(x_2)\bar{\psi}_{a_3}(x_3)\bar{\psi}_{a_4}(x_4)\right\rangle =&-\left\langle \psi_{a_1}(x_1)\bar{\psi}_{a_3}(x_3)\right\rangle\left\langle \psi_{a_2}(x_2)\bar{\psi}_{a_4}(x_4)\right\rangle\\ &+\left\langle \psi_{a_1}(x_1)\bar{\psi}_{a_4}(x_4)\right\rangle\left\langle \psi_{a_2}(x_2)\bar{\psi}_{a_3}(x_3)\right\rangle\\ =&-[-i\Upsilon_{a_1a_3}(x_1-x_3)][-i\Upsilon_{a_2a_4}(x_2-x_4)]\\ &+[-i\Upsilon_{a_1a_4}(x_1-x_4)][-i\Upsilon_{a_2a_3}(x_2-x_3)].\end{aligned}$$

Note that correlation functions at quadratic level (for the free theory) need to involve as many fields ψ as $\bar{\psi}$, otherwise they vanish. Similarly, ϕ must appear an even number of times. For mixed correlation functions one can easily separate ϕ from ψ and $\bar{\psi}$ at quadratic level, because $Z_2[\bar{\eta},\eta,J]$ factorizes. For example,

$$\left\langle \phi(x_1)\,\psi_a(x_2)\,\phi(x_3)\bar{\psi}_b(x_4)\right\rangle = [-i\Delta(x_1-x_3)][-i\Upsilon_{ab}(x_2-x_4)]. \tag{8.2}$$

Video: **Lecture17Video09.mp4**

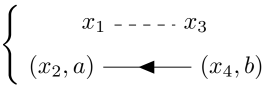
Graphical representation. It is useful to introduce also a graphical representation. We will represent the scalar propagator by a dashed line

$$-i\Delta(x-y) =$$ 

The Feynman propagator for the fermions will be represented by a solid line with arrow,

$$-i\Upsilon_{ab}(x-y) =$$ 

We can represent correlation functions graphically, for example, the mixed correlation function in eqn. (8.2) for the free theory would be

$$\langle\phi(x_1)\psi_a(x_2)\phi(x_3)\bar{\psi}_b(x_4)\rangle = \Big\{$$ 

$$= [-i\Delta(x_1-x_3)]\,[-i\Upsilon_{ab}(x_2-x_4)]\,.$$

Video: **Lecture17Video10.mp4**

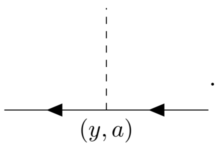
Perturbation theory in g. Let us now also consider the interaction terms in the action. In the functional integral it contributes according to

$$e^{iS[\bar{\psi},\psi,\phi]} = e^{iS_2[\bar{\psi},\psi,\phi]}\,\exp\left[-ig\int d^4x\phi(x)\bar{\psi}_a(x)\psi_a(x)\right].$$

We can assume that g is small and simply expand the exponential where it appears. This will add field factors $\sim \phi(x)\bar{\psi}_a(x)\psi_a(x)$ to correlation functions with an integral over x and an implicit sum over the spinor index a. The resulting expression involving correlation functions can then be evaluated as in the free theory. For example,

$$\begin{aligned}\big\langle\phi(x_1)\psi_b(x_2)\bar{\psi}_c(x_3)\big\rangle &= \big\langle\phi(x_1)\psi_b(x_2)\bar{\psi}_c(x_3)\big\rangle_0 \\ &\quad + \left\langle\phi(x_1)\psi_b(x_2)\bar{\psi}_c(x_3)\left[-ig\int_y \phi(y)\bar{\psi}_a(y)\psi_a(y)\right]\right\rangle_0 + \ldots\end{aligned}$$

The index 0 indicates that the correlation functions get evaluated in the free theory. Graphically, we can represent the interaction term as a vertex

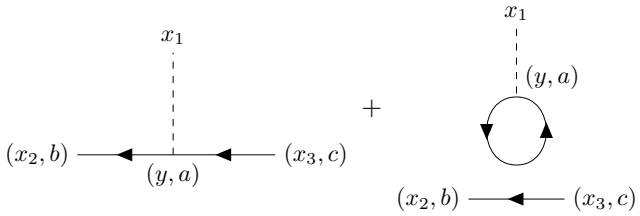
$$-ig\int_y\sum_a =$$ 

For each such vertex we need to include a factor $-ig$ as well as an integral over the space-time variable y and the spinor index a.

Video: **Lecture17Video11.mp4**

Three point function. To order g, we find for the example above

$$\langle\phi(x_1)\psi_b(x_2)\bar\psi_c(x_3)\rangle =$$



$$= -ig\int_y [-i\Delta(x_1-y)][-i\Upsilon_{ba}(x_2-y)][-i\Upsilon_{ac}(y-x_3)]$$
$$+ig\int_y [-i\Delta(x_1-y)][-i\Upsilon_{bc}(x_2-x_3)][-i\Upsilon_{aa}(y-y)].$$

The sign in the last line is due to an interchange of Grassmann fields. The last expression involves the fermion propagator for vanishing argument

$$\Upsilon_{ab}(0) = \delta_{ab}\int\frac{d^4p}{(2\pi)^4}\frac{1}{-p^0+\frac{\vec p^2}{2m}+V_0-i\epsilon} = i\theta(0)\delta_{ab}\delta^{(3)}(0).$$

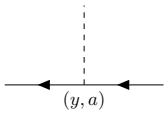
We will set here $\theta(0)=0$ so that the corresponding contribution vanishes. In other words, we will interpret

$$\Upsilon_{ab}(0) = \lim_{\Delta t\to 0}\Upsilon_{ab}(-\Delta t,\vec 0) = 0.$$

Although this is a little ambiguous at this point, it turns out that this is the right way to proceed.

Video: **Lecture17Video12.mp4**

Feynmann rules in position space. To calculate a field correlation function in position space we need to

- have a scalar line ending on x for a factor $\phi(x)$, x - - - - - -
- have a fermion line ending on x for a factor $\psi_a(x)$, (x,a) ⟵
- have a fermion line starting on x for a factor $\bar\psi_a(x)$, (x,a) ⟶
- include a vertex $-ig\int_y$ for every order g,  with integral over y.

- connect lines with propagators $[-i\Delta(x-y)]$ or $[-i\Upsilon_{ab}(x-y)]$
- determine the overall sign for interchanges of fermionic fields.

Video: **Lecture17Video13.mp4**

S-matrix elements from amputated correlation functions. To calculate S-matrix elements from correlation functions, we need to use the LSZ formula. For an outgoing fermion, we need to apply the operator

$$i\left[-i\partial_t - \frac{\vec{\nabla}^2}{2m} + V_0\right]\langle\cdots\psi_a(x)\cdots\rangle$$

and also go to momentum space by a Fourier transform

$$\int_x e^{+i\omega_p x^0 - i\vec{p}\vec{x}}.$$

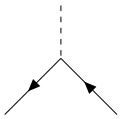
The operator simply removes the propagator leading to x, because of

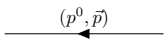
$$i\left[-i\partial_{x^0} - \frac{\vec{\nabla}_x^2}{2m} + V_0\right]\ [-i\Upsilon_{ab}(x-y)] = \delta_{ab}\int\frac{d^4p}{(2\pi)^4}e^{ip(x-y)}\frac{-p^0+\frac{\vec{p}^2}{2m}+V_0}{-p^0+\frac{\vec{p}^2}{2m}+V_0} = \delta_{ab}\delta^{(4)}(x-y).$$

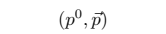
One says that the correlation function is "amputated" because the external propagator has been removed.

Video: **Lecture17Video14.mp4**

Feynman rules for S-matrix elements in momentum space. Moreover, all expressions are brought back to momentum space. One can formulate Feynmann rules directly for contributions to $i\mathcal{T}$ as follows.

- Incoming fermions are represented by an incoming line (to be read from right to left) associated with a momentum $\vec{p}$ and energy $\omega_{\vec{p}} = \frac{\vec{p}^2}{2m} + V_0$.
- Outgoing fermions are represented by an outgoing line
- Incoming or outgoing bosons are represented by and respectively.
- Vertices  contribute a factor $-ig$.
- Internal lines that connect two vertices are represented by Feynmann propagators in momentum space, e. g.



$$= \frac{-i\delta_{ab}}{-p^0+\frac{\vec{p}^2}{2m}+V_0},$$


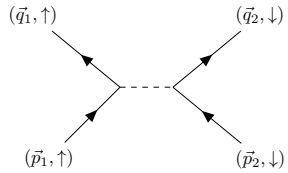
$$= \frac{-i}{-(p^0)^2+\vec{p}^2+M^2}$$

- Energy and momentum conservation are imposed on each vertex.
- For tree diagrams, all momenta are fixed by energy and momenta conservation. For loop diagrams one must include an integral over the loop momentum l_j with measure $\frac{d^4l_j}{(2\pi)^4}$.
- Some care is needed to fix overall signs for fermions.
- Some care is needed to fix overall combinatoric factors from possible interchanges of lines or functional derivatives.

For the last two points it is often useful to go back to the algebraic expressions or to have some experience. We will later discuss very useful techniques based on generating functionals.

Video: **Lecture17Video15.mp4**

Fermion-fermion scattering. We will now discuss an example, the scattering of (spin polarized) fermions of each other. The tree-level diagram is



Because the interaction with the scalar field does not change the spin, the outgoing fermion with momentum $\vec{q}_1$ will have spin $\uparrow$, the one with momentum $\vec{q}_2$ will have spin $\downarrow$. By momentum conservation the scalar line carries the four momentum

$$(\omega_{\vec{p}_1} - \omega_{\vec{q}_1},\ \vec{p}_1 - \vec{q}_1) = \left(\tfrac{\vec{p}_1^2}{2m} - \tfrac{\vec{q}_1^2}{2m}, \vec{p}_1 - \vec{q}_1\right) = (\omega_{\vec{q}_2} - \omega_{\vec{p}_2},\ \vec{q}_2 - \vec{p}_2).$$

The last equality follows from overall momentum conservation, $p_1 + p_2 = q_1 + q_2$. The Feynmann rules give

$$i\mathcal{T} = (-ig)^2 \frac{-i}{-(\omega_{\vec{p}_1} - \omega_{\vec{q}_1})^2 + (\vec{p}_1 - \vec{q}_1)^2 + M^2}.$$

In the center-of-mass frame, one has $\omega_{\vec{p}_1} = \omega_{\vec{p}_2} = \omega_{\vec{q}_1} = \omega_{\vec{q}_2}$ and thus

$$\mathcal{T} = \frac{g^2}{(\vec{p}_1 - \vec{q}_1)^2 + M^2}.$$

Video: **Lecture17Video16.mp4**

Limits of large and small mass. Note that for $g^2 \to \infty$, $M^2 \to \infty$ with g^2/M^2 finite, $\mathcal{T}$ becomes independent of momenta. This resembles closely the $\lambda(\phi^*\phi)^2$ interaction we discussed earlier for bosons. More, generally, one can write

$$(\vec{p}_1 - \vec{q}_1)^2 = 2|\vec{p}_1|^2(1 - \cos(\vartheta)) = 4|\vec{p}_1|^2 \sin^2(\vartheta/2),$$

where we used $|\vec{p}_1| = |\vec{q}_1|$ in the center of mass frame and ϑ is the angle between $\vec{p}_1$ and $\vec{q}_1$ (incoming and outgoing momentum of the spin $\uparrow$ particle). For the differential cross-section

$$\frac{d\sigma}{d\Omega_{q_1}} = \frac{|\mathcal{T}|^2 m^2}{16(\pi)^2},$$

we find

$$\frac{d\sigma}{d\Omega_{q_1}} = \frac{g^4 m^2}{16\pi^2}\left[\frac{1}{4\vec{p}_1^2 \sin^2(\vartheta/2) + M^2}\right]^2.$$

Another interesting limit is $M^2 \to 0$. One has then

$$\frac{d\sigma}{d\Omega_{q_1}} = \frac{g^4 m^2}{64\pi^2 |\vec{p}_1|^4} \frac{1}{\sin^4(\vartheta/2)}.$$

This is the differential cross-section form found experimentally by Rutherford. It results from the exchange of a massless particle or force carrier which is here the scalar boson ϕ and in the case of Rutherford experiment (scattering of α-particles on Gold nuclei) it is the photon. This cross section has a strong peak at forward scattering $\vartheta \to 0$, and for $\vec{p}^2 \to 0$. These are known as colinear and soft singularities. Note that they are regulated by a small, nonvanishing mass $M > 0$.

9 Lorentz symmetry and the Dirac equation

Video: **Lecture18Video01.mp4**

Symmetries are basic concepts for the construction of a model. Particle physics in flat Minkowski space is invariant under Lorentz transformations. Even though the cosmological solutions are not Lorentz invariant, Lorentz invariance holds to a very good approximation on length and time scales that are small compared to the "size" (inverse Hubble parameter) of the universe. The functional integral formulation makes the implementation of symmetries easy. One imposes that the action S is invariant under the symmetry transformations. This is sufficient if the functional measure is also invariant. All symmetry properties follow from the invariance of S and the functional measure.

A given model is specified by the action S that appears in the functional integral. Symmetries restrict the possible form that the action can take. Lorentz symmetry is therefore an important guiding principle for establishing possible models for particle physics. Together with other symmetries as gauge symmetries and the requirement that only a given finite number of derivatives of the fields can appear – this is related to renormalizability – the form of the action is often uniquely determined once a given set of fields is chosen. Because of this crucial importance we will discuss the Lorentz transformations in some detail.

The discussion of Lorentz transformations also introduces in a practical way some elements of the theory of Lie groups that are useful for the present lecture. A more profound knowledge of Lie groups is useful, but not required for the present lecture.

9.1 Lorentz transformations and invariant tensors

Video: **Lecture18Video02.mp4**

Lorentz metric. The cartesian coordinates of space and time are t and $\mathbf{x}$. They are denoted as the contravariant vector

$$x^\mu = (t, \mathbf{x}), \qquad t = x^0.$$

The corresponding covariant vector is

$$x_\mu = (-t, \mathbf{x}) = (-x^0, \mathbf{x}).$$

We can lower and raise indices with the metric tensor $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$, which have the same matrix representation,

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}.$$

As raising and lowering are inverse operations, the multiplication of both matrices is the identity,

$$\eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu_\rho.$$

The sign of the eigenvalues of a metric are called the "signature of the metric". In our case the signature is $(-,+,+,+)$. The explicit transformation rules are given by

$$x_\mu = \eta_{\mu\nu}x^\nu \qquad \text{and} \qquad x^\mu = \eta^{\mu\nu}x_\nu.$$

Video: **Lecture18Video03.mp4**

Lorentz transformations. From the contravariant and covariant four-vectors one can form the scalar product

$$x^\mu x_\mu = -t^2 + \vec{x}^2.$$

The Lorentz transformations can be defined as those linear transformations of x^μ that leave $x^\mu x_\mu$ invariant. In other words, for the transformation

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

we want to find the matrices $\Lambda^\mu{}_\nu$ that leave $x^\mu x_\mu$ invariant. We require that the expression

$$x'^\mu x'_\mu = x'^\mu x'^\nu \eta_{\mu\nu} = \Lambda^\mu{}_\rho x^\rho \Lambda^\nu{}_\sigma x^\sigma \eta_{\mu\nu}$$

is equal to $x^\mu x_\mu$. This results in the condition

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma}. \tag{9.1}$$

Equation (9.1) is the defining equation for Λ. All transformations that fulfill (9.1) are called *Lorentz transformations*. A sequence of two Lorentz transformations leaves $x^\mu x_\mu$ invariant and is therefore again a Lorentz transformation. The inverse Lorentz transformation as well as the identity transformation exists. The Lorentz transformations form a group, the "Lorentz group", which is often denoted by SO(1,3). The group elements are the matrices $\Lambda^\mu{}_\nu$.

So-called *proper, orthochronous Lorentz transformations* can be obtained as a sequence of infinitesimal transformations. Particle physics is invariant under the proper orthochronous Lorentz transformations for which we often use the shorthand "Lorentz transformations". The general transformations (9.1), which we will often call "extended Lorentz transformations", comprise discrete transformations like parity and time reversal. Particle physics is not invariant under those discrete transformations.

Video: **Lecture18Video04.mp4**

Transformation of tensors. Let us consider the contravariant and covariant four-momenta

$$p^\mu = (E, \mathbf{p})$$

$$p_\mu = (-E, \mathbf{p})$$

As we already discussed, we can raise and lower indices of vectors with the metric tensor $\eta_{\mu\nu}$ and the inverse $\eta^{\mu\nu}$. The Lorentz transformation of all contravariant vectors is defined as

$$p'^\mu = \Lambda^\mu{}_\nu p^\nu.$$

The corresponding transformation rule for covariant vectors can be found by expressing both sides in terms of covariant vectors

$$\eta^{\mu\rho} p'_\rho = \Lambda^\mu{}_\nu \eta^{\nu\sigma} p_\sigma.$$

Multiplication with an inverse metric yields

$$p'_\kappa = \eta_{\kappa\mu} \Lambda^\mu{}_\nu \eta^{\nu\sigma} p_\sigma,$$

where we recognize a Lorentz matrix with different index positions,

$$\Lambda_\kappa{}^\sigma = \eta_{\kappa\mu} \Lambda^\mu{}_\nu \eta^{\nu\sigma}.$$

In consequence, covariant vectors transform as

$$p'_\mu = \Lambda_\mu{}^\nu p_\nu.$$

A contravariant second rank tensor $T^{\mu\nu}$ is an object with two upper indices. It has the same transformation property as a product of two contravariant vectors

$$T'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T^{\mu\nu}. \tag{9.2}$$

An example is the energy-momentum tensor that plays an important role in gravity. The relation (9.2) ensures that product relations as

$$T^{\mu\nu} = a^\mu b^\nu$$

are "covariant". This means that the same product relation holds after the Lorentz transformation

$$T'^{\mu\nu} = a'^\mu b'^\nu.$$

This generalises to tensors with an arbitrary number of upper and lower indices. An example for the Lorentz transformation of a more complicated tensor is

$$A'^{\mu\nu\rho}{}_{\sigma\tau} = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} \Lambda^\rho{}_{\rho'} \Lambda_\sigma{}^{\sigma'} \Lambda_\tau{}^{\tau'} A^{\mu'\nu'\rho'}{}_{\sigma'\tau'}.$$

Video: **Lecture18Video05.mp4**

Contractions. The product of a covariant and a contravariant vector is a scalar: It is invariant under Lorentz transformations,

$$\begin{aligned} & s = a^\mu b_\mu, \\ \Rightarrow \quad & s' = \Lambda^\mu{}_\rho a^\rho \Lambda_\mu{}^\sigma b_\sigma = a^\rho \underbrace{\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\tau}_{\eta_{\rho\tau}} \eta^{\tau\sigma} b_\sigma = a^\rho b_\rho = s. \end{aligned}$$

The summation over an upper and lower index is called an "index contraction". Contracted Lorentz indices do not contribute to transformations:

$$(A')_{\mu\rho} (B')^{\rho\nu} = \Lambda_\mu{}^\sigma \Lambda^\nu{}_\tau A_{\sigma\rho} B^{\rho\tau},$$

such that $C_\mu{}^\nu = A_{\mu\rho} B^{\rho\nu}$ transforms as a 2-tensor. Similar to matrices, the order of indices matters for all tensors. For example, one has

$$C^\nu{}_\mu = \eta^{\nu\rho} C_\rho{}^\sigma \eta_{\sigma\mu},$$

which differs from $C_\mu{}^\nu$.

Video: **Lecture18Video06.mp4**

Invariant tensors. The metric $\eta_{\mu\nu}$ is a symmetric invariant tensor under Lorentz transformations

$$\eta'_{\mu\nu} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \eta_{\rho\sigma} = \eta_{\mu\nu}. \tag{9.3}$$

This follows from the defining relation (9.1) for the Lorentz transformations, that we can write in the form

$$\Lambda_{\nu\rho}\Lambda^{\nu\sigma} = (\Lambda^T)_{\rho\nu}\Lambda^{\nu\sigma} = (\Lambda^T)_\rho{}^\nu \Lambda_\nu{}^\sigma = \delta^\sigma_\rho,$$

where the transposed Lorentz matrix Λ^T obeys

$$(\Lambda^T)_{\rho\nu} = \Lambda_{\nu\rho}, \quad (\Lambda^T)^{\rho\nu} = \Lambda^{\nu\rho}.$$

We can identify $(\Lambda^T)_\rho{}^\sigma$ as the inverse matrix of the matrix $\Lambda_\rho{}^\sigma$. With $\Lambda^T\Lambda = 1$ implying $\Lambda\Lambda^T = 1$ we also have the relation

$$\Lambda_\rho{}^\nu (\Lambda^T)_\nu{}^\sigma = \delta^\sigma_\rho,$$

which is equivalent to eq. (9.3). We can equivalently use eq. (9.3) as the defining relation for the Lorentz transformations. From the standpoint of group theory this is a more natural definition since the group elements are defined as transformation matrices that leave the particular tensor $\eta_{\mu\nu}$ invariant. The naming SO(1,3) refers to the signature of $\eta_{\mu\nu}$. It is straightforward to see that $\eta^{\mu\nu}$ and δ^ν_μ are also invariant tensors.

There is only one more tensor that is invariant under Lorentz transformations. This is the totally antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$, the relativistic generalization of the Levi-Civita tensor ϵ_{ijk}. The Levi-Civita symbol with four indices $\epsilon_{\mu\nu\rho\sigma}$ is defined by total antisymmetry and

$$\epsilon_{0123} = 1.$$

It equals 1 for all cyclic permutations of $(0,1,2,3)$, and -1 for all anti-cyclic permutations. The ϵ-tensor with raised indices, $\epsilon^{\mu\nu\rho\sigma}$ has the opposite signs, e. g. $\epsilon^{0123} = -1$.

Let us prove that $\epsilon_{\mu\nu\rho\sigma}$ is invariant under Lorentz transformations. In the following lines we will use the short hand notation $\Lambda_\mu{}^\nu \to \Lambda$, and $\eta_{\mu\nu} \to \eta$. With this notation, the defining relation (9.3) reads

$$\Lambda\eta\Lambda^T = \eta, \tag{9.4}$$

If we compute the determinant on both sides, we find, using $\det(\Lambda)\det(\Lambda^T) = (\det(\Lambda))^2$,

$$\det(\Lambda) = \pm 1.$$

The determinant of Λ can also be calculated by

$$\det(\Lambda) = \frac{1}{4!}\Lambda_{\mu_1}{}^{\nu_1}\Lambda_{\mu_2}{}^{\nu_2}\Lambda_{\mu_3}{}^{\nu_3}\Lambda_{\mu_4}{}^{\nu_4}\epsilon_{\nu_1\nu_2\nu_3\nu_4}\epsilon^{\mu_1\mu_2\mu_3\mu_4} = \frac{1}{4!}\epsilon'_{\mu_1\mu_2\mu_3\mu_4}\epsilon^{\mu_1\mu_2\mu_3\mu_4}.$$

Here ϵ' is the Lorentz transformed tensor. We can verify that $\epsilon'_{\mu\nu\rho\sigma}$ is totally antisymmetric, thus $\epsilon'_{\mu\nu\rho\sigma} = c\,\epsilon_{\mu\nu\rho\sigma}$ with constant c. Using $\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = 4!$ we obtain $\det(\Lambda) = c$ or

$$\epsilon'_{\mu_1\mu_2\mu_3\mu_4} = \det(\Lambda)\epsilon_{\mu\nu\rho\sigma} = \pm\epsilon_{\mu_1\mu_2\mu_3\mu_4}.$$

Only Lorentz transformations with $\det(\Lambda) = +1$ will leave the ϵ-tensor invariant (they are called *proper* Lorentz transformations). The Lorentz transformations which are continuously related to the unit transformation obey $\det(\Lambda) = 1$.

Video: **Lecture18Video07.mp4**

Analogy to Rotations. Equation (9.4) looks very similar to orthogonal transformations $O_\mu{}^\nu$ with

$$O\mathbb{1}O^T = OO^T = \mathbb{1}, \qquad \mathbb{1}_{\mu\nu} = \delta_{\mu\nu},$$

where $\mathbb{1}$ is the unit matrix. The invariant tensor of the orthogonal transformations is the "euclidean metric" $\delta_{\mu\nu}$. In eq. (9.4) the euclidean metric is replaced by the metric tensor $\eta_{\mu\nu}$ for Minkowski space. In short,

- Orthogonal transformations : $\delta_{\mu\nu}$ invariant.
- Lorentz transformation: $\eta_{\mu\nu}$ invariant.
- Analytic continuation: $\delta_{\mu\nu} \to \eta_{\mu\nu}$.

The group of orthogonal transformations in four dimensions is denoted O(4). The analogy that we just discussed motivates the name *Pseudo orthogonal transformations* O(1, 3) where the separated 1 indicates the special role of time in special relativity. The "special orthogonal transformations" SO(4) have in addition the property $\det(\Lambda) = 1$. Similarly, for SO(1,3) one has $\det(\Lambda) = 1$.

Video: **Lecture18Video08.mp4**

Derivatives. The derivative with respect to a contravariant vector is a covariant vector,

$$\partial_\mu = \frac{\partial}{\partial x^\mu}.$$

We will discuss later the Lorentz - transformations of fields more carefully. For the moment we just generalize what we know from three-dimensional rotations, namely that the divergence of a vector field transforms as a scalar field. With $\partial_\mu a^\mu(x)$ transforming as a scalar field, ∂_μ has to compensate the transformation acting on the indices a^μ. For example one has

$$\partial_\mu x^\mu = 4.$$

The momentum operator is

$$\hat{p}_\mu = -i\partial_\mu.$$

It transforms as a covariant vector.

Video: **Lecture18Video09.mp4**

Four-dimensional Fourier transformation. The four-dimensional Fourier transformation of a function $\psi(x)$ is defined as

$$\psi(x) = \int_p e^{ip_\mu x^\mu} \psi(p).$$

With $p_\mu = (-\omega, \vec{p})$ and $p_\mu x^\mu = -\omega t + \vec{p}\vec{x}$ this reads

$$\psi(t, \vec{x}) = \int_\omega \int_{\vec{p}} e^{-i\omega t + i\vec{p}\vec{x}} \psi(\omega, \vec{p}).$$

Note that $p_\mu x^\mu$ is Lorentz invariant.

Video: **Lecture18Video10.mp4**

Covariant equations. For a covariant equation the left hand side and right hand side have the same transformation properties. An example is

$$\partial_\mu F^{\mu\nu} = J^\nu.$$

These are two of the four Maxwell equations. The other two are

$$\partial_\mu \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0.$$

Thus the Maxwell equations are Lorentz-covariant. We will later derive them from a Lorentz-invariant action.

9.2 Lorentz group

Video: **Lecture18Video11.mp4**

Group structure. If we have two elements g_1, g_2 that are elements of a group $\mathscr{G}$, the product of these two elements will still be an element of the group

$$g_3 = g_2 g_1 \in \mathscr{G}.$$

In particular, we can write for matrices

$$(\Lambda_3)^\mu{}_\nu = (\Lambda_2)^\mu{}_\rho (\Lambda_1)^\rho{}_\nu.$$

A group contains always a unit element e such that

$$ge = eg = g$$

for every group element g. For matrices, this unit element is $\delta^\mu{}_\nu$. Furthermore the inverse element g^{-1} exists. Every matrix Λ^μ_ν has an inverse matrix because the determinant of Λ is ± 1. Finally, for a group the multiplication law has to be associative, which is trivial for matrix multiplications.

Video: **Lecture18Video12.mp4**

Discrete symmetries. The (extended) Lorentz transformations contain some discrete symmetries that we discuss next.

Space reflection (parity). The space reflection transformation changes the sign of all space coordinates, $x^j \to -x^j$ for $j \in \{1,2,3\}$ while time stays invariant $t \to t$. The corresponding matrix is

$$P = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The determinant is $\det(P) = -1$. The metric tensor $\eta_{\mu\nu}$ is kept invariant under a space reflection, $P\eta P^T = \eta$.

Video: **Lecture18Video13.mp4**

Time reflection. The time reflection transformation is $x^j \to x^j$ for $j \in \{1,2,3\}$ and $t \to -t$. The corresponding matrix is

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}.$$

The determinant of T is the same as for P, $\det(T) = \det(P) = -1$. Both transformations change the sign of the ϵ-tensor and are therefore improper Lorentz transformations. Again, the metric tensor is invariant under $T\eta T^T = \eta$.

Space-time reflection. The combination of both space and time reflection is

$$PT = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This time the determinant is $+1$.

Video: **Lecture18Video14.mp4**

Video: **Lecture18Video15.mp4**

Continuous Lorentz Transformations. A continuous Lorentz transformation can be obtained as a *product of infinitesimal* transformations. We use *Lorentz transformation* for the continuous Lorentz transformations. Since no jumps are possible, the continuous Lorentz transformations have a determinant $+1$, so we can immediately conclude that the discrete transformations P and T can't be described by continuous Lorentz transformations. As the product PT has a determinant $+1$, one could first think that this may be obtained by continuous transformations, but this is not the case. The reason is that infinitesimal transformations will never change the sign in front of time variable, but actually, PT does exactly this. However, a discrete transformation that can be obtained by infinitesimal ones is the reflection of x and y. The product P_1P_2 with

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad P_1P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

can be obtained as a continuous transformation, as familiar from rotations in two-dimensional space.

9.3 Generators and Lorentz Algebra

Video: **Lecture18Video16.mp4**

Infinitesimal Lorentz Transformations. Let us consider the difference δp_μ between a four-momentum and the transformed four-momentum,

$$\delta p_\mu = p'_\mu - p_\mu = (\Lambda_\mu{}^\nu - \delta_\mu^\nu) p_\nu = \delta\Lambda_\mu{}^\nu p_\nu,$$

with

$$\Lambda_\mu{}^\nu = \delta_\mu^\nu + \delta\Lambda_\mu{}^\nu.$$

In a matrix representation, the infinitesimal Lorentz transformation is given by $\Lambda = \mathbb{1} + \delta\Lambda$. The defining relation of a Lorentz transformation ($\Lambda\eta\Lambda^T = \eta$) then leads to constraints for $\delta\Lambda$ as follows:

$$\begin{aligned} &\Lambda\eta\Lambda^T = \eta \\ &\Leftrightarrow (1+\delta\Lambda)\eta(1+\delta\Lambda)^T = \eta \\ &\Leftrightarrow \delta\Lambda\,\eta + \eta\,\delta\Lambda^T = 0. \end{aligned}$$

In this last line we neglected the 2nd order term in $\delta\Lambda$. If we write down this equation in the index notation of eq. (9.1), we have

$$\begin{aligned} \delta\Lambda_\rho{}^\mu \eta_{\mu\sigma} + \delta\Lambda_\sigma{}^\nu \eta_{\rho\nu} &= 0, \\ \delta\Lambda_{\rho\sigma} + \delta\Lambda_{\sigma\rho} &= 0. \end{aligned}$$

This equation tells us that $\delta\Lambda_{\mu\nu}$ is antisymmetric, while $\delta\Lambda_\mu{}^\nu$ is not antisymmetric. The matrices have six independent elements, what is obvious for $\delta\Lambda_{\mu\nu} = -\delta\Lambda_{\nu\mu}$. The number of independent elements in an antisymmetric matrix equals the number of linearly independent antisymmetric matrices. The six matrices represent *three infinitesimal rotations* and *three infinitesimal boosts*.

Video: **Lecture18Video17.mp4**

Generators. Let us write the infinitesimal transformation of the momentum vector in the following way,

$$\delta p_\mu = i\epsilon_z (T_z)_\mu{}^\nu p_\nu, \qquad z = 1 \ldots 6,$$

where a sum over z is implied. Any infinitesimal Lorentz transformation can be represented as a linear combination in this form

$$\delta\Lambda_\mu{}^\nu = i\epsilon_z (T_z)_\mu{}^\nu. \tag{9.5}$$

Video: **Lecture18Video18.mp4**

For the six independent generators we choose

$$\text{rotations}: \qquad (T_1)_{\mu\nu} = (T_1)_\mu^\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \tag{9.6}$$

$$(T_2)_{\mu\nu} = (T_2)_\mu^\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad (T_3)_{\mu\nu} = (T_3)_\mu^\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{9.7}$$

$$\text{boosts}: \qquad (T_4)_{\mu\nu} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (T_4)^{\nu}_{\mu} = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(T_5)^{\nu}_{\mu} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (T_6)^{\nu}_{\mu} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}. \tag{9.8}$$

Some remarks may be useful:

- T_1 is a rotation around the x-axis (only y and z components change). Similarly T_2 is a rotation around the y-axis and T_3 a rotation around the z-axis.
- For the rotation matrices, raising and lowering of indices doesn't change anything. The reason is that the metric tensor has a -1 only in the zero component and the rotation matrices are zero in the first row and column.
- For the boost matrices, raising of the first index changes the sign of the first row of the matrix (see T_4). After raising the index, the boost matrices are not any longer antisymmetric. Explicitly, one has

$$(T_4)^{\mu}{}_{\nu} = \eta^{\mu\rho}(T_4)_{\rho\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -(T_4)^{\nu}_{\mu}.$$

- In order to see that T_1, T_2 and T_3 are the generators of rotations, we may compare them to the infinitesimal rotations in two dimensions. A general rotation matrix,

$$R = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix},$$

  reads for an infinitesimal angle $\phi = \epsilon$

$$R = \begin{pmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{pmatrix}.$$

  The difference to the identity is

$$\delta R = \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix},$$

  This is equivalent to eq. (9.5) if we multiply the generator (9.6) with $i\epsilon$. The i in the definition of the generators is chosen such that T_1, T_2, T_3 are hermitian matrices.
- The matrices T_4, T_5 and T_6 generate boosts in x, y and z direction.

Video: **Lecture19Video01.mp4**

Lorentz algebra. The product of two group elements is again a group element. From this we can conclude that the commutator of two generators must again be a generator. In general we can therefore write

$$[T_x, T_y] = i f_{xyz} T_z, \tag{9.9}$$

where the sum over z is implied. The quantities f_{xyz} are called the *structure constants* of a group. The structure constants are central elements for Lie groups. They encode the algebraic properties. For the example of the three-dimensional rotation group SO(3) with $z = 1..3$ one has $f_{xyz} = \epsilon_{xyz}$. This tells us that the order matters for a sequence of rotations around different axes.

The central relation (9.9) can be shown as follows. Let us write two transformations Λ_1 and Λ_2 as

$$\Lambda_1 = e^{iA}, \quad \Lambda_2 = e^{iB}, \qquad A = \epsilon_z^{(A)} T_z, \quad B = \epsilon_y^{(B)} T_y.$$

The combined transformation,

$$\Lambda_1^{-1} \Lambda_2^{-1} \Lambda_1 \Lambda_2 = e^{-iA} e^{-iB} e^{iA} e^{iB} = e^{iC},$$

is again an element of the group and therefore $C = \epsilon_w^{(C)} T_w$. We use

$$e^{iA} e^{iB} = e^{iB} e^{iA} + [B, A] + \dots$$

and expand the combined transformation in lowest non-trivial order ϵ^2

$$\begin{aligned} \mathbb{1} + [B, A] &= \mathbb{1} + iC, \quad [B, A] = iC, \\ -\epsilon_y^{(B)} \epsilon_z^{(A)} [T_y, T_z] &= i \tilde{\epsilon}_w^{(C)} T_w. \end{aligned} \tag{9.10}$$

Here $\tilde{\epsilon}_w^{(C)}$ are real parameters of the order ϵ^2. The matrix equation (9.10) can only be true if the commutators $-i[T_z, T_y]$ are linear combinations of generators,

$$-i[T_z, T_y] = c_w^{(zy)} T_w.$$

The coefficients $c_w^{(zy)} = f_{zyw}$ can be identified with the structure constants.

Video: **Lecture19Video02.mp4**

Example. As an example, we consider a rotation in three dimensional space. We want to rotate a system

- by an angle α around the y-axis,
- by an angle β around the x-axis,
- by an angle $-\alpha$ around the y- axis,
- and finally by an angle $-\beta$ around the x-axis.

The result of a product of infinitesimal rotations is again an infinitesimal rotation,

$$\begin{aligned} &\left(1 - i\beta T_x - \tfrac{1}{2}\beta^2 T_x^2\right) \left(1 - i\alpha T_y - \tfrac{1}{2}\alpha^2 T_y^2\right) \left(1 + i\beta T_x - \tfrac{1}{2}\beta^2 T_x^2\right) \left(1 + i\alpha T_y - \tfrac{1}{2}\alpha^2 T_y^2\right) \\ &= 1 - \alpha\beta(T_x T_y - T_y T_x) = 1 - i\alpha\beta T_z \end{aligned}$$

The first order is zero, and the terms $\propto T_x^2$ and $\propto T_y^2$ cancel, too. The product $\alpha\beta$ is the parameter of the resulting infinitesimal transformation.

For the special case of a rotation in three dimensional space the commutation relation

$$[T_1, T_2] = iT_3$$

follows directly from the multiplication of the matrices specified in eqs.(9.6), (9.7). More generally, the generators of rotations obey

$$[T_k, T_l] = i\epsilon_{klm} T_m \quad \text{for} \quad k, l, m \in \{1, 2, 3\}.$$

The calculation of this example gives us already some commutation relations of the generators of the Lorentz group, if we consider the T_i as 4 x 4 matrices with zeroes in all elements of the first column and row. This is of course not surprising, as the three dimensional rotations are a subgroup of the Lorentz group. The other structure constants f_{xyz} where one element of x, y, z is 0 can also be found from the specified matrices. We will indicate them later explicitly.

9.4 Representations of the Lorentz group

Video: **Lecture19Video03.mp4**

The spin matrices $s_k = \frac{1}{2}\tau_k$ are given by the Pauli matrices τ_k. The commutation relations for the Pauli matrices imply that the spin matrices obey precisely the same commutation relations as the generators of the rotation group $SO(3)$,

$$[\tau_k, \tau_l] = 2i\epsilon_{klm}\tau_m, \quad [s_k, s_l] = i\epsilon_{klm} s_m.$$

We can define 3×3-matrices $\tilde{T}_k$ by omitting in eqs.(9.6), (9.7) the first row and column. The 3×3-matrices $\tilde{T}_k$ and the spin matrices s_k obey the same commutation relation. They correspond to *different representations of the rotation group*. The matrices $\tilde{T}_m$ are a three-dimensional and the matrices $\tau_m/2$ are a two-dimensional representation of the group $SO(3)$. Different representations of a group correspond to different sets of matrices that obey the same product rules for their multiplication, as given by the product rule for the abstract group elements. The dimension of the different matrix-representations can differ. For continuous transformations the transformation rules of a Lie group are encoded in the commutator relations for the generators. Two different sets of generators with the same commutation relations generate two different representations of the group.

d-dimensional representation. For a d-dimensional representation the set of $d \times d$- matrices T_z obeys the commutation relations of a given group. In the physics literature, one often uses the representation to design (somewhat improperly) also a d-component object on which the matrices T_z act.

Representations of the Lorentz group. Let us summarize what we know about the Lorentz group: It is $SO(1,3)$ and is generated by a set of 6 independent matrices T_z, which obey the commutation relations

$$[T_x, T_y] = if_{xyz} T_z.$$

For $x, y, z \in 1, 2, 3$ we know already that $f_{xyz} = \epsilon_{xyz}$. The dimension of the matrices T_z depends on the representation of the group: If we have a d-dimensional representation, the matrices will be $d \times d$.

Video: **Lecture19Video04.mp4**

Example: Energy momentum tensor. For a vector, the dimension is $d = 4$ because we have three space and one time coordinate. The generators in the four-dimensional representation are given by the six 4×4-matrices in eqs. (9.6)-(9.8). They describe the infinitesimal transformations of covariant vectors. Infinitesimal transformations of contravariant vectors are given by different 4×4-matrices. The corresponding generators obey the same commutation relations as for the transformation of covariant vectors. They form an equivalent, but different four-dimensional representation.

We next investigate the representations for other objects as tensors. Consider the symmetric energy-momentum-tensor $T^{\mu\nu} = T^{\nu\mu}$. We know that it has 10 independent elements: 4 diagonal and 6 off-diagonal ones. Let us write all independent elements into a 10 dimensional vector ψ^α. The generator T_z that transforms this vector into a new vector $\psi^\alpha + \delta\psi^\alpha$ must now be a 10×10 matrix:

$$\delta\psi^\alpha = i\epsilon_z (T_z)^\alpha_{\ \beta} \psi^\beta .$$

The elements of ψ are the elements of the energy-momentum tensor $T^{\mu\nu}$ and we therefore know the Lorentz transformations.

$$\delta T^{\mu\nu} = i\epsilon_z (T_z)^{\mu\nu}_{\ \ \mu'\nu'} T^{\mu'\nu'} . \tag{9.11}$$

Here $(\mu\nu) = (\nu\mu)$ is considered as a double index, $\alpha = (\mu\nu)$. Historically, the symbol T is used both for generators and the energy-momentum tensors. This leads to the unfortunate labelling of two different objects by the same symbol in eq. (9.11). Generators and energy momentum tensor should not be confused. The elements of generators of $(T_z)^{\mu\nu}_{\ \ \mu'\nu'}$ can easily be computed from the Lorentz transformation of a tensor established before.

Video: **Lecture19Video05.mp4**

Irreducible Representations. We can decompose T into the trace and the remaining traceless part $\tilde{T}$:

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} - \tfrac{1}{4}\theta\eta^{\mu\nu}, \quad \eta_{\mu\nu}\tilde{T}^{\mu\nu} = 0.$$

Here θ is the trace of the energy-momentum tensor and $\tilde{T}^{\mu\nu}$ is the traceless part. The trace of the energy-momentum tensor is defined as

$$\theta = T^\mu_\mu = \eta_{\nu\mu} T^{\mu\nu} .$$

The trace is a scalar and therefore invariant under Lorentz transformations:

$$T'^\mu_\mu = T^\mu_\mu .$$

Furthermore, the traceless tensor $\tilde{T}$ remains traceless when it is transformed. It has nine independent components. In this way, we have reduced the ten-dimensional representation into a nine-dimensional representation $(\tilde{T}^{\mu\nu})$ and a one-dimensional representation (θ), i.e. $10 = 9 + 1$. The transformation of traceless, symmetric tensors is represented by 9×9 matrices as generators. One-dimensional representations correspond to invariants. If a representation cannot be decomposed further into separate representations it is called "irreducible". The nine-dimensional representation associated to $\tilde{T}^{\mu\nu}$ is irreducible. The antisymmetric tensors form a six-dimensional irreducible representation.

Video: **Lecture19Video06.mp4**

Irreducible representations so far. At this stage we may summarize our present findings about irreducible representations of the Lorentz group

Representation	Dimension
scalar	1
vector	4
symmetric and traceless tensors	9
antisymmetric tensors	6
spinor	?

The spinor-representation will generalize the two-dimensional representation of $SO(3)$ by the spin matrices s_z. It will be a key element for the description of fermions.

Video: **Lecture19Video07.mp4**

9.5 Transformation of Fields

So far we have discussed the transformation of simple objects as x^μ or p_μ. In quantum field theory the basic objects are fields, as scalar fields $\varphi(x)$ or vector fields $A_\mu(x)$. Their transformation involves a part related to the transformation of the coordinates on which they depend, and another part related to the Lorentz-indices of these fields as for $A_\mu(x)$. In quantum field theory the energy-momentum tensor $T^{\mu\nu}(x)$ is also a field, and we will have to supplement the transformation discussed before by a part arising from the transformation of coordinates. The transformations discussed above refer, strictly speaking, to the transformation of x-independent energy-momentum tensors.

Scalar Field $\varphi(x)$. We start with the questions how scalar fields $\varphi(x)$ transform? For a scalar field, the value of the transformed field φ' at the transformed coordinate x' is the same as the field value before the transformation.

$$\varphi'(x') = \varphi(x). \tag{9.12}$$

We concentrate on an infinitesimal transformation and recall the transformation of the space-time vector x^μ:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad x'^\mu = x^\mu + \delta x^\mu, \quad \delta x^\mu = \delta\Lambda^\mu{}_\nu x^\nu.$$

Since $x - \delta x$ is transformed to x we find from eq. (9.12) the transformed field value φ' at the coordinate x

$$\varphi'(x) = \varphi(x - \delta x).$$

We can visualise this by the picture shown in 5. We want to consider field transformations at fixed

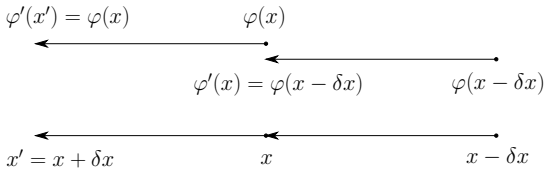


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Figure 5. Transformation of a scalar field

coordinates and therefore employ

$$\varphi'(x) = \varphi(x) + \delta\varphi(x) = \varphi(x - \delta x).$$

The transformation of φ at fixed x is called an **active transformation**, and we will employ this formulation. In contrast, leaving φ fixed and changing coordinates would be a **passive transformation**. (The combination of both does not change the field, $\varphi'(x') = \varphi(x)$.)

The infinitesimal change of the field $\delta\varphi$ at fixed coordinates obeys

$$\delta\varphi = \varphi(x - \delta x) - \varphi(x) = -\partial_\mu \varphi(x)\ \delta x^\mu.$$

We assume here that $\varphi(x)$ is a differentiable function, such that the second line reflects the definition of partial derivatives.

Video: **Lecture19Video08.mp4**

For Lorentz transformations we insert $\delta x^\mu = \delta\Lambda^\mu{}_\nu x^\nu$ and find

$$\begin{aligned}\delta\varphi &= -\delta\Lambda^\mu{}_\nu x^\nu \partial_\mu \varphi(x) \\ &= -i\epsilon_z (T_z)^\mu{}_\nu x^\nu \partial_\mu \varphi(x) \\ &= i\epsilon_z L_z \varphi(x).\end{aligned}$$

For the last identity we have defined the generators for scalar fields

$$L_z = -(T_z)^\mu{}_\nu x^\nu \partial_\mu.$$

The generators L_z contain a differential operator. Fields are infinite dimensional representations in a strict sense.

Video: **Lecture19Video09.mp4**

The letter L was not chosen arbitrary, as L_1, L_2 and L_3 are the angular momenta. For instance L_1 can be written as

$$L_1 = -(T_1)^\mu{}_\nu x^\nu \partial_\mu.$$

T_1 has only two non-zero elements: $(T_1)^2{}_3 = -i$ and $(T_1)^3{}_2 = i$, implying

$$L_1 = -ix^2 \tfrac{\partial}{\partial x^3} + ix^3 \tfrac{\partial}{\partial x^2} = -i(y\partial_z - z\partial_y).$$

Associating derivatives with momenta the three generators L_1, L_2 and L_3 have the structure of angular momentum as we know it from classical mechanics: $L = r \times p$. In quantum mechanics, this is precisely the structure of the angular momentum operator. You may recall the commutation relation $[L_k, L_l] = i\epsilon_{klm} L_m$ which shows that we deal indeed with a representation of $SO(3)$. Notice, however, that we have not used any operator formalism here. The generator L_z and their relations arise directly from the transformation properties of "classical" fields.

We conclude that the transformation of fields with Lorentz indices will have two ingredients. The first arises from the transformation of coordinates, the second is related to the Lorentz indices. For scalars one has only the coordinate part.

Video: **Lecture19Video10.mp4**

Vector Field $A^\mu(x)$. Contravariant vectors transform as

$$A^\mu(x) \quad \to \quad A'^\mu(x) = A^\mu(x) + \delta A^\mu(x),$$

where

$$\delta A^\mu(x) = \delta\Lambda^\mu_{\ \nu} A^\nu(x) + x^\rho \delta\Lambda_\rho^{\ \sigma} \partial_\sigma A^\mu(x).$$

Here, $\delta\Lambda^\mu_{\ \nu} A^\nu$ is the usual transformation law for covariant vectors. The part $x^\rho \delta\Lambda_\rho^{\ \sigma} \partial_\sigma A^\mu$ reflects the change of the coordinates. It is the same as for scalar fields, using

$$x^\rho \delta\Lambda_\rho^{\ \sigma} \partial_\sigma A^\mu = -\delta\Lambda^\sigma_{\ \rho} x^\rho \partial_\sigma A^\mu,$$

where we employ the antisymmetry of $\delta\Lambda_{\sigma\rho}$. Since the Minkowski metric is Lorentz invariant the upper and lower positions of contracted indices can be exchanged, i.e. $x^\rho \delta\Lambda_\rho^{\ \sigma} = x_\rho \delta\Lambda^{\rho\sigma}$, $\delta\Lambda_\rho^{\ \sigma}\partial_\sigma = \delta\Lambda_{\rho\sigma}\partial^\sigma$ etc. The part arising from the transformation of coordinates is always the same, no matter what kind of field we are transforming.

Covariant vectors transform as:

$$\delta A_\mu(x) = \delta\Lambda_\mu^{\ \nu} A_\nu(x) + x^\rho \delta\Lambda_\rho^{\ \sigma} \partial_\sigma A^\mu(x).$$

The covariant derivative transforms as

$$\partial_\mu\varphi(x) \quad \to \quad (\partial_\mu\varphi)'(x) = \partial_\mu(\varphi(x) + \delta\varphi(x)) = \partial_\mu\varphi(x) + \delta\partial_\mu\varphi(x).$$

One infers

$$\delta\partial_\mu\varphi(x) = \partial_\mu\delta\varphi(x) = \partial_\mu(x^\rho\delta\Lambda_\rho^{\ \sigma}\partial_\sigma\varphi(x)) = \delta\Lambda_\mu^{\ \sigma}\partial_\sigma\varphi(x) + (x^\rho\delta\Lambda_\rho^{\ \sigma}\partial_\sigma)(\partial_\mu\varphi(x)).$$

Thus $\partial_\mu\varphi(x)$ indeed transforms as a covariant vector field. With a similar argument one finds that the contravariant derivative transforms as a contravariant vector field. This implies

$$\delta(\partial^\mu\varphi(x)\partial_\mu\varphi(x)) = (x^\rho\delta\Lambda_\rho^{\ \sigma}\partial_\sigma)(\partial^\mu\varphi(x)\partial_\mu\varphi(x)),$$

and $\partial^\mu\varphi(x)\partial_\mu\varphi(x)$ therefore transforms as a scalar field.

Video: **Lecture19Video11.mp4**

Invariant Action Our aim is the construction of a Lorentz-invariant action as the starting point of formulating the functional integral for a quantum field theory. This is a central piece of our lecture. With all the machinery we have developed it is almost trivial. Now our works pays off. The basic construction principle is that an invariant action obtains as a space-time integral over a quantity that transforms as a scalar field. This scalar field is typically a composite expression, as $\partial^\mu\varphi(x)\partial_\mu\varphi(x)$.

Let $f(x)$ be some (composite) scalar field with infinitesimal Lorentz transformation

$$\delta f = x^\rho\delta\Lambda_\rho^{\ \sigma}\partial_\sigma f.$$

Examples constructed from scalar fields $\varphi(x)$ are $f = \varphi^2$ or $f = V(\varphi)$ or $f = \partial^\mu\varphi\partial_\mu\varphi$. It follows that an action constructed as

$$S = \int d^4x\ f(x)$$

is Lorentz invariant, $\delta S = 0$. The proof of $\delta S = 0$ employs the vanishing of an integral over total derivatives,

$$\begin{aligned}\delta S &= \int d^4x\ \delta f(x) = \int d^4x\ x^\rho \delta\Lambda_\rho{}^\sigma \partial_\sigma f \\ &= \int d^4x\ \partial_\sigma(x^\rho \delta\Lambda_\rho{}^\sigma f) - \int d^4x\ \delta^\rho_\sigma \delta\Lambda_\rho{}^\sigma \partial_\sigma f = 0.\end{aligned}$$

More precisely, the first integral is zero because we always assume the absence of boundary contributions. Then total derivatives in $\mathscr{L}$ can be neglected, $\int d^4x \partial_\mu A = 0$. The second integral is zero because of the antisymmetry of $\Lambda_{\rho\sigma}$:

$$\delta^\rho_\sigma \delta\Lambda_\rho{}^\sigma = \eta^{\rho\sigma}\delta\Lambda_{\rho\sigma} = 0.$$

Video: **Lecture19Video12.mp4**

Examples. It is now rather straight forward to construct possible actions for quantum field theories. Lorentz invariant actions can be written as sums of invariant terms.

$$S = \int d^4x \sum_k \mathscr{L}_k(x),$$

where $\mathscr{L}_k$ are (composite) scalar fields. Let us give some examples:

$$\bullet \mathscr{L} = \partial^\mu\varphi^*\partial_\mu\varphi + m^2\varphi^*\varphi.$$

This is the action for a **free charged scalar field**. It describes particles with mass m like e.g. pions $\pi^\pm$ with interactions neglected.

$$\bullet \mathscr{L} = \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

This is the action for free photons. The antisymmetric tensor field $F_{\mu\nu}(x) = -F_{\nu\mu}(x)$ is the **electromagnetic field strength**. As in classical electrodynamics the components F_{0k} describe the electric field, while $\epsilon_{ijk}F_{jk}$ denotes the components of the magnetic field.

$$\bullet \mathscr{L} = (\partial^\mu + ieA^\mu)\varphi^*(\partial_\mu - ieA_\mu)\varphi + \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}.$$

This describes a charged scalar field interacting with photons. A model with the corresponding action is called **scalar QED** where QED stands for quantum electrodynamics. The QED for electrons needs the Lorentz invariant description of fermions in order to account for the spin of the electrons and positrons.

9.6 Functional Integral, Correlation Functions

Video: **Lecture19Video13.mp4**

Measure and partition function. The partition function $Z = \int D\varphi\ e^{-S}$ is invariant if the measure $\int D\varphi$ and the action S are invariant. For a scalar field $\varphi(x)$ the measure is indeed invariant

$$\int D\varphi(x) = \int D\varphi'(x).$$

This follows from the equivalence of active and passive transformations,

$$\varphi'(x) = \varphi(\Lambda^{-1}x).$$

Since one integrates over φ at every point x, this is equivalent to an integration at every point $\Lambda^{-1}x$. Similarly, for vectors one has

$$\int DA^\mu = \int DA'^\mu \times \text{Jacobian}.$$

The Jacobian is a product over all x of factors $\det\Lambda$. Since $\det\Lambda = 1$. The functional measure is again invariant.

Regularisation. The product over all positions is not always well defined. One typically aims for a regularisation, which defines the functional integral as some limiting process of finite-dimensional integrals, as we have done it in the beginning of this lecture. One possibility is a lattice regularisation, for which the continuous manifold of points is replaced by the sites of a lattice – typically *a* d-dimensional hypercubic lattice. This regularisation has the advantage that gauge symmetries can be implemented rather easily. A lattice of points does not admit continuous Lorentz transformations. For such a regularisation the functional measure is not Lorentz invariant. In this case one expects Lorentz invariance to show up only in the continuum limit for which the lattice distance gets very small as compared to all length scales of interest. Lorentz invariance of the continuum limit has to be proven. For the models that are investigated this is indeed realised. We will simply assume here that a Lorentz-invariant measure exists and use this property implicitly.

Correlation Function. For a Lorentz invariant action and measure the correlation functions have covariant transformation properties. For example, the correlation function

$$\langle\varphi(x)\varphi(x')\rangle = Z^{-1}\int D\varphi\varphi(x)\varphi(x')e^{-S}$$

transforms in the same way as the product $\varphi(x)\varphi(x')$. This *covariant construction* makes it easy to construct an invariant S-matrix. Then scattering cross sections and similar quantities are Lorentz invariant.

Summary Explicit Lorentz covariance is an important advantage of the functional integral formulation of quantum field theories. In the operator formalism the implementation of Lorentz symmetry can sometimes be more complicated. The basic reason is that the operator formalism is a Hamiltonian formalism. The Hamiltonian is not Lorentz-invariant – it singles out a time direction. In contrast, the action is a four-dimensional object for which Lorentz invariance can be implemented in a very straightforward way.

9.7 Spinor representations

Video: **Lecture20Video01.mp4**

Spinor representations of the Lorentz group. As a final building block for the construction of quantum electrodynamics (QED) or quantum chromodynamics (QCD) for the strong interactions, or the standard model (SM) of particle physics which unifies the weak and electromagnetic interactions, we need a field for fermions and its transformation with respect to the Lorentz group. Electrons have half-integer spin. With respect to the rotation group electrons transform according to a two-dimensional representation, the spinor representation. Before generalising to the Lorentz

group, we first investigate the spinor representation of the rotation group $SO(3)$, which is a subgroup of the Lorentz group. For *nonrelativistic* electrons this subgroup is all that matters.

The two-dimensional spinor representation of the rotation group involves two complex fields $\chi_1(x)$ and $\chi_2(x)$, that we order in a two-component complex vector field:

$$\chi(x) = \begin{pmatrix} \chi_1(x) \\ \chi_2(x) \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}.$$

The $SO(3)$-rotations act on this field as

$$\delta\chi = i\epsilon_z T_z \chi + \delta'\chi, \quad \delta'\chi(x) = x^\rho \delta\Lambda_\rho{}^\sigma \partial_\sigma \chi(x).$$

We will omit δ' in the notation from now on since this universal contribution from the change of coordinates is the same for all fields. It is implicitly added if we transform fields. The spinor representation of $SO(3)$ is two-dimensional. The three 2×2 matrices T_z are given by the Pauli matrices,

$$T_z = \tfrac{1}{2}\tau_z, \quad z = 1, 2, 3.$$

The generators T_z of the two-dimensional representations are identical to the spin-matrices introduced earlier. We use here a common symbol T for generators in arbitrary representations. Since the Pauli matrix τ_2 is purely imaginary, the two component field $\chi(x)$ has to be complex. No real two-component representation of $SO(3)$ exists. The fermion fields are Grassmann variables. This is not relevant for symmetry transformations.

For a relativistic quantum field theory for electrons or protons, neutrons or neutrinos we have to answer two questions

- What are the spinor representations of the Lorentz group, i.e. what are the generators T_z for $z = 1, \ldots, 6$?

- Are there two-dimensional representations, i.e. are there six 2×2 matrices that obey

$$[T_x, T_y] = i f_{xyz} T_z ?$$

Two-dimensional representations are the minimal setting, since already the rotation-subgroup requires a two-component complex field. We will see that neutrinos can be described by such a two-dimensional representation. For charged fermions we will find a four-dimensional complex representation, the so called "Dirac spinors".

A systematic construction of the spinor representations of the Lorentz group belongs to the mathematical field of **representation** theory. We do not attempt here to follow a systematic construction principle. We rather follow the results of Dirac, Pauli and Weyl, indicate the generators in the appropriate representations, and verify that they obey the commutation relations of the Lorentz group.

Video: **Lecture20Video02.mp4**

Dirac Spinors. Dirac spinors are four-dimensional complex fields

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}.$$

It is convenient to label the six generators in the corresponding four-dimensional spinor representation by

$$i\epsilon_z T_z = \tfrac{i}{2}\varepsilon_{\hat{\mu}\hat{\nu}} T^{\hat{\mu}\hat{\nu}}, \quad T^{\hat{\mu}\hat{\nu}} = -T^{\hat{\nu}\hat{\mu}}, \quad \varepsilon_{\hat{\mu}\hat{\nu}} = -\varepsilon_{\hat{\nu}\hat{\mu}}.$$

With $\hat{\mu}, \hat{\nu} = 0, 1, 2, 3$ the six generators $T^{\hat{\mu}\hat{\nu}}$ are now labelled by $\hat{\mu}\hat{\nu}$ instead of z. The factor $\frac{1}{2}$ accounts for $\frac{1}{2}(\varepsilon_{12}T^{12} + \varepsilon_{21}T^{21}) = \varepsilon_{12}T^{12}$ etc. The pairs $(\hat{\mu}\hat{\nu})$ are just labels of the six generators and we have put the hats on $\hat{\mu}$ and $\hat{\nu}$ in order to avoid confusion: the matrices $T^{\hat{\mu}\hat{\nu}}$ are fixed 4×4 matrices and Lorentz transformations do not act on $(\hat{\mu}\hat{\nu})$ as they do on fields. As an example T^{12} is itself a 4×4 matrix with elements $(T^{12})^{\mu}_{\ \nu}$ or $(T^{12})_{\mu}^{\ \nu}$. Thus $\hat{\mu}\hat{\nu} = 12$ is just a convenient label for this matrix, which we could also have labelled equivalently by $z = 3$.

Video: **Lecture20Video03.mp4**

Dirac matrices. The matrices $T^{\hat{\mu}\hat{\nu}}$ are obtained as the commutators of the Dirac matrices $\gamma^{\hat{\mu}}$

$$T^{\hat{\mu}\hat{\nu}} = -\tfrac{i}{4}\left[\gamma^{\hat{\mu}}, \gamma^{\hat{\nu}}\right]. \tag{9.13}$$

The **Dirac matrices** $\gamma^{\hat{\mu}}$ are four complex 4×4 matrices, given explicitly by

$$\gamma^0 = \begin{pmatrix} 0 & -i\mathbf{1} \\ -i\mathbf{1} & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & -i\tau_k \\ i\tau_k & 0 \end{pmatrix}, \quad k = 1, 2, 3 \quad , \tag{9.14}$$

with τ_k, $k = 1, 2, 3$ the Pauli matrices. In the following, we often omit the hat for γ^{μ}, or $T^{\mu\nu}$. We should always recall, however, that Lorentz transformations act only on fields, whereas the matrices γ^{μ} are kept fixed.
An explicit computation of the generators (9.13) from the Dirac matrices (9.14) is a good exercise. One finds that they are of the block-diagonal form

$$T^{\mu\nu} = \begin{pmatrix} T_+^{\mu\nu} & 0 \\ 0 & T_-^{\mu\nu} \end{pmatrix} \tag{9.15}$$

where the $T_{\pm}^{\mu\nu}$ are 2×2 matrices. The ij-components are rotations,

$$T_+^{ij} = T_-^{ij} = \tfrac{1}{2}\varepsilon^{ijk}\tau_k, \quad i, j, k \in \{1, 2, 3\}.$$

For a rotation around the z-axis ($\varepsilon_{12} = -\varepsilon_{21} \equiv \varepsilon_3$), one has

$$\begin{aligned} \varepsilon_3 T_3 &\equiv \tfrac{1}{2}(\varepsilon_{12}T^{12} + \varepsilon_{21}T^{21}), \\ &= \varepsilon_3 T^{12} = \varepsilon_3(\tfrac{1}{2}\varepsilon^{123}\tau_3) = \varepsilon_3 \tfrac{\tau_3}{2}, \end{aligned}$$

confirming $T_3 = \frac{\tau_3}{2}$. If we denote

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \psi_L, \quad \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} = \psi_R, \quad \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \psi,$$

then ψ_L and ψ_R transform as 2-component spinors with respect to rotations.

Video: **Lecture20Video04.mp4**

The generators T^{0k} are boosts,

$$T^{0k}_{+} = -T^{0k}_{-} = -\tfrac{i}{2}\tau_k.$$

The boost generators are not hermitian. They act on ψ_L and ψ_R with different signs.

Video: **Lecture20Video05.mp4**

Commutation relation of generators. The commutation relations can be computed as

$$[T^{\mu\nu}, T^{\rho\sigma}] = i\left(\eta^{\mu\rho}T^{\nu\sigma} - \eta^{\mu\sigma}T^{\nu\rho} + \eta^{\nu\sigma}T^{\mu\rho} - \eta^{\nu\rho}T^{\mu\nu}\right). \tag{9.16}$$

These are indeed the commutation relations of the Lorentz group.
We can compare with the defining vector representation by the identification

$$T_1 = T^{23}, \quad T_2 = T^{31}, \quad T_3 = T^{12},$$
$$T_4 = T^{01}, \quad T_5 = T^{02}, \quad T_6 = T^{03}.$$

The contravariant vector representation obtains from the covariant vector representation given by eqs. (9.6) - (9.8) by raising the first index and lowering the second, which changes the sign of T_4, T_5 and T_6. In this representation the generators are given explicitly by

$$\left(T^{\hat{\mu}\hat{\nu}}\right)^{\mu}{}_{\nu} = -i\left(\eta^{\hat{\mu}\mu}\delta^{\hat{\nu}}_{\nu} - \eta^{\hat{\nu}\mu}\delta^{\hat{\mu}}_{\nu}\right).$$

Examples are

$$(T_1)^{\mu}{}_{\nu} = (T^{23})^{\mu}{}_{\nu} = -i\left(\delta^{2\mu}\delta^{3}_{\nu} - \delta^{3\mu}\delta^{2}_{\nu}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

or

$$(T_4)^{\mu}{}_{\nu} = (T^{01})^{\mu}{}_{\nu} = i\left(\delta^{0\mu}\delta^{1}_{\nu} + \delta^{1\mu}\delta^{0}_{\nu}\right).$$

In this representation the commutation relation (9.16) is easily established.

Video: **Lecture20Video06.mp4**

Weyl Spinors. The matrices $T^{\mu\nu}$ in eq. (9.15) are block-diagonal. This implies that Dirac spinors are reducible representations of the continuous Lorentz group. The irreducible representations are the two-dimensional representations ψ_L and ψ_R, which do not mix under Lorentz transformations. Mathematically, there are two invariant subspaces, and the Dirac representation is therefore reducible. The Weyl representation is the two-dimensional irreducible representation (irrep). The decomposition of the Dirac representation can be formulated in a four-component notation

$$\psi_L = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix}, \qquad \psi_R = \begin{pmatrix} 0 \\ 0 \\ \psi_3 \\ \psi_4 \end{pmatrix}.$$

With only a slight abuse of notation we employ the same notation also for the two-component Weyl spinors

$$\psi_L = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \psi_R = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}.$$

Weyl spinors describe neutrinos. The naming of left-handed Weyl spinors ψ_L and right-handed Weyl spinors ψ_R will be understood later. Neutrinos are left-handed Weyl spinors, while the complex conjugated field describes right-handed antineutrinos. Electrons, quarks and other charged fermions are described by Dirac spinors. This is related to the fact that the symmetries related to conserved charges cannot be implemented for Weyl spinors. Dirac spinors describe particles and their antiparticles, e.g. electrons and positrons. Also the parity transformation maps between ψ_L and ψ_R.

Video: **Lecture20Video10.mp4**

Parity Transformation. The parity transformation is defined by

$$\psi(x) \to \gamma^0 \psi(Px), \quad Px = (x^0, -\vec{x}).$$

For the action on Weyl spinors we observe

$$\gamma^0 \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = -i \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix},$$

and therefore

$$(\psi')_L = -i\psi_R, \quad (\psi')_R = -i\psi_L.$$

Parity exchanges left-handed and right-handed Weyl spinors. This is indeed one of the reasons why one needs a left-handed and a right-handed Weyl spinor to describe electrons. Since neutrinos are described only by a left-handed Weyl spinor, they necessarily violate parity!

Video: **Lecture20Video07.mp4**

Projection Matrix. A projection from the Dirac to the Weyl representation can be defined in terms of the matrix γ^5 by

$$\psi_L = \tfrac{1}{2}(1+\gamma^5)\psi,$$
$$\psi_R = \tfrac{1}{2}(1-\gamma^5)\psi.$$

In our representation of Dirac matrices γ^5 has the simple form

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{9.17}$$

where the 1 represents a 2×2-unit-matrix. One can check easily the relations

$$[\gamma^5, T^{\mu\nu}] = 0, \qquad (\gamma^5)^2 = 1. \tag{9.18}$$

For the Dirac matrices (9.14) one verifies by explicit computation the anticommutation relation

$$\{\gamma^\mu, \gamma^5\} = 0. \tag{9.19}$$

This anticommutation can be used for a definition of γ^5 in an arbitrary representation of the Dirac matrices. It implies the relations (9.18) . For a proof we use that γ^5 commutes with a product of two Dirac matrices,

$$\gamma^5 T^{\mu\nu} = -\tfrac{i}{4}\gamma^5(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \tfrac{i}{4}(\gamma^\mu\gamma^5\gamma^\nu - \gamma^\nu\gamma^5\gamma^\mu) = -\tfrac{i}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma^5 = T^{\mu\nu}\gamma^5.$$

The anticommutation relation (9.19) is obeyed by the definition

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3.$$

which yields eq. (9.17) for our particular representation of the Dirac matrices. In our particular representation the projectors are very simple

$$\frac{1+\gamma^5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \frac{1-\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

or

$$\gamma^5\psi_L = \psi_L, \quad \gamma^5\psi_R = -\psi_R.$$

Video: **Lecture20Video08.mp4**

Dirac Matrices. The defining property for Dirac matrices is given by anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.$$

This is known as the Clifford algebra. From this relation one can derive all the commutator relations for the $T^{\mu\nu}$ and γ^5. In particular, one has $(\gamma^k)^2$ =1, k= 1,2,3 and $(\gamma^0)^2 = -1$. Different books on quantum field theory will use different representations of the Clifford algebra. Different representations are related by a similarity transformation

$$\gamma^\mu \to \gamma'^\mu = A\gamma^\mu A^{-1}.$$

For any regular matrix A this transformation does not change the anticommutator relations:

$$\{\gamma'^\mu, \gamma'^\nu\} = A\{\gamma^\mu, \gamma^\nu\}A^{-1} = 2A\eta^{\mu\nu}A^{-1} = \eta^{\mu\nu}.$$

10 Quantum electrodynamics

Video: **Lecture20Video11.mp4**

We are now ready to construct the action for quantum electrodynamics (QED). Charged fermions as electrons, muons or quarks are described by Grassmann variables in the Dirac representation of the Lorentz group. We start with free electrons, and add the interactions with photons subsequently.

10.1 Invariant action for free electrons

Kinetic term. We want to use the spinor representation discussed in the previous section to establish a Lorentz invariant action for fermions. We can construct a kinetic term with only one derivative:

$$S = \int d^4x\ \mathscr{L}, \qquad \mathscr{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi = i\bar{\psi}_\alpha(\gamma^\mu)_{\alpha\beta}\partial_\mu\psi_\beta. \tag{10.1}$$

As usual, ψ denotes a column vector and $\bar{\psi}$ is a line vector,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \qquad \bar{\psi} = \left(\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3, \bar{\psi}_4\right).$$

The Dirac indices or spinor indices $\alpha, \beta = 1, 2, 3, 4$ should not be confused with Lorentz-indices $\mu = 0, 1, 2, 3$. For Weyl spinors we use only two spinor indices, and in other dimensions the dimension of the Dirac representation differs from d. The kinetic term for fermions involves only one derivative.

A priori, ψ_α and $\bar{\psi}_\alpha$ are independent Grassmann variables, and Grassmann variables are neither real nor complex numbers. In contrast to the kinetic term for scalars, which requires two time-derivatives, the formulation with only one time-derivative is closer to the formulation for non-relativistic particles for which we have derived the functional integral from the operator formalism. Under an infinitesimal Lorentz transformation, ψ and $\bar{\psi}$ transform as

$$\begin{aligned} \delta\psi &= \tfrac{i}{2}\epsilon_{\mu\nu}T^{\mu\nu}\psi, \\ \delta\bar{\psi} &= -\tfrac{i}{2}\epsilon_{\mu\nu}\bar{\psi}T^{\mu\nu}. \end{aligned} \tag{10.2}$$

One can introduce a complex structure in the Grassmann algebra by defining ψ^* through

$$\bar{\psi} = \psi^\dagger\gamma^0 = (\psi^*)^T\gamma^0.$$

Video: **Lecture20Video12.mp4**

This is the defining relation for ψ^* in terms of $\bar{\psi}$. One can check the consistency of complex conjugation with Lorentz transformations,

$$\delta\psi^* = -\tfrac{i}{2}\epsilon_{\mu\nu}T^{\mu\nu}\psi^*, \qquad \delta\bar{\psi} = (\delta\psi)^\dagger\gamma^0.$$

Having defined ψ^*, one could define real and imaginary parts $\psi_{\rm R} = \frac{1}{2}(\psi+\psi^*)$ and $\psi_{\rm I} = -\frac{i}{2}(\psi-\psi^*)$ and use those as independent Grassmann variables.

Video: **Lecture20Video13.mp4**

Transformation of Spinor Bilinears. We next have to verify the Lorentz-invariance of the Dirac action (10.1). For this purpose we compute the behavior of general bilinear forms of spinors under Lorentz transformations. It is sufficient to consider infinitesimal Lorentz transformations.

As a first relation we proof the invariance of $\bar{\psi}\psi$,

$$\delta(\bar{\psi}\psi) = 0.$$

Insertion of eq. (10.2) yields directly

$$\delta(\bar{\psi}\psi) = \delta\bar{\psi}\psi + \bar{\psi}\delta\psi = -\tfrac{i}{2}(\bar{\psi}T^{\mu\nu}\psi - \bar{\psi}T^{\mu\nu}\psi) = 0.$$

We recall that there is an additional contribution from the transformation of coordinates that we do not display explicitly. Thus $\bar{\psi}\psi$ transforms as a scalar field under Lorentz transformations.

Next we show that $\bar{\psi}\gamma^\mu\psi$ transforms as a contravariant vector under Lorentz transformations.

$$\delta(\bar{\psi}\gamma^\mu\psi) = \delta\Lambda^\mu_{\ \nu}(\bar{\psi}\gamma^\nu\psi) = \epsilon^\mu_{\ \nu}\bar{\psi}\gamma^\nu\psi.$$

Video: **Lecture20Video14.mp4**

This can be seen in three steps. First we note that

$$\delta(\bar\psi\gamma^\rho\psi) = \delta\bar\psi\gamma^\rho\psi + \bar\psi\gamma^\rho\delta\psi = -\tfrac{i}{2}\epsilon_{\mu\nu}(\bar\psi T^{\mu\nu}\gamma^\rho\psi - \bar\psi\gamma^\rho T^{\mu\nu}\bar\psi) = -\tfrac{i}{2}\epsilon_{\mu\nu}\bar\psi[T^{\mu\nu},\gamma^\rho]\psi.$$

Second, we employ the identity

$$\gamma^\mu\gamma^\nu\gamma^\rho = \gamma^\mu\{\gamma^\nu,\gamma^\rho\} - \gamma^\mu\gamma^\rho\gamma^\nu = 2\eta^{\nu\rho}\gamma^\mu - \gamma^\mu\gamma^\rho\gamma^\nu.$$

in order to establish the commutator

$$\begin{aligned}[T^{\mu\nu},\gamma^\rho] &= -\tfrac{i}{4}(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\nu\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\mu\gamma^\nu + \gamma^\rho\gamma^\nu\gamma^\mu)\\ &= -\tfrac{i}{4}(2\eta^{\nu\rho}\gamma^\mu - \gamma^\mu\gamma^\rho\gamma^\nu - 2\eta^{\mu\rho}\gamma^\nu + \gamma^\nu\gamma^\rho\gamma^\mu - 2\eta^{\mu\rho}\gamma^\nu\\ &\quad + \gamma^\mu\gamma^\rho\gamma^\nu + 2\eta^{\nu\rho}\gamma^\mu - \gamma^\nu\gamma^\rho\gamma^\mu)\\ &= -i(\eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu).\end{aligned}$$

Video: **Lecture21Video01.mp4**

Third, the insertion of this commutation relation yields

$$\delta(\bar\psi\gamma^\rho\psi) = -\tfrac{i}{2}\bar\psi\epsilon_{\mu\nu}(-i)(\eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu)\psi = -\tfrac{1}{2}\bar\psi(\epsilon_\mu{}^\rho\gamma^\mu - \epsilon^\rho{}_\nu\gamma^\nu)\psi = \epsilon^\rho{}_\nu\bar\psi\gamma^\nu\psi.$$

Since we also know the transformation properties of ∂_ρ, we can easily check that $\bar\psi\gamma^\rho\partial_\rho\psi$ transforms as a scalar field,

$$\delta(\bar\psi\gamma^\rho\partial_\rho\psi) = \epsilon^\rho{}_\nu\bar\psi\gamma^\nu\partial_\rho\psi + \bar\psi\gamma^\rho\epsilon_\rho{}^\nu\partial_\nu\psi = \epsilon_{\rho\nu}\bar\psi\gamma^\nu\partial^\rho\psi + \epsilon_{\nu\rho}\bar\psi\gamma^\nu\partial^\rho\psi = 0.$$

Video: **Lecture21Video02.mp4**

Electrons with mass m. Free electrons are massive particles. A mass term in the action involves a fermion bilinear without derivatives that transforms as a scalar field. We have already established that $\bar\psi\psi$ has these properties. Extending the action (10.1) by a mass term the action for free electrons involves

$$\mathscr{L} = i\bar\psi\gamma^\mu\partial_\mu\psi + im\bar\psi\psi.$$

We will see that this Lorentz invariant action describes a quantum field theory for free electrons and positrons.

10.2 Dirac equation

Video: **Lecture21Video03.mp4**

Dirac Equation. The functional variation of the associated action S with regard to $\bar{\psi}$ leads to the famous Dirac equation

$$\frac{\delta S}{\delta \bar{\psi}} = 0 \Rightarrow (\gamma^\mu \partial_\mu + m)\psi = 0.$$

Since $\mathscr{L}$ is invariant this is a covariant equation. For a single particle state, this is also the Schrödinger equation, with ψ interpreted as a wave function. In this case ψ is a complex function (not a Grassmann variable). This interpretation of the Dirac equation as a relativistic Schrödinger equation for a one particle state does not hold in the presence interactions. A generalisation by adding an external electromagnetic is possible, however. In the following we will first concentrate on the interpretation of the Dirac equation as a relativistic Schrödinger equation. The physical properties found there will be useful once we later turn to the interacting quantum field theory for electromagnetism.

Energy-Momentum Relation. A free electron should obey the energy momentum relation for a relativistic particle. This can be established if we square the Dirac equation

$$\gamma^\nu \partial_\nu \gamma^\mu \partial_\mu \psi = m^2 \psi.$$

Using the anticommutator relation for the γ matrices

$$\tfrac{1}{2}\{\gamma^\nu, \gamma^\mu\}\partial_\nu \partial_\mu \psi = \eta^{\nu\mu}\partial_\nu \partial_\mu \psi = \partial^\mu \partial_\mu \psi = m^2 \psi,$$

we find the Klein-Gordon equation $(\partial^\mu \partial_\mu - m^2)\psi = 0$. All solutions of the Dirac equation (10.2) have to solve this equation. The general solution of the Klein-Gordon equation is a superposition of plane waves $\psi = \psi_0\, e^{ip_\mu x^\mu} = \psi_0\, e^{-i(Et - \mathbf{px})}$. This implies indeed the relativistic energy-momentum relation,

$$(E^2 - \mathbf{p}^2 - m^2)\psi = 0 \Rightarrow E^2 = \mathbf{p}^2 + m^2.$$

We observe the existence of solutions for both signs of the energy, $E = \pm\sqrt{p^2 + m^2}$ and we have to interpret the meaning of the solution with negative energy.

Video: **Lecture21Video04.mp4**

Hamiltonian Formulation. In order to obtain the usual form of the Schrödinger equation we multiply eq. (10.2) with $-i\gamma^0$,

$$-i\gamma^0\gamma^\mu\partial_\mu\psi = -i(\gamma^0)^2\partial_0\psi - i\gamma^0\gamma^k\partial_k\psi = i\gamma^0 m\psi,$$

and introduce

$$\alpha^k = -\gamma^0\gamma^k = \gamma^k\gamma^0 = \begin{pmatrix} -\tau_k & 0 \\ 0 & \tau_k \end{pmatrix}, \qquad \beta = i\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This leads to the standard form of the Schrödinger equation

$$i\dot{\psi} = -i\alpha^k\partial_k\psi + m\beta\psi = H\psi.$$

One can check that the Hamiltonian H is indeed a hermitian operator. For a free fermion the different momentum modes evolve independently, and one finds in the momentum basis,

$$i\dot{\psi} = H\psi \qquad \text{with} \qquad H = \alpha^k p_k + m\beta.$$

It is instructive to consider the rest frame of the particle ($\mathbf{p} = 0$). For the Hamiltonian one gets

$$H = m \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

This matrix mixes the Weyl spinors ψ_L and ψ_R

$$i\partial_t \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = m\beta \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = m \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}.$$

We can verify that H has two eigenvectors with positive energy ($E = +m$), and another two with negative energy ($E = -m$).

Video: **Lecture21Video05.mp4**

Positrons. We can construct linear combinations of ψ_L and ψ_R, which are mass eigenstates

$$\psi_\pm = \tfrac{1}{\sqrt{2}}(\psi_L \pm \psi_R) \qquad \text{and} \qquad i\dot{\psi}_\pm = \pm m\psi_\pm.$$

By conjugating the equation for ψ_-,

$$-i\dot{\psi}^*_- = -m\psi^*_- \Rightarrow i\dot{\psi}^*_- = m\psi^*_-,$$

one finds that ψ^*_- is an eigenstate of the Hamiltonian with positive eigenvalue $E = +m$. This field can be interpreted as the field for a new particle, called the positron. The positron is the antiparticle to the electron. We will see that ψ^*_- has electric charge $-e$, while ψ_+ has charge e. We use ψ_+ for electrons and therefore $e < 0$. The existence of antiparticles is a direct consequence of the Dirac equation, which has predicted the positron before its experimental discovery. In turn, it is a consequence of Lorentz symmetry, combined with conserved electric charge. The latter makes it impossible to describe the electron by a Weyl spinor. The complex conjugation in the definition of ψ^*_- is part of a so called "charge conjugation" operation C, which is a discrete symmetry similar to parity P and time reversal T. General arguments that we will not discuss here have shown that Lorentz symmetry implies the invariance of the action under the combined symmetry CPT. Individual discrete symmetries can be violated, as for neutrinos which violate P and C.

Video: **Lecture21Video06.mp4**

Video: **Lecture21Video07.mp4**

Electrons and Positrons in the Electromagnetic Field. We next investigate the dynamics of electrons and positrons in an electromagnetic field. For this purpose we first construct the piece of the action which describes the interaction of electrons with photons or electromagnetic fields. Taking fixed "external" electromagnetic fields the action for the spinor field remains quadratic. We can therefore interpret the Dirac equation as a Schrödinger equation for the one-particle wave function also for this case. In particular, we will see that the positron has the opposite charge of the electron.

As for classical electrodynamics, the electromagnetic field is given by $A_\mu = (-\phi, \mathbf{A})$, and the covariant Lagrangian by

$$\mathscr{L} = i\bar{\psi}\gamma^\mu(\partial_\mu - ieA_\mu)\psi + im\bar{\psi}\psi. \tag{10.3}$$

Since A_μ transforms as a covariant vector this action is Lorentz invariant.

The interaction term $\sim \bar{\psi}\gamma^\mu\psi A_\mu$ is dictated by the gauge symmetry of the electromagnetic interactions. This gauge symmetry replaces partial derivatives by covariant derivatives, according to

$$\partial_t\psi \to (\partial_0 + ie\phi)\psi,$$
$$\partial_k\psi \to (\partial_k - ieA_k)\psi,$$

Varying the action with respect to $\bar{\psi}$ yields the Dirac equation in an electromagnetic field,

$$i\dot{\psi} = \left(\alpha^k(\hat{p}_k - eA_k) + e\phi + \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}\right)\psi, \quad \hat{p}_k = -i\partial_k.$$

With

$$\alpha^k \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} -\tau_k & 0 \\ 0 & \tau_k \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} -\tau_k\psi_L \\ \tau_k\psi_R \end{pmatrix},$$

Video: **Lecture21Video08.mp4**

the action of α^k on the linear combinations (10.2) reads

$$\alpha^k\psi_+ = -\tau_k\psi_-,$$
$$\alpha^k\psi_- = -\tau_k\psi_+.$$

For the mass eigenstates ψ_+, ψ_- the Dirac equation becomes

$$i\dot{\psi}_+ = (m + e\phi)\psi_+ + i(\partial_k - ieA_k)\tau_k\psi_-,$$
$$i\dot{\psi}_- = (-m + e\phi)\psi_- + i(\partial_k - ieA_k)\tau_k\psi_+.$$

The Schrödinger equation for the positron obtains by complex conjugation of the equation for ψ_-,

$$i\dot{\psi}_-^* = (m - e\phi)\psi_-^* + i(\partial_k + ieA_k)\tau_k^*\psi_+^*.$$

The positrons described by ψ_-^* have indeed the opposite charge as the electrons described by ψ_+.

One observes that for non-zero momentum $\hat{p}_k$ or non-zero vector potential A_k the components ψ_+ and ψ_- mix. The eigenfunctions for the electrons and positrons are therefore not simply ψ_+ and ψ_-^* as for the free fermions at rest. In the absence of interactions the eigenfunctions of the Hamiltonian for non-zero momentum can be obtained from the ones in the rest frame by applying a suitable Lorentz boost. In the presence of electromagnetic fields the eigenfunctions are more complicated. The one-particle description ceases to be valid if the fields are strong enough such that the creation of electron-positron pairs becomes possible.

Video: **Lecture21Video09.mp4**

Quantum electrodynamics. For quantum electrodynamics (QED) the electromagnetic field is treated on the same level as the other fields in the functional integral for a quantum field theory. Thus the functional measure includes an integration over the fields $A_\mu(x)$. This contrasts to the fixed "external fields" discussed before. The excitations of the electromagnetic field are the photons. Similar to the case of scalar fields, photons are propagating particles. Their number is not conserved.

We can discuss processes as photon-electron scattering, or the production of electron-positron pairs by the annihilation of a photon pair. In the presence of the electromagnetic interaction the electric charge remains a conserved quantity. This is guaranteed by the gauge symmetry. The number of electrons is no longer conserved, however.

A Lorentz invariant action for the electromagnetic field employs the field strength $F_{\mu\nu}$,

$$\mathscr{L}_F = \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{10.4}$$

Since $F_{\mu\nu}$ is gauge invariant, this action also preserves the gauge symmetry. Combining eq. (10.3) and (10.4) yields a Lorentz- and gauge invariant action for electrons and photons. (In the following we often understand by "electrons" the description of both electrons and positrons.)

In summary, QED is defined by the functional integral for the partition function

$$\begin{aligned} Z &= \int D\varphi \ \exp\left(-i \int_x \mathscr{L}_{QED}\right), \\ \int D\varphi &= \int D\psi D\bar{\psi} D A_\mu, \\ \mathscr{L}_{QED} &= i\bar{\psi}\gamma^\mu(\partial_\mu - ieA_\mu)\psi + im\bar{\psi}\psi + \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu}. \end{aligned}$$

The integral over the fermion fields is a Grassmann functional integral. The integral over the gauge fields A_μ should preserve both the Lorentz symmetry and the gauge symmetry. It needs a regularisation, to which later parts of the QFT-lecture will turn in detail. For our purpose it is sufficient to know that a suitable functional measure exists.

From the functional integral all correlation functions can, in principle, be computed. They can be compared with precise measurements for many processes. In the case of QED the fine structure constant $\alpha = e^2/(4\pi) \sim 1/137$ is a small parameter. A perturbative computation amounts to an expansion in α. It can be performed to rather high order. Precise computations with many decimal places agree perfectly with observation.

Video: **Lecture21Video10.mp4**

Gauge symmetry. The action of QED is invariant under local gauge transformations.

$$\psi'(x) = e^{i\alpha(x)\psi(x)},$$

$$A'_\mu(x) = A_\mu(x) + \tfrac{1}{e}\partial_\mu \alpha(x).$$

For a local transformation the transformation parameter $\alpha(x)$ depends on x. This contrasts to the Lorentz-transformations which are "global" symmetry transformations. The local gauge transformations change ψ at every x independently. If the action (and measure) is invariant under a local transformation one speaks about a "local symmetry", and often simply about a "gauge symmetry".

In order to verify the gauge invariance of the action for QED we first note that the free fermion kinetic term is not gauge invariant,

$$\bar{\psi}\gamma^\mu \partial_\mu \psi \to \bar{\psi}\gamma^\mu \partial_\mu \psi + i\partial_\mu \alpha \bar{\psi}\gamma^\mu \psi.$$

Also the interaction term $\sim \bar{\psi}\gamma^\mu \psi A_\mu$ alone is not gauge invariant

$$-ie\bar{\psi}\gamma^\mu A_\mu \psi \to -ie\bar{\psi}\gamma^\mu A_\mu \psi - i\partial_\mu \alpha \bar{\psi}\gamma^\mu \psi.$$

Only the combination into a covariant derivative $D_\mu = \partial_\mu - ieA_\mu$ yields an invariant expression

$$i\bar{\psi}\gamma^\mu D_\mu\psi = i\bar{\psi}\gamma^\mu(\partial_\mu - ieA_\mu)\psi.$$

The gauge invariance of the field strength $F_{\mu\nu}$ follows from the commutativity of partial derivatives.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu A_\nu - \partial_\nu A_\mu + \tfrac{1}{e}\partial_\mu\partial_\nu\alpha - \tfrac{1}{e}\partial_\nu\partial_\mu\alpha = F_{\mu\nu}.$$

Local gauge invariance is an important principle for constructing the action of a quantum field theory. It is closely related to renormalizability.

Video: **Lecture21Video11.mp4**

Renomalizability. Gauge symmetry is a powerfull restriction for the choice of the action. Is it sufficient? Consider a possible term

$$\Delta\mathscr{L} = \tfrac{b}{m}\bar{\psi}[\gamma^\mu,\gamma^\nu]\psi F_{\mu\nu}.$$

This term is Lorentz invariant and gauge invariant. If we add it with an unknown coefficient b, predictions will depend on this coefficient. Predictivity of QED, which only involves m and $\alpha = e^2/4\pi$, would be lost. The reason for the absence of such a term will be discussed in later parts of the QFT-lecture related to renormalizability.

Video: **Lecture21Video12.mp4**

Non-relativistic limit of Dirac equation. For the quantum mechanics of atoms one uses the Schrödinger equation for a complex two-component electron wave function $\chi(x)$. This equation is not covariant under Lorentz-transformations – it is a non-relativistic equation. It contains various terms, as a coupling between spin and angular momentum. A fundamental theory as QED should fix all such couplings. Since QED contains only two parameters, namely m^2 and α, all aspects of the non-relativistic Schrödinger equation should follow from the Dirac equation. The way of deriving the usual Schrödinger equation for electrons proceeds by taking the non-relativistic limit of the Dirac equation.

The Schrödinger equation for the two-component spinor χ for the electron is given by

$$i\partial_t\chi = H\chi = \frac{1}{2m}(\vec{p} - e\vec{A})^2 + e\varphi - \frac{e}{m}\vec{S}\vec{B}, \qquad \vec{S} = \tfrac{1}{2}\vec{\tau}, \tag{10.5}$$

with spin operator $\vec{S}$, momentum operator $\vec{p}_k = -i\vec{\nabla}_k$ and magnetic field $\vec{B}$. We recall that we use units with $\hbar = 1$. One usually deals with small electromagnetic fields for which one linearizes in $\vec{A}$. In particular, for a constant magnetic field $\vec{B}$ one takes $\vec{A} = -\frac{1}{2}\vec{r}\times\vec{B}$ and obtains

$$\tfrac{1}{2m}(\vec{p} - e\vec{A})^2 = \frac{\vec{p}^2}{2m} - \frac{e}{2m}\vec{L}\vec{B},$$

with $\vec{L}$ the angular momentum operator. One recognises the Schrödinger equation for atomic physics. There are indeed no new free couplings.

The magnetic field couples to a linear combination of angular momentum and spin different from the total angular momentum, with a term

$$(\vec{L} + g\vec{S})\vec{B}, \qquad g = 2.$$

The Dirac equation predicts the relative coupling $g = 2$. For QED, the Dirac equation is not an exact equation. As a one-particle equation it assumes implicitly that the particle number is conserved. We have discussed before that the number of electrons is not conserved in QED since electron-positron pairs can be created or annihilated without violation of charge conservation. One expects possible corrections to the prediction of the Dirac equation due to fluctuation effects. Indeed, the QED corrections from fluctuations yield a small correction to $g - 2$, which is computed to many decimal places. For the electron one finds

$$\frac{g_e}{2} = 1.001159652181643(764),$$

to be compared with the experimental values

$$\frac{g_e^{exp}}{2} = 1.00115965218073(28),$$

with errors on the last digits in brackets. The impressive agreement on the level 10^{-12} is a strong indication for the correctness of QED.

Video: **Lecture21Video13.mp4**

Video: **Lecture21Video14.mp4**

The derivation of the non-relativistic limit of the Dirac equation can be done in several steps.
Step 1: Square the Dirac equation,

$$\gamma^\nu(\partial_\nu - ieA_\nu)\gamma^\mu(\partial_\mu - ieA_\mu)\psi = m^2\psi.$$

Step 2: From the Dirac algebra we use $[\gamma^\mu, \gamma^\nu] = 4iT^{\mu\nu}$ and obtain

$$\left((\partial^\mu - ieA^\mu)(\partial_\mu - ieA_\mu) + eT^{\mu\nu}F_{\mu\nu} - m^2\right)\psi = 0.$$

Step 3: We use $T^{\mu\nu}F_{\mu\nu} = \frac{1}{2}B_k\tau_k + \frac{i}{2}E_k\tau_k\gamma^5$, with $\tau_k = \begin{pmatrix} \tau_k & 0 \\ 0 & \tau_k \end{pmatrix}$. Also using $\psi_\pm$, one obtains

$$\left\{(\partial^\mu - ieA^\mu)(\partial_\mu - ieA_\mu) - m^2 + eB_k\tau_k\right\}\psi_+ = -ieE_k\tau_k\psi_-.$$

Step 4: One neglects the positrons by setting $\psi_- = 0$. The resulting equation for ψ_+ has only two independent components. The lower two and upper two components obey an identical Schrödinger equation. We take the upper two components.
Step 5: We introduce the non-relativistic wave function χ by

$$\psi_+ = e^{-imt}\chi.$$

We formally write

$$i\partial_t\chi = H\chi = (E - m)\chi,$$

by putting all terms that are not linear in the time derivative χ on the right hand side.
Step 6: The non-relativistic limit is given by $|H| \ll m$. In this limit one can neglect

$$\frac{\partial_t^2}{m}, \quad \frac{A_0\partial_t}{m}, \quad \frac{(\partial_t A_0)}{m}, \quad \frac{A_0^2}{m}.$$

Omitting these terms in H yields the above non-relativistic result. The quantity H becomes the non-relativistic Hamiltonian in eq. (10.5).

Video: **Lecture22Video01.mp4**

10.3 Functional integral for photons

For photons, the field one integrates over in the functional integral is the gauge field $A_\mu(x)$. The field theory is described by the partition function

$$\begin{aligned} Z_2[J] &= \int DA \ \exp\left[iS_2[A] + i\int J^\mu A_\mu\right] \\ &= \int DA \ \exp\left[i\int d^4x \ \left\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu\right\}\right]. \end{aligned}$$

One can go to momentum space as usual

$$A_\mu(x) = \int \frac{d^4p}{(2\pi)^4} \ e^{ipx} A_\mu(p),$$

and finds for the term in the exponential

$$\begin{aligned} &\int_x \left\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu\right\} \\ &= \frac{1}{2}\int \frac{d^4p}{(2\pi)^4} \left\{-A_\mu(-p)\left(p^2\eta^{\mu\nu} - p^\mu p^\nu\right) \ A_\nu(p) + J^\mu(-p)A_\mu(p) + A_\mu(-p)J^\mu(p)\right\}. \end{aligned}$$

Video: **Lecture22Video02.mp4**

Attempt to invert the inverse propagator and gauge fixing. The next step would now be to perform the Gaussian integral over A_μ by completing the square. However, a problem arises here: The "inverse propagator" for the gauge field

$$p^2\eta^{\mu\nu} - p^\mu p^\nu = p^2 \mathscr{P}^{\mu\nu}(p),$$

is not invertible. We wrote it here in terms of

$$\mathscr{P}_\mu{}^\nu(p) = \delta_\mu{}^\nu - \frac{p_\mu p^\nu}{p^2},$$

which is in fact a projector to the space orthogonal to p_ν

$$\mathscr{P}_\mu{}^\nu(p)\mathscr{P}_\nu{}^\rho(p) = \mathscr{P}_\mu{}^\rho(p).$$

As a projector matrix it has eigenvalues 0 and 1, only. However,

$$\mathscr{P}_\mu{}^\nu(p) \ p_\nu = 0.$$

The field $A_\nu(p)$ can be decomposed into two parts,

$$A_\nu(p) = \frac{i}{e}p_\nu\beta(p) + \hat{A}_\nu(p),$$

with

$$\hat{A}_\nu(p) = \mathscr{P}_\nu{}^\rho(p)A_\rho(p),$$

such that $p^\nu\hat{A}_\nu(p) = 0$. Moreover

$$\beta(p) = \frac{e}{ip^2}p^\nu A_\nu(p).$$

When acting on $\hat{A}_\nu(p)$, the projector $\mathscr{P}_\mu{}^\nu(p)$ is simply the unit matrix.

Recall that gauge transformations shift the field according to

$$A_\mu(x) \to \frac{1}{e}\partial_\mu \alpha + A_\mu(x)$$

or in momentum space

$$A_\mu(p) \to \frac{i}{e} p_\mu \alpha(p) + A_\mu(p).$$

One can therefore always perform a gauge transformation such that $\beta(p) = 0$ or

$$\partial^\mu A_\mu(x) = 0.$$

This is known as Lorenz gauge or Landau gauge. We will use this gauge in the following and restrict the functional integral to field configurations that fulfil the gauge condition.

Video: **Lecture22Video03.mp4**

Quadratic partition function. Now we can easily perform the Gaussian integral,

$$\begin{aligned} Z_2[J] = \int DA\ \exp\left[\frac{i}{2}\int_p \left\{ -\left(A_\mu(-p) - J_\rho(-p)\frac{\mathscr{P}^\rho{}_\mu}{p^2}\right) p^2\, \mathscr{P}^{\mu\nu} \left(A_\nu(p) - \frac{\mathscr{P}_\nu{}^\sigma}{p^2} J_\sigma(p)\right)\right\}\right] \\ \times \exp\left[\frac{i}{2}\int_p J^\mu(-p)\frac{\mathscr{P}_{\mu\nu}(p)}{p^2} J^\nu(p)\right] \\ = \text{const} \times \exp\left[\frac{i}{2}\int_{x,y} J^\mu(x)\Delta_{\mu\nu}(x-y)J^\nu(y)\right]. \end{aligned}$$

In the last line we used the photon propagator in position space (in Landau gauge)

$$\Delta_{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)} \frac{\mathscr{P}_{\mu\nu}(p)}{p^2 - i\epsilon}.$$

In the last step we have inserted the $i\epsilon$ term as usual.

Video: **Lecture22Video04.mp4**

Photon propagator in position space. In the free theory one has

$$\langle A_\mu(x) A_\nu(x)\rangle = \frac{1}{i^2}\left(\frac{1}{Z[J]} - \frac{\delta^2}{\delta J^\mu(x)\delta J^\nu(y)} Z[J]\right)_{J=0} = \frac{1}{i}\Delta_{\mu\nu}(x-y).$$

We use the following graphical notation

$$(x,\mu) \sim\sim\sim (y,\nu) = \frac{1}{i}\Delta_{\mu\nu}(x-y),$$

or with sources $iJ^\mu(x)$ at the end points

$$\bullet\sim\sim\sim\bullet = \frac{1}{2}\int_{x,y} iJ^\mu(x)\,\frac{1}{i}\Delta_{\mu\nu}(x-y)\,iJ^\nu(y).$$

Video: **Lecture22Video05.mp4**

Free solutions. To describe incoming and outgoing photons we need to discuss free solutions for the gauge field. In momentum space, and for the gauge-fixed field (Landau gauge), the linear equation of motion (Maxwell's equation) is simply

$$p^2 \mathscr{P}_\mu{}^\nu(p) \hat{A}_\nu(p) = p^2 \hat{A}_\mu(p) = 0.$$

Non-trivial solutions satisfy $p^2 = 0$. Without loss of generality we assume now $p^\mu = (E, 0, 0, E)$; all other light like momenta can be obtained from this via Lorentz-transformations.

Video: **Lecture22Video06.mp4**

Polarizations. Quite generally, a four-vector can be written as

$$\hat{A}_\nu(p) = \left(b, \frac{a_1 + a_2}{\sqrt{2}}, \frac{-ia_1 + ia_2}{\sqrt{2}}, c \right).$$

From the Landau gauge condition $p^\nu \hat{A}_\nu = 0$ it follows that $b = -c$, so that one can write

$$\hat{A}_\nu(p) = \tilde{c} \times (-E, 0, 0, E) + a_1 \epsilon_\nu^{(1)} + a_2 \epsilon_\nu^{(2)},$$

with

$$\epsilon_\nu^{(1)} = \left(0, \frac{1}{\sqrt{2}}, \frac{-i}{\sqrt{2}}, 0 \right), \qquad \epsilon_\nu^{(2)} = \left(0, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 0 \right).$$

However, the term $\sim \tilde{c}$ is in fact proportional to $p_\nu = (-E, 0, 0, E)$. We can do another gauge transformation such that $\tilde{c} = 0$. This does not violate the Landau gauge condition because of $p^\nu p_\nu = 0$. In other words, the photon field has only two independent polarization states, chosen here as positive and negative circular polarizations, or helicities.

Video: **Lecture22Video07.mp4**

Mode expansion. In summary, we can expand free solutions of the photon field like

$$A_\mu(x) = \sum_{\lambda=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_{\vec{p},\lambda}\, \epsilon_\mu^{(\lambda)}(p)\, e^{ipx} + a_{\vec{p},\lambda}^\dagger\, \epsilon_\mu^{(\lambda)*}(p)\, e^{-ipx} \right\},$$

where $E_p = |\vec{p}|$ is the energy of a photon. The index λ labels the two polarization states.

In the current setup, $a_{\vec{p},\lambda}$ and $a_{\vec{p},\lambda}^\dagger$ are simply expansion coefficients, while they become annihilation and creation operators in the operator picture. The non-trivial commutation relation becomes then

$$\left[a_{\vec{p},\lambda}, a_{\vec{p}',\lambda'}^\dagger \right] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}') \delta_{\lambda\lambda'}.$$

Video: **Lecture22Video08.mp4**

Video: **Lecture22Video09.mp4**

LSZ reduction formula for photons. We also need a version of the Lehmann-Symanzik-Zimmermann reduction formula for photons. Recall that for non-relativistic bosons we could replace for the calculation of the interacting part of the S-matrix

$$a_{\vec{q}}(\infty) \to i\left[-q^0 + \tfrac{\vec{q}^2}{2m} + V_0\right]\varphi(q),$$

$$a^\dagger_{\vec{q}}(-\infty) \to i\left[-q^0 + \tfrac{\vec{q}^2}{2m} + V_0\right]\varphi^*(q).$$

For relativistic fields this is in general somewhat more complicated because of renormalization. This will be discussed in more detail in the second part of the course. In the following we will discuss only tree level diagrams where this plays no role. For photons one can replace for outgoing states

$$\sqrt{2E_p}\; a_{\vec{p},\lambda}(\infty) \to i\epsilon^{\nu *}_{(\lambda)}(p)\int d^4x\; e^{-ipx}[-\partial_\mu\partial^\mu]A_\nu(x)$$

$$\sqrt{2E_p}\; a^\dagger_{\vec{p},\lambda}(-\infty) \to i\epsilon^{\nu}_{(\lambda)}(p)\int d^4x\; e^{ipx}[-\partial_\mu\partial^\mu]A_\nu(x).$$

These formulas can be used to write S-matrix elements as correlation functions of fields. Note that $[-\partial_\mu\partial^\nu]$ is essentially the inverse propagator in Landau gauge.

Video: **Lecture22Video10.mp4**

Mode expansion for Dirac fields. We also need a mode expansion for free Dirac fields in order to describe asymptotic (incoming and outgoing) fermion states. We write the fields as

$$\psi(x) = \sum_{s=1}^{2}\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{b_{\vec{p},s}\; u_s(p)\; e^{ipx} + d^\dagger_{\vec{p},s}\; v_s(p)\; e^{-ipx}\right\},$$

$$\bar\psi(x) = \sum_{s=1}^{2}\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{-i\; b^\dagger_{\vec{p},s}\; \bar u_s(p)\; e^{-ipx} - id^\dagger_{\vec{p},s}\; \bar v_s(p)\; e^{ipx}\right\}.$$

Again, $b_{\vec{p},s}$, $d_{\vec{p},s}$ etc. can be seen as expansion coefficients and become operators in the operator picture.

Video: **Lecture22Video11.mp4**

Solutions of Dirac equation. The Dirac equation

$$(\gamma^\mu\partial_\mu + m)\psi(x) = 0,$$

becomes for the plane waves

$$(i\not{p} + m)\; u_s(\vec{p}) = 0,$$
$$(-i\not{p} + m)\; v_s(\vec{p}) = 0,$$

with $\not{p} = \gamma^\mu p_\mu$. We consider this first in the frame where the spatial momentum vanishes, $\vec{p} = 0$, such that $p_\mu = (-m, 0, 0, 0)$,

$$\not{p} = -\gamma^0 m = im\begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix}.$$

The last equation holds in the chiral basis where

$$\gamma^\mu = -i \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}.$$

with $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})$. For the spinor u_s one has the equation

$$(i\not{p} + m)u_s = m \begin{pmatrix} +\mathbb{1} & -\mathbb{1} \\ -\mathbb{1} & +\mathbb{1} \end{pmatrix} u_s = 0.$$

The two independent solutions are

$$u_1^{(0)} = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad u_2^{(0)} = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

The normalization has been chosen for later convenience. Similarly

$$(-i\not{p} + m)v_s(0) = m \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} v_s(0) = 0$$

has the two independent solutions

$$v_1^{(0)} = \sqrt{m} \begin{pmatrix} 0 \\ +1 \\ 0 \\ -1 \end{pmatrix}, \qquad v_2^{(0)} = \sqrt{m} \begin{pmatrix} -1 \\ 0 \\ +1 \\ 0 \end{pmatrix}.$$

We see here that the Dirac equation has two independent solutions (for spin up and and down with respect to some basis) for particles and two more for anti-particles. One can now go to an arbitrary reference frame by performing a Lorentz transformation. That gives

$$u_s(\vec{p}) = \begin{pmatrix} \sqrt{-p_\mu \sigma^\mu}\, \xi_s \\ \sqrt{-p_\mu \bar{\sigma}^\mu}\, \xi_s \end{pmatrix}, \qquad v_s(\vec{p}) = \begin{pmatrix} \sqrt{-p_\mu \sigma^\mu}\, \xi_s \\ -\sqrt{-p_\mu \bar{\sigma}^\mu}\, \xi_s \end{pmatrix},$$

with a two-dimensional orthonormal basis ξ_s such that

$$\xi_s^\dagger \xi_r = \delta_{rs}, \qquad \sum_{s=1}^{2} \xi_s \xi_s^\dagger = \mathbb{1}_2.$$

Other identities involving $u_s(\vec{p})$, $v_s(\vec{p})$ as well as

$$\bar{u}_s(\vec{p}) = u_s^\dagger(\vec{p}) i\gamma^0 = u_s^\dagger(p) \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix},$$
$$\bar{v}_s(\vec{p}) = v_s^\dagger(\vec{p}) i\gamma^0 = v_s^\dagger(p) \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix},$$

have been discussed in exercises. They will be mentioned here once they are needed.

Video: **Lecture22Video12.mp4**

LSZ reduction for Dirac fermions. Finally, let us give the LSZ reduction formulas for Dirac fermions (again neglecting renormalization effects)

$$\sqrt{2E_p}b_{\vec{p},s}(\infty) \to i\int d^4x\, e^{-ipx}\bar{u}_s(\vec{p})(\gamma^\mu\partial_\mu + m)\psi(x),$$
$$\sqrt{2E_p}d^\dagger_{\vec{p},s}(-\infty) \to -i\int d^4x\, e^{-ipx}\bar{v}_s(\vec{p})(\gamma^\mu\partial_\mu + m)\psi(x),$$
$$\sqrt{2E_p}d_{\vec{p},s}(\infty) \to -i\int d^4x\, i\bar{\psi}_s(x)(-\gamma^\mu\overleftarrow{\partial}_\mu + m)v_s(x)\, e^{-ipx},$$
$$\sqrt{2E_p}b^\dagger_{\vec{p},s}(-\infty) \to i\int d^4x\, i\bar{\psi}_s(x)(-\gamma^\mu\overleftarrow{\partial}_\mu + m)u_s(x)\, e^{ipx}.$$

The left-pointing arrows indicate here that these derivatives act to the left (on the field $\bar{\psi}_s(x)$). These relations have been obtained as part of the exercises.

10.4 Feynman rules and Feynman diagrams

Video: **Lecture22Video13.mp4**

Action and partition function. We are now ready to formulate the Feynman rules for a perturbative treatment of quantum electrodynamics. The microscopic action is

$$\begin{aligned} S &= \int d^4x \left\{ -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\bar{\psi}\gamma^\mu(\partial_\mu - ieA_\mu)\psi - im\bar{\psi}\psi \right\} \\ &= S_2[\bar{\psi},\psi,A] - \int d^4x\, e\bar{\psi}\gamma^\mu A_\mu\psi. \end{aligned}$$

The last term is cubic in the fields $\bar{\psi}, \psi$ and A_μ, while all others terms are quadratic. We will perform a perturbative expansion in the electric charge e.

Let us write the partition function as

$$Z[\bar{\eta},\eta,J] = \int D\bar{\psi}D\psi DA\ \exp\left[iS[\bar{\psi},\psi,A] + i\int \left\{ \bar{\eta}\psi + \bar{\psi}\eta + J^\mu A_\mu \right\} \right]$$

with $\bar{\eta}\psi = \bar{\eta}_\alpha\psi_\alpha$ where $\alpha = 1,\ldots,4$ sums over spinor components. Formally, one can write

$$Z[\bar{\eta},\eta,J] = \exp\left[-e\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\right)\left(i\frac{\delta}{\delta\eta_\alpha(x)}\right)(\gamma^\mu)_{\alpha\beta}\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}_\beta(x)}\right)\right] Z_2[\bar{\eta},\eta,J],$$

with quadratic partition function

$$\begin{aligned} Z_2 &= \int D\bar{\psi}D\psi DA\ \exp\left[iS_2[\bar{\psi},\psi,A] + i\int \left\{ \bar{\eta}\psi + \bar{\psi}\eta + J^\mu A_\mu \right\} \right] \\ &= \exp\left[i\int d^4x d^4y\ \bar{\eta}(x)S(x-y)\eta(y) \right] \times \exp\left[\frac{i}{2}\int d^4x d^4y\ J^\mu(x)\Delta_{\mu\nu}(x-y)J^\nu(y) \right]. \end{aligned}$$

Video: **Lecture22Video14.mp4**

Propagator for Dirac fermions. We have introduced here also the propagator for Dirac fermions, which is in fact a matrix in spinor space,

$$\begin{aligned} S_{\alpha\beta}(x-y) &= -i\int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}(ip_\mu\gamma^\mu + m)^{-1}_{\alpha\beta} \\ &= -i\int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\frac{(-i\not{p} + m\mathbb{1})_{\alpha\beta}}{p^2 + m^2 - i\epsilon}. \end{aligned}$$

We can now calculate S-matrix elements by first expressing them as correlation functions which get then evaluated in a perturbative expansion of the functional integral. These perturbative expressions have an intuitive graphical representation as we have briefly discussed before. We concentrate here on tree diagrams for which renormalization is not needed yet.

The correlation function of two Dirac fields can also be expressed in terms of the Dirac propagator,

$$\langle \psi_\alpha(x)\bar{\psi}_\beta(y)\rangle = \frac{1}{Z_2}\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}_\alpha(x)}\right)\left(i\frac{\delta}{\delta\eta_\beta(y)}\right) Z_2\Big|_{\bar{\eta}=\eta=J=0} = \frac{1}{i}S_{\alpha\beta}(x-y).$$

We introduce a graphical representation for thus, as well,

$$(x,\alpha)\longrightarrow(y,\beta) = \frac{1}{i}S_{\alpha\beta}(x-y).$$

With sources $i\bar{\eta}_\alpha(x)$ and $i\eta_\beta(y)$ at the end this would be

$$\bullet\longrightarrow\bullet = \int_{x,y} i\bar{\eta}_\alpha(x)\,\frac{1}{i}S_{\alpha\beta}(x-y)\,i\eta_\beta(y) = i\int_{x,y}\bar{\eta}(x)S(x-y)\eta(y).$$

The conventions are such that the arrow points away from the source η and to the source $\bar{\eta}$. It can also be seen as denoting the direction of fermions while anti-fermions move against the arrow direction. The Dirac indices α,β are sometimes left implicit when there is no doubt about them.

Video: **Lecture23Video01.mp4**

Expanding out exponentials. We now consider the full partition function and expand out the exponentials,

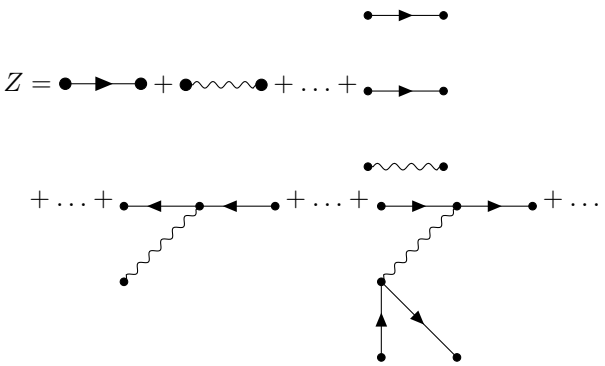
$$\begin{aligned} Z[\bar{\eta},\eta,J] = &\sum_{V=0}^{\infty}\frac{1}{V!}\left[\int_x\left(\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\right)\left(i\frac{\delta}{\delta\eta_\alpha(x)}\right)\left(-e\gamma^\mu_{\alpha\beta}\right)\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}_{\beta(x)}}\right)\right]^V \\ &\times\sum_{F=0}^{\infty}\frac{1}{F!}\left[\int_{x',y'} i\bar{\eta}_\alpha(x')\left(\frac{1}{i}S_{\alpha\beta}(x'-y')\right)i\eta_\beta(y')\right]^F \\ &\times\sum_{p=0}^{\infty}\frac{1}{P!}\left[\frac{1}{2}\int_{x'',y''} iJ^\mu(x'')\left(\frac{1}{i}\Delta_{\mu\nu}(x''-y'')\right)iJ^\nu(y'')\right]^P. \end{aligned}$$

The index F counts the number of fermion propagators (corresponding to fermion lines in a graphical representation), the index P counts the number of photon propagators (photon lines). The index V counts vertices that connect fermion and photon in a specific way. More specifically, each power of this term removes one of each kind of sources and introduces $-e\gamma^\mu_{\alpha\beta}$ to connect the lines in the graphical representation.

Video: **Lecture23Video02.mp4**

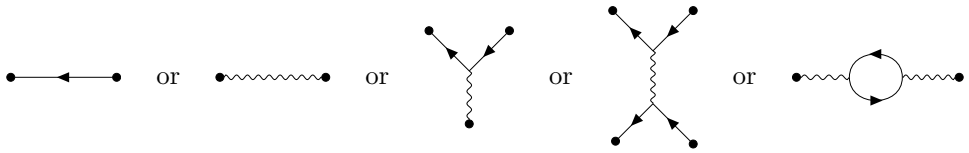
Graphical representation for partition function. In the full expression for $Z[\bar{\eta},\eta,J]$ many terms are present, in fact all graphs one can construct with fermion lines, photon lines and the

vertex. For example



Video: **Lecture23Video03.mp4**

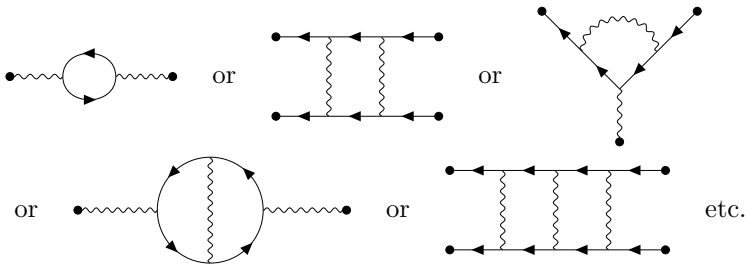
Connected and disconnected diagrams. One distinguishes connected diagrams where all endpoints are connected with lines to each other, for example



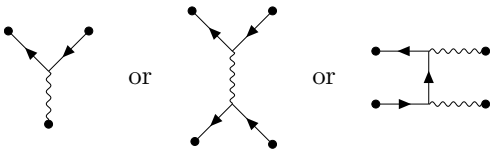
Disconnected diagrams can be decomposed into several connected diagrams.

Video: **Lecture23Video04.mp4**

Tree and loop diagrams. One also distinguishes tree diagrams and loop diagrams. Loop diagrams have closed loops of particle flow, for example

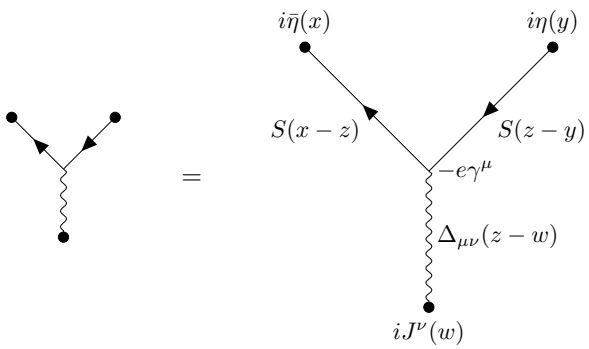


Tree diagrams have no closed loop, for example



Video: **Lecture23Video05.mp4**

Corresponding algebraic expressions. To each of these diagrams with sources one can associate an expression, for example



$$= \int_{x,y,z,w} i\bar{\eta}(x) \left[\frac{1}{i} S(x-z)\right] (-e\gamma^\mu) \left[\frac{1}{i} S(z-y)\right] i\eta(y) \left[\frac{1}{i} \Delta_{\mu\nu}(z-w)\right] iJ^\nu(w).$$

To calculate S-matrix elements we are mainly interested in the connected diagrams because disconnected diagrams describe events where not all particles scatter. Also, we concentrate here on tree diagrams. Loop diagrams will be discussed somewhat later.

Video: **Lecture23Video06.mp4**

S-matrix elements. Now that we have seen how to represent $Z[\bar{\eta}, \eta, J]$, let us discuss how to obtain S-matrix elements. For example, for an outgoing photon we had the LSZ rule

$$\sqrt{2E_p}\, a_{\vec{p},\lambda}(\infty) \to i\epsilon^{\nu *}_{(\lambda)}(p) \int d^4x\, e^{-ipx} \left[-\partial_\mu \partial^\mu\right] A_\nu(x).$$

To obtain the field $A_\nu(x)$ under the functional integral we can use

$$A_\nu(x) \to \frac{1}{i} \frac{\delta}{\delta J^\nu(x)},$$

acting on $Z[\bar{\eta}, \eta, J]$. Moreover, $i[-\partial_\mu \partial^\mu]$ will remove one propagator line for the outgoing photon,

$$i\left[-\partial_\mu \partial^\mu\right] \frac{1}{i} \Delta_{\rho\sigma}(x-y) = \left[-\partial_\mu \partial^\mu\right] \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{\mathscr{P}_{\rho\sigma}(p)}{p^2 - i\epsilon}$$
$$= \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)}\, \mathscr{P}_{\rho\sigma}(p) \to \eta_{\rho\sigma} \delta^{(4)}(x-y).$$

The projector has no effect if the photon couples to conserved currents and the result is simply $\eta_{\rho\sigma}\delta^{(4)}(x-y)$. What remains is to multiply with the polarization vector

$$\epsilon^*_{(\lambda)\mu}(p)$$

for the out-going photon with momentum p. Also, the Fourier transform brings the expression to momentum space. The out-going momentum is on-shell, i. e. it satisfies $p_\mu p^\mu = 0$ for photons. Similarly, for incoming photons we need to remove the external propagator line and contract with

$$\epsilon_{(\lambda)\mu}(p),$$

instead.

For out-going electrons we need to remove the external fermion propagator and multiply with $\bar{u}_s(\vec{p})$ where p is the momentum of the out-going electron satisfying $p^2 + m^2 = 0$ and s labels its spin state. Similarly, for an incoming electron we need to contract with $iu_s(p)$.

For out-going positrons we need to contract with $iv_s(p)$ (and include here one factor i because $i\bar{\psi}$ appears in the LSZ rule in our conventions). For an incoming positron the corresponding external spinor is $\bar{v}_s(p)$.

Video: **Lecture23Video07.mp4**

Propagators in momentum space. Working now directly in momentum space, the photon propagator is represented by

$$-i\frac{\mathscr{P}_{\mu\nu}(p)}{p^2 - i\epsilon} = -i\frac{\eta_{\mu\nu} - \frac{p_\mu\, p_\nu}{p^2}}{p^2 - i\epsilon}.$$

The fermion propagator is

$$-i\frac{-i\not{p} + m}{p^2 + m^2 - i\epsilon}.$$

The vertex is as before $-e\gamma^\mu$. Momentum conservation must be imposed at each vertex. Together these rules constitute the Feynman rules of QED. One can work with the graphical representation and then translate to formula at a convenient point. However, when in doubt, one can always go back to the functional representation.

10.5 Elementary scattering processes

We are now ready to use the formalism of quantum field theory, specifically quantum electrodynamics, to determine actually scattering amplitudes and cross section. The incoming and outgoing states can consist of photons, electrons and positrons but also muons or anti-muons and more generally any charged particles. When the charged particles are scalar bosons, one would use a variant of the theory called scalar electrodynamics, but we are here concerned with charged spin-1/2 particles which are described by standard spinor electrodynamics.

In the following we will bring together several of the elements we have discussed before, such as

- the Lagrangian of spinor quantum electrodynamics,
- the idea of perturbation theory as an expansion in the coupling constant e,
- the graphical representation in terms of Feynman diagrams,
- solutions to the free Dirac equation for incoming or outgoing electrons and positrons (or muons and anti-muons),
- the propagators for Dirac femions and for photons,

It might be a good idea to go back and revise these topics if you feel uncertain about them. We will see on the way that we need some additional technical knowledge, specifically

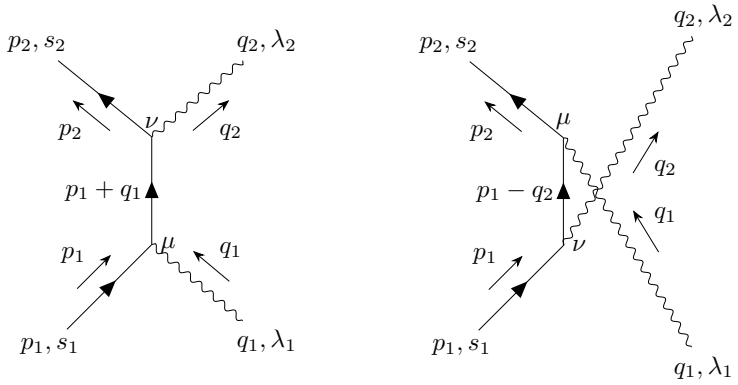
- how to do spin sums,
- how to calculate traces of gamma matrices
- how Mandelstam variables are defined and how one can work with them.

These points will also be discussed in the exercises.

We will then start to look at the elastic scattering of a photon and an electron, a process known as Compton scattering. We will write down the Feynman diagrams and the corresponding algebraic expressions. For another process, namely the scattering of an electron-positron pair to a muon-anti-muon pair we will do this, as well, but then also go on and evaluate the expressions further until we arrive at a nice and compact result for the scattering cross-section.

Video: **Lecture23Video09.mp4**

Compton Scattering. As a first example let us consider Compton scattering $e^-\gamma \to e^-\gamma$



These are two diagrams at order e^2, as shown above. The first diagram corresponds to the expression

$$\bar{u}_{s_2}(p_2)(-e\gamma^\nu)\left(-i\frac{-i(\not{p}_1+\not{q}_1)+m}{(p_1+q_1)^2+m^2}\right)(-e\gamma^\mu)\; iu_s(p_1)\;\epsilon_{(\lambda_1)\mu}(q_1)\;\epsilon^*_{(\lambda_2)\nu}(q_2).$$

Similarly, the second diagram gives

$$\bar{u}_{s_2}(p_2)(-e\gamma^\mu)\left(-i\frac{-i(\not{p}_1-\not{q}_1)+m}{(p_1-q_1)^2+m^2}\right)(-e\gamma^\nu)\; iu_s(p_1)\;\epsilon_{(\lambda_1)_\mu}(q_1)\;\epsilon^*_{(\lambda_2)\nu}(q_2).$$

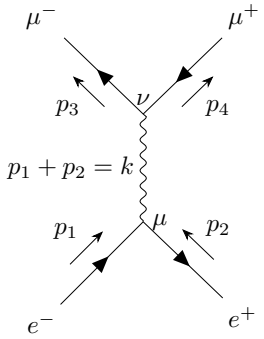
Combining terms and simplifying a bit leads to

$$i\mathcal{T} = e^2\epsilon_{(\lambda_1)\mu}(q_1)\;\epsilon^*_{(\lambda_2)\nu}(q_2)\;\bar{u}_{s_2}(p_2)\left[\gamma^\nu\;\frac{-i(\not{p}_1+\not{q}_1)+m}{(p_1+q_1)^2+m^2}\gamma^\mu+\gamma^\mu\frac{-i(\not{p}_1-\not{q}_2)+m}{(p_1-q_2)^2+m^2}\gamma^\nu\right]u_{s_1}(p_1).$$

Video: **Lecture24Video01.mp4**

Electron-positron to muon-anti-muon scattering. As another example for an interesting process in QED we consider $e^-e^+ \to \mu^-\mu^+$. From the point of view of QED, the muon behaves like the electron but has a somewhat larger mass. Diagrams contributing to this process are (we keep

the polarizations implicit)



The corresponding expression is

$$i\mathcal{T} = \bar{v}(p_2)(-e\gamma^\mu)\, iu(p_1)\, \left(-i\frac{\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}}{(k^2)}\right) \bar{u}(p_3)\, (-e\gamma^\nu)\, iv(p_4),$$

with $k = p_1 + p_2 = p_3 + p_4$.

Video: **Lecture24Video02.mp4**

On-shell conditions. The external momenta are on-shell and the spinors $u(p_1)$ etc. satisfy the Dirac equation,

$$(i\not{p}_1 + m_e)u(p_1) = 0, \qquad (-i\not{p}_4 + m_\mu)v(p_4) = 0,$$
$$\bar{u}(p_3)(i\not{p}_3 + m_\mu) = 0, \qquad \bar{v}(p_2)(-i\not{p}_2 + m_e) = 0.$$

This allows to write

$$i\bar{v}(p_2)\,\gamma^\mu k_\mu\, u(p_1) = i\bar{v}(p_2)\,(\not{p}_1 + \not{p}_2)\, u(p_1) = \bar{v}(p_2)\,(-m_e + m_e)\, u(p_1) = 0,$$
$$i\bar{u}(p_3)\,\gamma^\nu k_\nu\, v(p_4) = i\bar{u}(p_3)\,(\not{p}_3 + \not{p}_4)\, v(p_4) = \bar{u}(p_3)\,(-m_\mu + m_\mu)\, v(p_4) = 0.$$

These arguments show that the term $\sim k_\mu k_\nu$ can be dropped. This is essentially a result of gauge invariance.

Video: **Lecture24Video03.mp4**

Complex conjugate and squared amplitudes. We are left with

$$\mathcal{T} = \frac{e^2}{k^2}\bar{v}(p_2)\gamma^\mu u(p_1)\, \bar{u}(p_3)\gamma_\mu v(p_4).$$

To calculate $|\mathcal{T}|^2$ we also need $\mathcal{T}^*$ which follows from hermitian conjugation

$$\mathcal{T}^* = \frac{e^2}{k^2}v^\dagger(p_4)\gamma_\mu^\dagger\bar{u}^\dagger(p_3)\; u^\dagger(p_1)\gamma^{\mu\dagger}\bar{v}^\dagger(p_2).$$

Recall that $\bar{u}(p) = u(p)^\dagger\beta$ with $\beta = i\gamma^0$. With the explicit representation

$$\gamma^\mu = \begin{pmatrix} & -i\bar{\sigma}^\mu \\ -i\sigma^\mu & \end{pmatrix},$$

it is also easy to prove $\beta\gamma^{\mu\dagger}\beta = -\gamma^\mu$. By inserting $\beta^2 = \mathbb{1}$ at various places we find thus

$$\mathcal{T}^* = \frac{e^2}{k^2}\bar{v}(p_4)\gamma_\mu u(p_3)\ \bar{u}(p_1)\gamma^\mu v(p_2)$$

Putting together and using $s = -k^2 = -(p_1+p_2)^2$ we obtain

$$|\mathcal{T}|^2 = \frac{e^4}{s^2}\bar{u}(p_1)\gamma^\mu v(p_2)\ \bar{v}(p_2)\gamma^\nu u(p_1)\ \bar{u}(p_3)\gamma_\nu v(p_1)\ \bar{v}(p_4)\gamma_\mu u(p_3).$$

Video: **Lecture24Video04.mp4**

Spin sums and averages. To proceed further, we need to specify also the spins of the incoming and outgoing particles. The simplest case is the one of unpolarized particles so that we need to average the spins of the incoming electrons, and to sum over possible spins in the final state. Summing over the spins of the μ^+ can be done as follows (exercise)

$$\sum_{s=1}^{2} v_s(p_4)\bar{v}_s(p_4) = -i\not{p}_4 - m_\mu,$$

and similarly for μ^-

$$\sum_{s=1}^{2} u_s(p_3)\bar{u}_s(p_3) = -i\not{p}_3 + m_\mu.$$

We can therefore write

$$\bar{u}(p_3)\gamma_\nu v(p_4)\,\bar{v}(p_4)\gamma_\mu u(p_3) = \mathrm{tr}\left\{(-i\not{p}_3 + m_\mu)\gamma_\nu(-i\not{p}_4 - m_\mu)\gamma_\mu\right\}.$$

Spins of the electron and positron must be averaged instead,

$$\frac{1}{2}\sum_{s=1}^{2} u(p_1)\bar{u}(p_1) = \frac{1}{2}(-i\not{p}_1 + m_e),$$
$$\frac{1}{2}\sum_{s=1}^{2} v(p_2)\bar{v}(p_2) = \frac{1}{2}(-i\not{p}_2 - m_e).$$

This leads to

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{T}|^2 = \frac{e^4}{4s^2}\mathrm{tr}\left\{(-i\not{p}_1 + m_e)\gamma^\mu(-i\not{p}_2 - m_e)\gamma^\nu\right\} \times \mathrm{tr}\left\{(-i\not{p}_3 + m_\mu)\gamma_\nu(-i\not{p}_4 - m_\mu)\gamma_\mu\right\}.$$

In order to proceed further, we need to know how to evaluate traces of up to four gamma matrices.

Video: **Lecture24Video05.mp4**

Traces of gamma matrices. We need to understand how to evaluate traces of the form $\mathrm{tr}\{\gamma^{\mu_1}\cdots\gamma^{\mu_n}\}$ To work them out we can use $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$, $\gamma_5^2 = \mathbb{1}$ and $\{\gamma^\mu,\gamma_5\} = 0$. Also, $\mathrm{tr}\{\mathbb{1}\} = 4$. First we prove that traces of an odd number of gamma matrices must vanish,

$$\begin{aligned}\mathrm{tr}\{\gamma^{\mu_1}\cdots\gamma^{\mu_n}\} &= \mathrm{tr}\{\gamma_5^2\ \gamma^{\mu_1}\gamma_5^2\cdots\gamma_5^2\gamma^{\mu_n}\}\\ &= \mathrm{tr}\{(\gamma_5\gamma^{\mu_1}\gamma_5)\cdots(\gamma_5\gamma^{\mu_1}\gamma_5)\}\\ &= \mathrm{tr}\{(-\gamma_5^2\gamma_1^\mu)\cdots(-\gamma_5^2\gamma^{\mu_n})\}\\ &= (-1)^n\mathrm{tr}\{\gamma^{\mu_1}\cdots\gamma^{\mu_n}\}.\end{aligned}$$

This implies what we claimed.

Video: **Lecture24Video06.mp4**

Now for even numbers

$$\mathrm{tr}\{\gamma^\mu\gamma^\nu\} = \mathrm{tr}\{\gamma^\nu\gamma^\mu\} = \tfrac{1}{2}\mathrm{tr}\{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\} = \eta^{\mu\nu}\mathrm{tr}\{\mathbb{1}\} = 4\eta^{\mu\nu}.$$

From this it also follows that

$$\mathrm{tr}\{\not{p}\not{q}\} = 4p\cdot q.$$

Now consider $\mathrm{tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\}$. This idea is to commute γ^μ to the right using $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$. Thus

$$\begin{aligned}\mathrm{tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} &= -\mathrm{tr}\{\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\} + 2\eta^{\mu\nu}\ \mathrm{tr}\{\gamma^\rho\gamma^\sigma\}\\ &= \mathrm{tr}\{\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma\} - 2\eta^{\rho\mu}\mathrm{tr}\{\gamma^\nu\gamma^\sigma\} + 2\eta^{\mu\nu}\ \mathrm{tr}\{\gamma^\rho\gamma^\sigma\}\\ &= -\mathrm{tr}\{\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu\} + 2\eta^{\sigma\mu}\ \mathrm{tr}\{\gamma^\nu\gamma^\rho\} - 2\eta^{\rho\mu}\ \mathrm{tr}\{\gamma^\nu\gamma^\sigma\} + 2\eta^{\mu\nu}\ \mathrm{tr}\{\gamma^\rho\gamma^\sigma\}.\end{aligned}$$

But by the cyclic property of the trace

$$\mathrm{tr}\{\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu\} = \mathrm{tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\}$$

which is also on the left hand side. Bringing it to the left and dividing by 2 gives

$$\begin{aligned}\mathrm{tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} &= \eta^{\sigma\mu}\ \mathrm{tr}\{\gamma^\nu\gamma^\rho\} - \eta^{\rho\mu}\ \mathrm{tr}\{\gamma^\nu\gamma^\sigma\} + \eta^{\mu\nu}\mathrm{tr}\{\gamma^\rho\gamma^\sigma\}\\ &= 4(\eta^{\sigma\mu}\eta^{\nu\rho} - \eta^{\rho\mu}\eta^{\nu\sigma} + \eta^{\mu\nu}\eta^{\rho\sigma}).\end{aligned}$$

This is the result we were looking for. Clearly by using this trick we can in principle evaluate traces of an arbitrary number of gamma matrices.

Video: **Lecture24Video07.mp4**

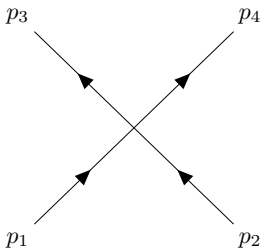
Result so far. Coming back to $e^-e^+ \to \mu^-\mu^+$ we find

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|\mathcal{T}|^2 =& \frac{4e^4}{s^2}\left[-p_1^\mu p_2^\nu - p_1^\nu p_2^\mu + (p_1\cdot p_2 - m_e^2)\eta^{\mu\nu}\right]\\ &\times\left[-(p_3)_\nu(p_4)_\mu - (p_3)_\mu(p_4)_\nu + (p_3\cdot p_4 - m_\mu^2)\eta^{\mu\nu}\right]\\ =& \frac{8e^4}{s^2}\left[(p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_3)(p_2\cdot p_4) - m_\mu^2(p_1\cdot p_2) - m_e^2(p_3\cdot p_4) + 2m_e^2m_\mu^2\right]\end{aligned}$$

This looks already quite decent but it can be simplified even further in terms of Mandelstam variables.

Video: **Lecture24Video08.mp4**

Mandelstam Variables. The Mandelstam variables for a $2 \to 2$ process



are given by

$$s = -(p_1 + p_2)^2 = -(p_3 + p_4)^2,$$
$$t = -(p_1 - p_3)^2 = -(p_2 - p_4)^2,$$
$$u = -(p_1 - p_4)^2 = -(p_2 - p_3)^2.$$

Together with the squares p_1^2, p_2^2, p_3^2, p_4^2, the Mandelstam variables can be used to express all Lorentz invariant bilinears in the momenta. Incoming and outgoing momenta are on-shell such that $p_1^2 + m_1^2 = 0$ etc. The sum of Mandelstam variables is

$$s + t + u = -(p_1^2 + p_2^2 + p_3^2 + p_4^2) = m_1^2 + m_2^2 + m_3^2 + m_4^2.$$

Using these variables for example through

$$p_1 \cdot p_4 = -\frac{1}{2}\left[(p_1 - p_4)^2 - p_1^2 - p_4^2\right] = \frac{1}{2}\left[u - m_e^2 + m_\mu^2\right],$$

one finds for $e^- e^+ \to \mu^- \mu^+$

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{T}|^2 = \frac{2e^4}{s^2}\left[t^2 + u^2 + 4s(m_e^2 + m_\mu^2) - 2(m_e^2 + m_\mu^2)^2\right].$$

Video: **Lecture24Video09.mp4**

Differential cross section. From the squared matrix element we can calculate the differential cross section in the center of mass frame. For relativistic kinematics of $2 \to 2$ scattering and the normalization conventions we employ here one has in the center of mass frame

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{p}_3|}{|\vec{p}_1|}\frac{1}{4}\sum_{\text{spins}} |\mathcal{T}|^2.$$

Let us express everything in terms of the energy E of the incoming particles and the angle θ between the incoming e^- electron momenta and outgoing μ^- muon.

$$|\vec{p}_1| = \sqrt{E^2 - m_e^2}, \qquad s = 4E^2,$$
$$|\vec{p}_3| = \sqrt{E^2 - m_\mu^2}, \qquad t = m_e^2 + m_\mu^2 - 2E^2 + 2\vec{p}_1 \cdot \vec{p}_3,$$
$$\vec{p}_1 \cdot \vec{p}_3 = |\vec{p}_1||\vec{p}_3|\cos\theta, \qquad u = m_e^2 + m_\mu^2 - 2E^2 - 2\vec{p}_1 \cdot \vec{p}_3.$$

With these relations we can express $\frac{d\sigma}{d\Omega}$ in terms of E and θ only.

Video: **Lecture24Video10.mp4**

Ultrarelativistic limit. Let us concentrate on the ultrarelativistic limit $E \gg m_e, m_\mu$ so that we can set $m_e = m_\mu = 0$. One has then $|\vec{p}_1| = |\vec{p}_3|$ and

$$t^2 + u^2 = 8E^4(1 + \cos^2\theta), \qquad \frac{2(t^2 + u^2)}{s^2} = 1 + \cos^2\theta,$$

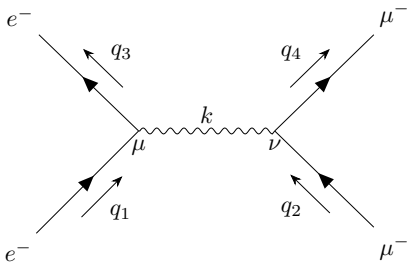
which leads to

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 s}(1 + \cos^2\theta) = \frac{\alpha^2}{4s}(1 + \cos^2\theta).$$

In the last equation we used $\alpha = e^2/(4\pi)$.

Video: **Lecture24Video11.mp4**

Electron-Muon Scattering. We can also consider the scattering process $e^-\mu^- \to e^-\mu^-$,



$$k = q_1 - q_3$$

$$i\mathcal{T} = \bar{u}(q_3)(-e\gamma^\mu)iu(q_1)\left(-i\frac{\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}}{k^2}\right)\bar{u}(q_4)(-e\gamma^\nu)iu(q_2).$$

By a similar argument as before the term $\sim k_\mu k_\nu$ drops out,

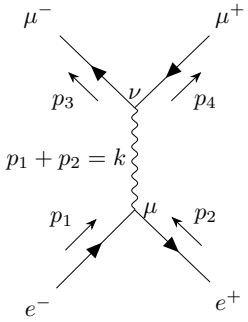
$$\mathcal{T} = \frac{e^2}{(q_1 - q_3)^2}\bar{u}(q_3)\gamma^\mu u(q_1)\bar{u}(q_4)\gamma_\mu u(q_2) \qquad (e^-\mu^- \to e^-\mu^-).$$

Video: **Lecture24Video12.mp4**

Comparison to electron to muon scattering. Compare this to what we have found for $e^-e^+ \to \mu^-\mu^+$

$$\mathcal{T} = \frac{e^2}{(p_1 + p_2)^2}\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(p_3)\gamma_\mu v(p_4),$$

where the conventions were according to



There is a close relation and the expressions agree if we put

$$\begin{aligned}
q_1 &= +p_1, & u(q_1) &= u(p_1),\\
q_2 &= -p_4, & u(q_2) &= u(-p_4) \to v(p_4),\\
q_3 &= -p_2, & \bar{u}(q_3) &= \bar{u}(-p_2) \to \bar{v}(p_2),\\
q_4 &= +p_3, & \bar{u}(q_4) &= \bar{u}(p_3).
\end{aligned}$$

Video: **Lecture25Video01.mp4**

Crossing symmetry. Recall that

$$(i\not{p} + m)\, u(p) = 0 \qquad \text{but} \qquad (-i\not{p} + m)\, v(p) = 0.$$

However one sign arises from the spin sums

$$\sum_{s=1}^{2} u_s(p)\bar{u}_s(p) = -i\not{p} + m,$$
$$\sum_{s=1}^{2} v_s(p)\, \bar{v}_s(p) = -i\not{p} - m = -\sum_{s} u_s(-p)\, \bar{u}_s(-p).$$

Because it appears twice, the additional sign cancels for $|\mathcal{T}|^2$ after spin averaging and one finds indeed the same result as for $e^-e^+ \to \mu^-\mu^+$ but with

$$\begin{aligned} s_q &= -(q_1 + q_2)^2 = -(p_1 - p_4)^2 = u_p, \\ t_q &= -(q_1 - q_3)^2 = -(p_1 + p_2)^2 = s_p, \\ u_q &= -(q_1 - q_4)^2 = -(p_1 - p_3)^2 = t_p. \end{aligned}$$

We can take what we had calculated but must change the role of s, $t$and u ! This is an example of *crossing symmetries*.

Electron-muon scattering in the massless limit. Recall that we found for $e^-e^+ \to \mu^-\mu^+$ in the massless limit $m_e = m_\mu = 0$ simply

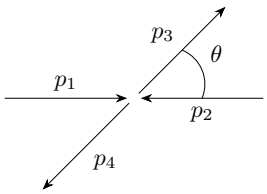
$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{T}|^2 = \frac{2e^4}{s^2}\left[t^2 + u^2\right].$$

For $e^-\mu^- \to e^-\mu^-$ we find after the replacements $u \to s$, $s \to t$, $t \to u$,

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{T}|^2 = \frac{2e^4}{t^2}\left[u^2 + s^2\right].$$

Video: **Lecture25Video03.mp4**

More on Mandelstam variables. To get a better feeling for s, t and u, let us evaluate them in the center of mass frame for a situation where all particles have mass m.

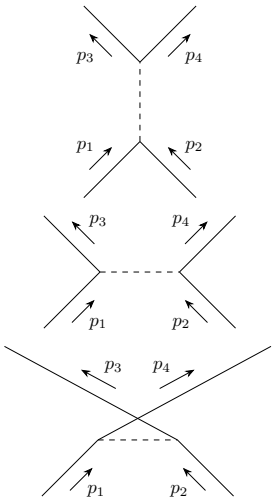


$$p_1^\mu = (E, \vec{p}), \qquad p_2^\mu = (E, -\vec{p}),$$
$$p_3^\mu = (E, \vec{p}'), \qquad p_4^\mu = (E, -\vec{p}').$$

While s measures the center of mass energy, t is a momentum transfer that vanishes in the soft limit $\vec{p}^2 \to 0$ and in the colinear limit $\theta \to 0$. Similarly, u vanishes for $\vec{p}^2 \to 0$ and for backward scattering $\theta \to \pi$.

Video: **Lecture25Video04.mp4**

s-, t- and u-channels. One speaks of interactions in different channels for tree diagrams of the following generic types,



$$\propto \frac{1}{-s+m^2} \qquad s\text{-channel}$$

$$\propto \frac{1}{-t+m^2} \qquad t\text{-channel}$$

$$\propto \frac{1}{-u+m^2} \qquad u\text{-channel}$$

Video: **Lecture25Video05.mp4**

Electron-muon scattering. For the cross section we find for $e^-\mu^- \to e^-\mu^-$ in the massless limit

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{1}{4}\sum_{\text{spins}}|\mathcal{T}|^2 = \frac{\alpha^2[4+(1+\cos\theta)^2]}{2s(1-\cos\theta)^2}$$

This diverges in the colinear limit $\theta \to 0$ as we had already seen for Yukawa theory in the limit where the exchange particle becomes massless.

Note that by the definition $s \geq 0$ while u and t can have either sign. Replacements of the type used for crossing symmetry are in this sense always to be understood as analytic continuation.

10.6 Relativistic scattering and decay kinematics

Video: **Lecture25Video06.mp4**

Covariant normalization of asymptotic states. For non-relativistic physics this we have used a normalization of single particle states in the asymptotic incoming and out-going regimes such that

$$\langle\vec{p}|\vec{q}\rangle = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}).$$

For relativistic physics this has the drawback that it is not Lorentz invariant. To see this let us consider a boost in z-direction

$$\begin{aligned} E' &= \gamma(E+\beta p^3), \\ p^{1\prime} &= p^1, \\ p^{2\prime} &= p^2, \\ p^{3\prime} &= \gamma(p^3+\beta E). \end{aligned}$$

Using the identity

$$\delta\left(f(x)-f(x_0)\right) = \frac{1}{|f'(x_0)|}\delta(x-x_0),$$

one finds

$$\begin{aligned}\delta^{(3)}(\vec{p}-\vec{q}) &= \delta^{(3)}(\vec{p}'-\vec{q}')\frac{dp^{3\prime}}{dp^3} = \delta^{(3)}(\vec{p}-\vec{q})\gamma\left(1+\beta\frac{dE}{dp^3}\right)\\ &= \delta^{(3)}(\vec{p}'-\vec{q}')\frac{1}{E}\gamma\left(E+\beta p^3\right)\\ &= \frac{E'}{E}\delta^{(3)}(\vec{p}'-\vec{q}').\end{aligned}$$

This shows, however, that $E\,\delta^{(3)}(\vec{p}-\vec{q})$ is in fact Lorentz invariant.

Video: **Lecture25Video07.mp4**

This motivates to change the normalization such that

$$|p;\mathrm{in}\rangle = \sqrt{2E_p}\,a^\dagger_{\vec{p}}(-\infty)|0\rangle = \sqrt{2E_{\vec{p}}}\,|\vec{p};\mathrm{in}\rangle.$$

Note the subtle difference in notation between $|p;\mathrm{in}\rangle$ (relativistic normalization) and $|\vec{p};\mathrm{in}\rangle$ (non-relativistic normalization). This implies for example

$$\langle p;\mathrm{in}|q;\mathrm{in}\rangle = 2E_p(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}).$$

With this normalization we must divide by $2E_p$ at the same places. In particular the completeness relation for single particle incoming states is

$$\mathbb{1}_{1-\mathrm{particle}} = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}|p;\mathrm{in}\rangle\langle p;\mathrm{in}|.$$

In fact, what appears here is a Lorentz invariant momentum measure. To see this consider

$$\int\frac{d^4p}{(2\pi)^4}\,(2\pi)\,\delta(p^2+m^2)\,\theta(p^0) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}.$$

The left hand side is explicitly Lorentz invariant and so is the right hand side.

Video: **Lecture25Video08.mp4**

Covariantly normalized S-matrix. We can use the covariant normalization of states also in the definition of S-matrix elements. The general definition is as before

$$S_{\beta\alpha} = \langle\beta;\mathrm{out}|\alpha;\mathrm{in}\rangle = \delta_{\beta\alpha} + i\,\mathcal{T}_{\beta\alpha}(2\pi)^4\delta^{(4)}(p^{\mathrm{in}}-p^{\mathrm{out}}).$$

But now we take elements with relativistic normalization, e.g. for $2\to 2$ scattering

$$S_{q_1q_2,p_1p_2} = \langle q_1,q_2;\mathrm{out}|p_1,p_2;\mathrm{in}\rangle.$$

We can calculate these matrix elements as before using the LSZ reduction formula to replace $\sqrt{2E_p}a^\dagger_{\vec{p}}(-\infty)$ by fields. For example, for relativistic scalar fields

$$\sqrt{2E_{\vec{p}}}\;a^\dagger_{\vec{p}}(-\infty) = \sqrt{2E_{\vec{p}}}\;a^\dagger_{\vec{p}}(\infty) + i\left[-(p^0)^2+\vec{p}^2+m^2\right]\phi^*(p).$$

This allows to calculate S-matrix elements through correlation functions.

Video: **Lecture25Video09.mp4**

Cross sections for $2 \to n$ scattering. Let us now generalize our discussion of $2 \to 2$ scattering of non-relativistic particles to a scattering $2 \to n$ of relativistic particles. The transition probability is as before

$$P = \frac{|\langle \beta; \text{out}|\alpha; \text{in}\rangle|^2}{\langle \beta; \text{out}|\beta; \text{out}\rangle \langle \alpha; \text{in}|\alpha; \text{in}\rangle}.$$

Rewriting the numerator in terms of $\mathcal{T}_{\beta\alpha}$ and going over to the transition rate we obtain as before

$$\dot{P} = \frac{V(2\pi)^4 \delta^{(4)}(p^{\text{out}} - p^{\text{in}})|\mathcal{T}|^2}{\langle \beta; \text{out}|\beta; \text{out}\rangle \langle \alpha; \text{in}|\alpha; \text{in}\rangle}. \tag{10.6}$$

But now states are normalized in a covariant way

$$\begin{aligned}
\langle p|p\rangle &= \lim_{q\to p} \langle p|q\rangle \\
&= \lim_{q\to p} 2E_p(2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \\
&= 2E_p(2\pi)^3 \delta^{(3)}(0) \\
&= 2E_p V
\end{aligned}$$

One has thus for the incoming state of two particles

$$\langle \alpha; \text{in}|\alpha; \text{in}\rangle = 4E_1 E_2 V^2.$$

For the outgoing state of n particles one has instead

$$\langle \beta; \text{out}|\beta; \text{out}\rangle = \prod_{j=1}^{n} \{2q_j^0 V\}.$$

The product goes over final state particles which have the four-momentum q_j^n. So, far we have thus

$$\dot{P} = \frac{V\ (2\pi)^4\ \delta^{(4)}(p^{\text{in}} - p^{\text{out}})|\mathcal{T}|^2}{4E_1 E_2 V^2\ \prod_{j=1}^{n} \{2q_j^0 V\}}.$$

Video: **Lecture25Video10.mp4**

Lorentz invariant phase space. To count final state momenta appropriately we could go back to finite volume and then take the continuum limit. This leads to an additional factor

$$\sum_{\vec{n}_j} \to V \int \frac{d^3q}{(2\pi)^3}$$

for each final state particle. The transition rate becomes

$$\dot{P} = \frac{|\mathcal{T}|^2}{4E_1 E_2 V} \left[(2\pi)^4\ \delta^{(4)}\Big(p^{\text{in}} - \sum_j q_j\Big) \prod_{j=1}^{n} \left\{ \frac{d^3 q_j}{(2\pi)^3 2q_j^0} \right\} \right]$$

The expression in square brackets is known as the *Lorentz-invariant phase space* measure (sometimes "LIPS").

Video: **Lecture25Video11.mp4**

Flux and differential cross section. To go from there to a differential cross section we need to divide by a flux of particles. There is one particle per volume V with velocity $v = v_1 - v_2$, so the flux is

$$\mathcal{F} = \frac{|v|}{V} = \frac{|v_1 - v_2|}{V} = \frac{\left|\frac{p_1^3}{p_1^0} - \frac{p_2^3}{p_2^0}\right|}{V}.$$

In the last equality we chose the beam axis to coincide with the z-axis. For the differential cross section we obtain

$$d\sigma = \frac{|\mathcal{T}|^2}{4E_1E_2|v_1 - v_2|}\ [\text{LIPS}].$$

The expression in the prefactor can be rewritten like

$$\frac{1}{E_1E_2|v_1 - v_2|} = \frac{1}{p_1^0p_2^0\left|\frac{p_1^3}{p_1^0} - \frac{p_2^3}{p_2^0}\right|} = \frac{1}{|p_2^0p_1^3 - p_1^0p_2^3|} = \frac{1}{|\epsilon_{\mu xy\nu}p_2^\mu p_1^\nu|}.$$

This is not Lorentz invariant in general but invariant under boosts in the z-direction. In fact it transforms as a two-dimensional area element as it should.

Video: **Lecture25Video12.mp4**

Differential cross section in the centre of mass frame. In the center of mass frame one has $p_2^3 = -p_1^3 = \pm|\vec{p}_1|$ and

$$\frac{1}{|p_2^0p_1^3 - p_1^0p_2^0|} = \frac{1}{|\vec{p}_1|(p_1^0 + p_2^0)} = \frac{1}{|\vec{p}_1|_{\text{COM}}\sqrt{s}}$$

This leads finally to the result for the differential cross section

$$d\sigma = \frac{|\tau|^2}{4|\vec{p}_1|_{\text{COM}}\sqrt{s}}\left[(2\pi)^4\delta^{(4)}\Big(p^{\text{in}} - \sum_j q_j\Big)\prod_{j=1}^n\left\{\frac{d^3q_j}{(2\pi)^3 2q_j^0}\right\}\right].$$

Video: **Lecture25Video13.mp4**

$2 \to 2$ scattering. For the case of $n = 2$ one can write the Lorentz invariant differential phase space element in the center of mass frame (exercise)

$$\left[(2\pi)^4\ \delta^{(4)}(p^{\text{in}} - q_1 - q_2)\frac{d^3q_1}{(2\pi)^3 2q_1^0}\ \frac{d^3q_2}{(2\pi)^3 q_2^0}\right] = \frac{|\vec{q}_1|}{16\pi^2\sqrt{s}}d\Omega$$

such that

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{q}_1|}{|\vec{p}_1|}|\mathcal{T}|^2.$$

Video: **Lecture25Video14.mp4**

Decay rate. Let us now consider the decay rate of a single particle, i. e. a process $1 \to n$. We can still use equation (10.6), but now the initial state is normalized like

$$\langle\alpha; \text{in}|\alpha; \text{in}\rangle = 2E_1V.$$

We find then for the differential transition or decay rate $d\Gamma = \dot{P}$

$$d\Gamma = \frac{|\mathcal{T}|^2}{2E_1}\left[(2\pi)^4\delta^{(4)}\left(p^{\text{in}} - \sum_j q_j\right)\prod_{j=1}^{n}\left\{\frac{d^3q_j}{(2\pi)^3 2q_j^0}\right\}\right]$$

In the center of mass frame one has $E_1 = m_1$. For the special case of $1 \to 2$ decay one finds in the center of mass frame or rest frame of the initial particle

$$d\Gamma = \frac{|\mathcal{T}|^2|\vec{q}_1|}{32\pi^2 m_1^2}d\Omega.$$

10.7 Higgs/Yukawa theory

Video: **Lecture25Video15.mp4**

In the following two lectures we will discuss a quantum field theoretic model that extends somewhat beyond quantum electrodynamics. We add to the theory a neutral massive scalar field that couples to the fermions through a Yukawa interaction. One may see that additional massive scalar particle as an analog of the Higgs boson, even though our model reflects only a few of the properties of the real electroweak standard model.

We discuss the model as a further example for an interesting quantum field theory and because we can nicely study there decay processes.

- A massive Higgs boson can decay into two fermions through the Yukawa interaction. This is a tree level process and rather easy to calculate.
- Interestingly a neutral and massive Higgs boson can also decay into two photons. This process is not allowed at tree level (because the Higgs boson is neutral), but it is induced by loop diagrams. This will be the first loop diagram we will calculate in detail.

In the second part of the lecture course loop diagrams and their physical consequences will be studied in much more detail. For the Higgs decay into photons we do not need renormalization yet, which simplifies the discussion. Nevertheless there will be some new elements to be discussed.

Video: **Lecture25Video16.mp4**

Action for Higgs/Yukawa theory and fermion mass. Let us consider the following extension of QED by a neutral scalar field (with $m = gv$)

$$S[\bar{\psi},\psi,A,\phi] = \int_x \left\{-i\bar{\psi}\gamma^\mu\left(\partial_\mu - ieA_\mu\right)\psi - im\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\phi\left(-\partial_\mu\partial^\mu + M^2\right)\phi - ig\phi\bar{\psi}\psi\right\}.$$

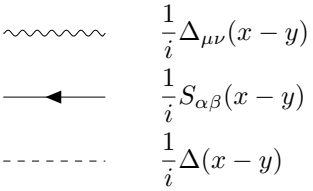
Note that a constant (homogeneous) scalar field ϕ modifies the fermion mass according to

$$m_{\text{eff}} = m + g\phi = g(v+\phi)$$

In fact, one can understand the massses of elementary fermions (leptons and quarks) in the standard model of elementary particle physics as being due to such a scalar field expectation value for the Higgs field.

Video: **Lecture25Video17.mp4**

Propagators and vertices. In the theory above we have now different propagators



$$\frac{1}{i}\Delta_{\mu\nu}(x-y)$$

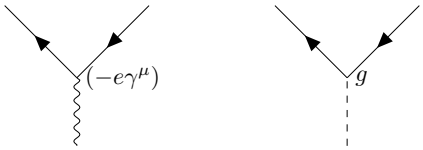
$$\frac{1}{i}S_{\alpha\beta}(x-y)$$

$$\frac{1}{i}\Delta(x-y)$$

with scalar propagator

$$\Delta(x-y)=\int_p e^{ip(x-y)}\frac{1}{p^2+M^2}.$$

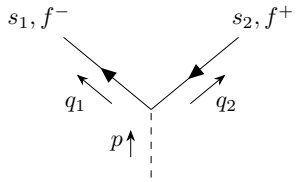
The vertices are



10.8 Higgs decay into fermions

Video: **Lecture26Video01.mp4**

Higgs decay to fermions. Let us discuss first the process $\phi \to f^- f^+$. The fermions could be leptons (e, μ, τ) or quarks (u, d, s, c, b, t). The Feynman diagram for the decay is simply



According to the Feynman rules we obtain

$$i\mathcal{T} = g\ \bar{u}_{s_1}(q_1) i v_{s_2}(q_2), \qquad \mathcal{T}^* = g\ \bar{v}_{s_2}(q_2) u_{s_1}(q_1).$$

For the absolute square one finds

$$|\mathcal{T}|^2 = g^2\ \bar{u}_{s_1}(q_1) v_{s_2}(q_2)\ \bar{v}_{s_2}(q_2) u_{s_1}(q_1).$$

Video: **Lecture26Video02.mp4**

Spin sums and Dirac traces. We will assume that the final spins are not observed and sum them

$$\sum_{\text{spins}} |\mathcal{T}|^2 = g^2\ \text{tr}\left\{(-i\not{q}_2 - m)(-i\not{q}_1 + m)\right\}$$

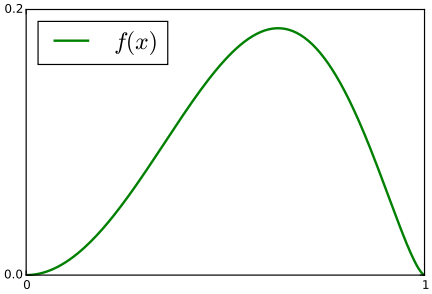
We used here again the spin sum formula

$$\sum_s v_s(p)\bar{v}_s(p) = -i\not{p} - m, \qquad \sum_s u_s(p)\bar{u}_s(p) = -i\not{p} + m.$$

Performing also the Dirac traces gives

$$\sum_{\text{spins}} |\mathcal{T}|^2 = g^2\left(-4q_1\cdot q_2 - 4m^2\right).$$

Video: **Lecture26Video03.mp4**



Kinematics in the Higgs boson rest frame. Let us now go into the rest frame of the decaying particle where

$$p = (M, 0, 0, 0), \qquad q_1 = \left(\tfrac{M}{2}, \vec{q}\right), \qquad q_2 = \left(\tfrac{M}{2}, -\vec{q}\right),$$

with

$$\vec{q}^2 = -m^2 + \tfrac{M^2}{4}, \qquad q_1 \cdot q_2 = -\frac{M^2}{4} - \vec{q}^2 = -\frac{M^2}{m^2},$$

and

$$\sum_{\text{spins}} |\mathcal{T}|^2 = 2\, g^2 M^2 \left(1 - 4\frac{m^2}{M^2}\right).$$

Note that the decay is kinematically possible only for $M > 2m$ so that the bracket is always positive.

Video: **Lecture26Video04.mp4**

Decay rate. For the particle decay rate we get

$$\frac{d\Gamma}{d\Omega} = \frac{|\vec{q}_1|}{32\pi^2 M^2} \sum_{\text{spins}} |\mathcal{T}|^2 = \frac{g^2 M}{32\pi^2} \left(1 - 4\frac{m^2}{M^2}\right)^{3/2}.$$

Because this is independent of the solid angle Ω one can easily integrate to obtain the decay rate

$$\Gamma = \frac{g^2 M}{8\pi} \left(1 - 4\frac{m^2}{M^2}\right)^{3/2}.$$

Video: **Lecture26Video05.mp4**

Dependence on fermion mass. If the scalar boson ϕ is the Higgs boson, the Yukawa coupling is in fact proportional to the fermion mass m,

$$g = \frac{m}{V}.$$

One has then

$$\Gamma = \frac{M^3}{32\pi v^2} f\left(\frac{2m}{M}\right)$$

where

$$f(x) = x^2(1 - x^2)^{3/2}$$

Decay into light fermions is suppressed because of small coupling while decay into very heavy fermions is suppressed by small phase space or even kinematically excluded for $2m > M$.

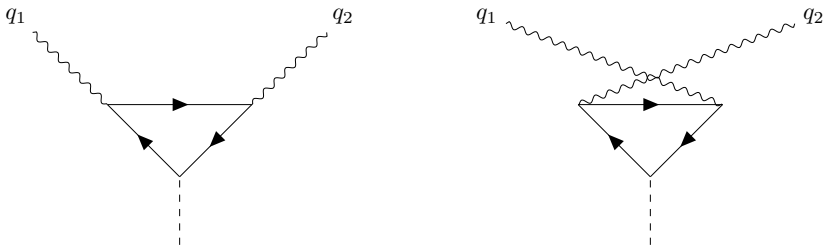
For Higgs boson mass of $M = 125$ GeV the largest decay rate to fermions is to $b\bar{b}$ (bottom quark and anti-quark). This corresponds to $m = 4.18$ GeV. The top quark would have larger coupling but is in fact too massive ($m = 172$ GeV). (The lepton with largest mass is the tauon τ with $m = 1.78$ GeV.)

10.9 Higgs decay into photons

Video: **Lecture26Video06.mp4**

Higgs decay into photons. A Higgs particle can also decay into photons and this is in fact how it was discovered. How is this possible? If we try to write down a diagram in the theory introduced above we realize that there is no tree diagram. However, there are loop diagrams!

Consider the diagrams



These terms arise from the expansion of the partition function if the fermion propagator appears 3 times and there are 2 fermion-photon and one fermion-scalar vertices.

Video: **Lecture26Video07.mp4**

Signs in fermion loops. Schematically, the vertices are derivatives

$$\left[(-e\gamma^\mu)\left(\frac{1}{i}\frac{\delta}{\delta J^\mu}\right)\left(i\frac{\delta}{\delta\eta}\right)\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}}\right)\right] \qquad \text{or} \qquad \left[g\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\left(i\frac{\delta}{\delta\eta}\right)\left(\frac{1}{i}\frac{\delta}{\bar{\eta}}\right)\right]$$

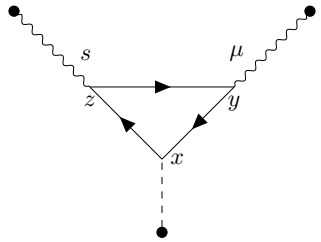
and they act here on a chain like

$$\left[(i\bar{\eta})\left(\frac{1}{i}S\right)(i\eta)\right]\left[(i\bar{\eta})\left(\frac{1}{i}S\right)(i\eta)\right]\left[(i\bar{\eta})\left(\frac{1}{i}S\right)(i\eta)\right].$$

Note that the derivative with respect to $\bar{\eta}$ can be commuted through the square brackets and acts on $\bar{\eta}$ from the left. Factors $1/i$ and i cancel. The derivative with respect to η receives an additional minus sign from commuting and this cancels against i^2. In this way the vertices can connect the elements of the chain. However, for a closed loop also the beginning and end of the chain must be connected. To make this work, one can first bring the $(i\eta)$ from the end of the chain to its beginning. This leads to one additional minus sign from anti-commuting Grassmann fields. This shows that *closed fermion lines have one more minus sign.*

Video: **Lecture26Video08.mp4**

Position space representation. In position space and including sources, the first diagram is

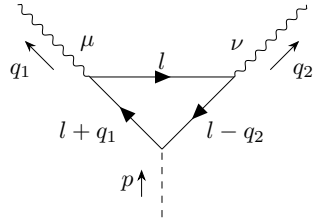


$$g(-1)\int_{x,y,z}\mathrm{tr}\left\{\left[\frac{1}{i}S(x-y)\right](-e\gamma^{\mu})\left[\frac{1}{i}S(y-z)\right](-e\gamma^{\nu})\left[\frac{1}{i}S(z-x)\right]\right\}$$
$$\times\int_{u,v,w}\left[\frac{1}{i}\Delta_{\mu\alpha}(y-u)\right](iJ^{\alpha}(u)\left[\frac{1}{i}\Delta_{\nu\beta}(z-v)\right](iJ^{\beta}(v)\left[\frac{1}{i}\Delta(x-w)\right](iJ(w))$$

The trace is for the Dirac matrix indices.

Video: **Lecture26Video09.mp4**

Momentum space representation for first diagram. If one translates this now to momentum space and considers the amputated diagram for an S-matrix element, one finds that momentum conservation constrains momenta only up to one free integration momentum or loop momentum. In fact, more generally, there is one integration momentum for every closed loop. The first diagram is then



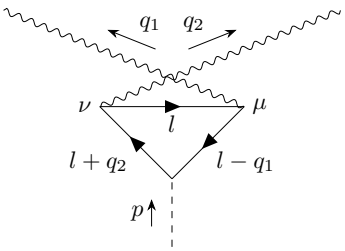
$$(-1)ge^2\,\epsilon^*_{\mu}(q_1)\epsilon^*_{\nu}(q_2)\int_l\frac{1}{[(l+q_1)^2+m^2-i\epsilon][l^2+m^2-i\epsilon][(l-q_2)^2+m^2+i\epsilon]}$$
$$\times\,\mathrm{tr}\left\{\left[-i(\not{l}+\not{q}_1)+m\right]\gamma^{\mu}\left[-i\not{l}+m\right]\gamma^{n}u\left[-i(\not{l}-\not{q}_2)+m\right]\right\}$$

Here we use here the abbreviation

$$\int_l=\int\frac{d^4l}{(2\pi)^4}.$$

Video: **Lecture26Video10.mp4**

Momentum space representation for second diagram. For the second diagram we can write



$$(-1)ge^2\,\epsilon^*_\mu(q_1)\,\epsilon^*_\nu(q_2)\int_l\ldots$$

where the integrand is the same up to the interchange $q_1 \leftrightarrow q_2$ and $\mu \leftrightarrow \nu$. We can therefore concentrate on evaluating the first diagram.

Video: **Lecture26Video11.mp4**

Analytic continuation and Dirac traces. The Feynman $i\epsilon$ terms allow to perform a Wick rotation to Euclidean space $l^0 = i\tilde{l}^0_E$ so that l^2 is then positive. Let us count powers of l. First, in the Dirac trace we have terms with up to 5 gamma matrices. However, only traces of an even number of gamma matrices are non-zero.

With a bit of algebra one finds for the Dirac trace

$$\begin{aligned}
&\mathrm{tr}\left\{\left[-i(\not{l}+\not{q}_1)+m\right]\gamma^\mu\left[-i\not{l}+m\right]\gamma^\nu\left[-i(\not{l}-\not{q}_2)+m\right]\right\}\\
&=-m\,\mathrm{tr}\left\{(\not{l}+\not{q}_1)\gamma^\mu\not{l}\gamma^\nu+(\not{l}+\not{q}_1)\gamma^\mu\gamma^\nu(\not{l}-\not{q}_2)+\gamma^\mu\not{l}\gamma^\nu(\not{l}-\not{q}_2)\right\}+m^3\,\mathrm{tr}\left\{\gamma^\mu\gamma^\nu\right\}\\
&=-4m\Big[(l+q_1)^\mu l^\nu+(l+q_1)^\nu l^\mu-(l+q_1)\cdot l\,\eta^{\mu\nu}\\
&\qquad+(l+q_1)^\mu(l-q_2)^\nu+(l+q_1)\cdot(l-q_2)\eta^{\mu\nu}-(l+q_1)^\nu(l-q_2)^\mu\\
&\qquad+l^\mu(l-q_2)^\nu+(l-q_2)^\mu l^\nu-\eta^{\mu\nu}l\cdot(l-q_2)\Big]+4\eta^{\mu\nu}m^3\\
&=-4m\left[4l^\mu l^\nu-l^2\eta^{\mu\nu}-l^2\eta^{\mu\nu}+2q_1^\mu l^\nu-2q_2^\nu l^\mu-q_1^\mu q_2^\nu+q_1^\nu q_2^\mu-(q_1\cdot q_2)\eta^{\mu\nu}\right]+4\eta^{\mu\nu}m^3.
\end{aligned}$$

Video: **Lecture26Video12.mp4**

Feynman parameters. Let us now consider the denominator. One can introduce so-called Feynman parameters to write

$$\begin{aligned}
&\frac{1}{[(l+q_1)^2+m^2][l^2+m^2][(l-q_2)^2+m^2]}\\
&=2!\int_0^1 du_1\cdots du_3\,\delta(u_1+u_2+u_3-1)\frac{1}{\left[u_1[(l+q_1)^2+m^2]+u_2[l^2+m^2]+u_3[(l-q_2)^2+m^2]\right]^3}\\
&=2\int_0^1 du_1\cdots du_3\frac{\delta(u_1+u_2+u_3-1)}{\left[l^2+2l(u_1q_1-u_3q_2)+u_1q_1^2+u_3q_2^2+m^2\right]^3}.
\end{aligned}$$

We have used here the identity (will be proven in the second part of the course QFT 2)

$$\frac{1}{p_1\cdots p_n}=(n-1)!\int_0^1 du_1\ldots du_n\frac{\delta(u_1+\ldots+u_n-1)}{[u_1A_1+\ldots+u_nA_n]^n}.$$

In a next step one commutes the integral over $u_1\ldots u_3$ with the integral over l.

Video: **Lecture26Video13.mp4**

Shifting momenta. It is useful to change integration variables according to

$$\begin{aligned}
l+u_1q_1-u_3q_2&\to k,\\
l&=k-u_1q_1+u_3q_2.
\end{aligned}$$

Collecting terms we find for the first diagram

$$(-1)ge^2\,\epsilon^*_\mu(q_1)\,\epsilon^*(q_2)\,2\int_0^1 du_1\cdots du_3\,\delta(u_1{+}u_2{+}u_3{-}1)\int\frac{d^4k}{(2\pi)^4}\frac{A^{\mu\nu}}{\left[k^2+u_1q_1^2+u_3q_2^2-(u_1q_1-u_3q_2)^2+m^2\right]^3},$$

where

$$\begin{aligned}A^{\mu\nu}=-4m\Big[&4k^\mu k^\nu-k^2\eta^{\mu\nu}+\text{terms linear in }k\\&+4(u_1q_1-u_3q_2)^\mu(u_1q_1-u_3q_2)^\nu-(u_1q_1-u_3q_2)^2\eta^{\mu\nu}\\&-q_1^\mu q_2^\nu+q_1^\nu q_2^\mu-(q_1\cdot q_2)\eta^{\mu\nu}-\eta^{\mu\nu}-\eta^{\mu\nu}m^2\Big].\end{aligned}$$

The integral over k is now symmetric around the origin.

Video: **Lecture26Video14.mp4**

A further cancelation. There is no contribution from linear terms in k and also the quadratic terms cancels. In fact, one can prove that

$$\lim_{d\to4}\int\frac{d^dk}{(2\pi)^d}\frac{4k^\mu k^\nu-(k^2+A)\eta^{\mu\nu}}{(k^2+A)^3}=0.$$

We will develop the techniques to prove this in QFT2.

Video: **Lecture26Video15.mp4**

Result so far. Taking this as well as $\epsilon^*_\mu(q_1)q_1^\mu=\epsilon^*_\nu(q_2)q_2^\nu=0$ and $q_1^2=q_2^2=0$ into account leads to

$$A^{\mu\nu}=-4m\left[1-4u_1u_2\right]\left[q_1^\mu q_2^\nu-(q_1\cdot q_2)\eta^{\mu\nu}\right].$$

Note that this is symmetric with respect to $(q_1,\mu)\leftrightarrow(q_2,\nu)$, so we can add the second diagram by multiplying with 2. We obtain

$$\begin{aligned}i\mathcal{T}=&8ge^2m\,\epsilon^*_\mu(q_1)\epsilon^*_\nu(q_2)\left[q_1^\nu q_2^\mu-(q_1\cdot q_2)\eta^{\mu\nu}\right]\\&\times2\int_0^1 du_1\cdots du_3\,\delta(u_1+u_2+u_3-1)[1-4u_1u_3]\int\frac{d^4k}{(2\pi)^4}\frac{1}{\left[k^2+2u_1u_3q_1\cdot q_2+m^2\right]^3}\end{aligned}$$

Video: **Lecture26Video16.mp4**

Momentum integral. To evaluate the integral over k we note that in the rest frame of the decaying scalar boson $p=q_1+q_2=(M,0,0,0)$ such that $p^2=2q_1\cdot q_2=-M^2$. If we concentrate on fermions that are very heavy such that $m\gg M$ we can expand in the term $u_1u_3q_1\cdot q_2$ in the integral over k. One finds to lowest order

$$\int\frac{d^4k}{(2\pi)^4}\frac{1}{[k^2+m^2]^3}=i\frac{1}{(4\pi)^2}\frac{1}{2m^2}.$$

This i is due to the Wick rotation $k^0=ik^0_E$.

Video: **Lecture26Video17.mp4**

Integral over Feynman parameters. Also the integral over Feynman parameters can now easily be performed

$$\begin{aligned}
&2\int_0^1 du_1\dots du_3\;\delta(u_1+u_2+u_3-1)[1-4u_1u_3]\\
&=2\int_0^1 du_1du_3\,\theta(1-u_1-u_3)\,[1-4u_1u_3]\\
&=2\int_0^1 du_1\int_0^{1-u_1} du_3\,[1-4u_1u_3]\\
&=2\int_0^1 du_1[(1-u_1)-4u_1\tfrac{1}{2}(1-u_1)^2]\\
&=2-3+\frac{8}{3}-1=\frac{2}{3}.
\end{aligned}$$

Collecting terms we find

$$i\mathcal{T}=i\frac{8ge^2}{3(4\pi)^2m}\;\epsilon^*_\mu(q_1)\;\epsilon^*_\nu(q_2)\,[q_1^\nu q_2^\mu-(q_1\cdot q_2)\eta^{\mu\nu}]\,.$$

Video: **Lecture26Video18.mp4**

Photon polarization sums and Ward identity. Before we continue we need to develop a method to perform the spin sums for photons. In the squared amplitude expressions like the following appear

$$\sum_{\text{polarizations}}|\mathcal{T}|^2=\sum_{\text{polarizations}}\epsilon^*_\mu(q)\epsilon_\nu(q)\mathcal{M}^\mu(q)\mathcal{M}^{\nu*}(q).$$

We have extended here the polarization vector of a photon from the amplitude by decomposing

$$\mathcal{T}=\epsilon^*_\mu(q)\mathcal{M}^\mu(q).$$

Let us choose without loss of generality $q^\mu=(E,0,0,E)$ and use the polarization vector introduced previously,

$$\epsilon^{(1)}_\mu=\left(0,\frac{1}{\sqrt{2}},-\frac{i}{\sqrt{2}},0\right),$$
$$\epsilon^{(2)}_\mu=\left(0,\frac{1}{\sqrt{2}},\frac{i}{\sqrt{2}},0\right),$$

such that

$$\epsilon^{*(1)}_\mu\;\epsilon^{(1)}_\nu+\epsilon^{*(2)}_\mu\;\epsilon^{(2)}_\nu=\begin{pmatrix}0&&&\\&1&&\\&&1&\\&&&0\end{pmatrix}.$$

This would give

$$\sum_{j=1}^{2}\epsilon^{*(j)}_\mu\;\epsilon^{(j)}_\nu\;\mathcal{M}^\mu\;\mathcal{M}^{*\nu}=|\mathcal{M}^1|^2+|\mathcal{M}^2|^2.$$

Video: **Lecture26Video19.mp4**

Ward identity. To simplify this one can use an identity we will prove later,

$$q_\mu \mathcal{M}^\mu(q) = 0.$$

This is in fact a consequence of gauge symmetry known as *Ward identity*. For the above choice of q^μ it follows

$$-\mathcal{M}^0 + \mathcal{M}^3 = 0$$

Accordingly, one can add $0 = -|\mathcal{M}^0|^2 + |\mathcal{M}^3|^2$ to the spin sum

$$\sum_{j=1}^{2} \epsilon_\mu^{*(j)}\, \epsilon_\nu^{(j)}\, \mathcal{M}^\mu\, \mathcal{M}^{*\nu} = -|\mathcal{M}^0|^2 + |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 + |\mathcal{M}^3|^2 = \eta_{\mu\nu}\mathcal{M}^\mu \mathcal{M}^{*\nu}.$$

In this sense we can use for external photons

$$\sum_{j=1}^{2} \epsilon_\mu^{*(j)}\, \epsilon_\nu^{(j)} \to \eta_{\mu\nu}.$$

Video: **Lecture26Video20.mp4**

Squared amplitude. With this we can now calculate the sums over final state photon polarizations

$$\begin{aligned}\sum_{\text{pol.}} |\mathcal{T}|^2 &= \left(\frac{8ge^2}{3\,(4\pi)^2\,m}\right)^2 \left[q_1^\nu q_2^\mu - (q_1\cdot q_2)\eta^{\mu\nu}\right]\left[q_1^\beta q_2^\alpha - (q_1\cdot q_2)\eta^{\alpha\beta}\right] \\ &\quad \times \sum_{\text{pol.}} \epsilon_\mu^*(q_1)\,\epsilon_\alpha(q_1) \sum_{\text{pol.}} \epsilon_\nu^*(q_2)\,\epsilon_\beta(q_2) \\ &= \left(\frac{8ge^2}{3\,(4\pi)^2\,m}\right)^2 2(q_1\cdot q_2)^2 = \frac{2g^2\alpha^2}{9\pi^2 m^2} M^4.\end{aligned}$$

In the last step we have used that the momentum of the incoming Higgs particle is $p = q_1 + q_2$. The square is given by the rest mass, $p^2 = -M^2 = 2(q_1 \cdot q_2)$. Here we also used that the photons are massless, $q_1^2 = q_2^2 = 0$. We also used the fine structure constant $\alpha = e^2/(4\pi)$.

Video: **Lecture26Video21.mp4**

Decay rate. For the differential particle decay rate $\varphi \to \gamma\gamma$ this gives in the rest frame of the Higgs particle with $|\vec{q}_1| = M/2$,

$$\frac{d\Gamma}{d\Omega} = \frac{|\vec{q}_1|}{32\pi^2 M^2} \sum_{\text{pol.}} |\mathcal{T}|^2 = \frac{g^2\alpha^2}{9\times 32\pi^4 m^2} M^3.$$

Finally, we integrate over solid angle $\Omega = (1/2)4\pi$ where the factor $(1/2)$ is due to the fact that the photons in the final state are indistinguishable. The decay rate for $\varphi \to \gamma\gamma$ through a heavy fermion loop is finally

$$\Gamma = \frac{g^2\alpha^2}{144\pi^3 m^2}\, M^3$$

Note that because of $g = m/v$ this is in fact independant of the heavy fermion mass m.