Remarks on 2PI formalisms and the fRG

Functional Renormalization

— from quantum gravity and dark energy to ultracold atoms and condensed matter

Heidelberg, March 10, 2017

Based on work done in collaboration with

J. Pawlowski and U. Reinosa

A paper to appear... soon !

Jean-Paul Blaizot, IPhT-Saclay

Motivations

Study (controlable ?) non perturbative methods in many-body physics and field theory

Two exact formulae

Thermodynamic potential as a functional of the propagator

$$\Omega[G] = \frac{1}{2}\mathrm{Tr}\log G^{-1} - \frac{1}{2}\mathrm{Tr}\Sigma G + \Phi[G]$$

Flow of the effective action

$$\partial_\kappa \hat{\Gamma}_\kappa[\phi] = \frac{1}{2}\int_q \partial_\kappa R_\kappa(q)\, G_\kappa(q,-q;\phi)$$

$$G_\kappa^{-1}(q,-q;\phi) = \Gamma_\kappa^{(2)}(q,-q;\phi) + R_\kappa(q)$$

These formulae are useful mostly for the approximations that they suggest

One can use one formalism to shed light on the other (this talk)

Present discussion limited to scalar field theory

(can be generalized)

$$S[\varphi] = \int d^d x \left\{ \frac{1}{2} (\partial\varphi(x))^2 + \frac{m^2}{2} \varphi^2(x) + \frac{\lambda}{4!} \varphi^4(x) \right\}$$

Some representative recent related works (not limited to scalar field)

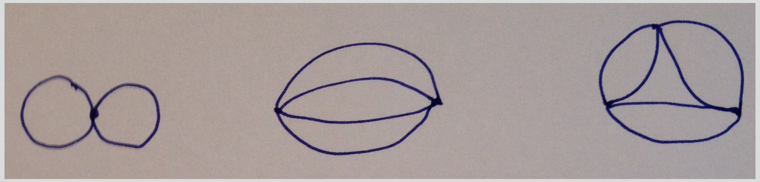
JPB, J. Pawlowski, U. Reinosa (2010)
M. Carrington et al (2014)
N. Dupuis (2013)
V. Meden et al (2016)

Basics of 2PI formalisms (1)

$$\Omega[G] = \tfrac{1}{2}\mathrm{Tr}\log G^{-1} - \tfrac{1}{2}\mathrm{Tr}\Sigma G + \Phi[G]$$

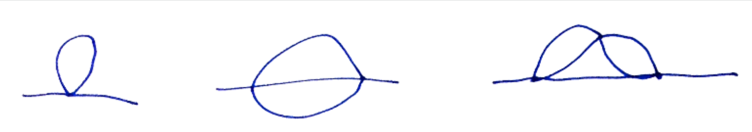
Luttinger-Ward functional

$$\Phi[G]$$



Self-energy

$$\Sigma(p) = 2\frac{\delta\Phi}{\delta G(p)}$$



Self-consistency condition

$$\Sigma[G] = G^{-1} - G_0^{-1}$$

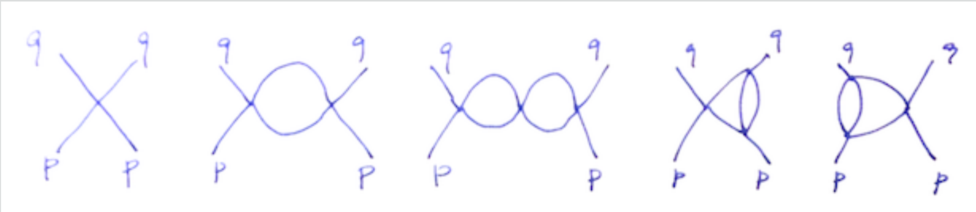
Stationarity property

$$\left.\frac{\delta\Omega[G]}{\delta G}\right|_{G_0} = 0$$

Basics of 2PI formalisms (2)

Irreducible kernel

$$\mathcal{I}(q,p) = 2\frac{\delta\Sigma(p)}{\delta G(q)} = 4\frac{\delta^2\Phi}{\delta G(q)\delta G(p)} = \mathcal{I}(p,q)$$



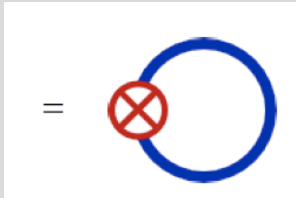
Bethe-Salpeter equation

$$\Gamma^{(4)}(q,p) = \mathcal{I}(q,p) - \frac{1}{2}\int_l \; \Gamma^{(4)}(q,l)\, G^2(l)\, \mathcal{I}(l,p)$$

Basics of functional RG

Flow equation (Wetterich)

$$\partial_\kappa \hat{\Gamma}_\kappa[\phi] = \frac{1}{2}\int_q \partial_\kappa R_\kappa(q)\, G_\kappa(q,-q;\phi)$$

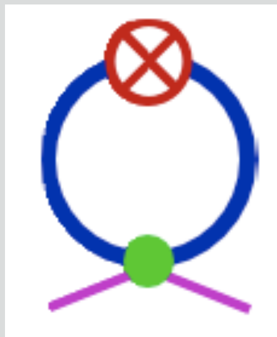


$$G_\kappa^{-1}(q,-q;\phi) = \Gamma_\kappa^{(2)}(q,-q;\phi) + R_\kappa(q)$$

Infinite hierarchy of coupled flow equations for the n-point functions

Equation for the 2-point function

$$\partial_\kappa \Gamma^{(2)}\kappa(p) = -\frac{1}{2}\int_q \partial_\kappa R_\kappa(q)\, G_\kappa^2(q)\, \Gamma_\kappa^{(4)}(q,p)$$



And so on…..

The theory in the presence of $R_\kappa(q)$

All formal relations between n-point functions hold for any κ

One can then take derivatives w.r.t. κ

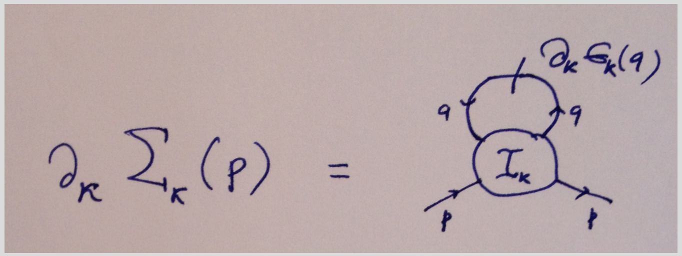
…. thereby obtaining flow equations

Equation for the 2-point function (or self-energy)

$$G_\kappa^{-1}(p) = p^2 + m^2 + \Sigma_\kappa(p) + R_\kappa(p)$$

$$\partial_\kappa \Sigma_\kappa(p) = 2\int_q \partial_\kappa G_\kappa(q)\, \left.\frac{\delta^2 \Phi[G]}{\delta G(q)\delta G(p)}\right|_{G_\kappa} = \frac{1}{2}\int_q \partial_\kappa G_\kappa(q)\, \mathcal{I}_\kappa(q,p)$$

This is NOT quite the usual flow equation



Solving the Bethe-Salpeter equation to get $\Gamma^{(4)}(q,p)$

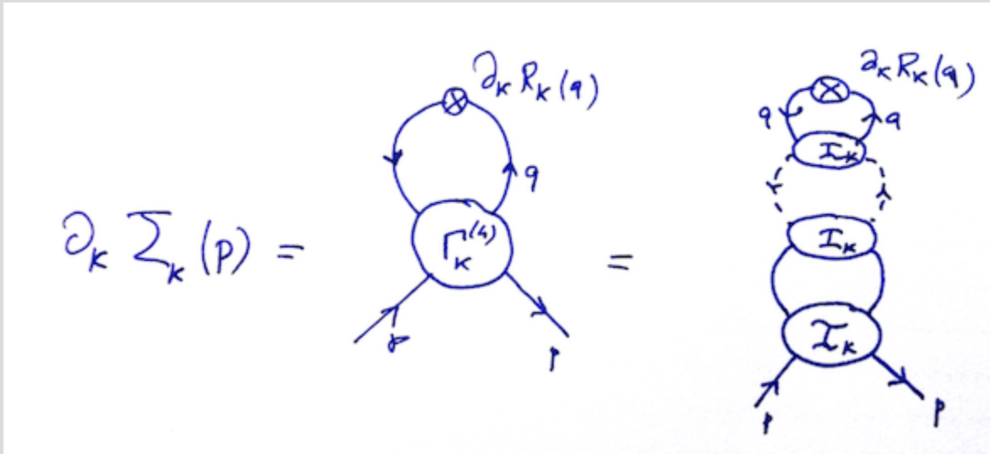
$$\Gamma^{(4)}_\kappa(q,p) = \mathcal{I}_\kappa(q,p) - \frac{1}{2}\int_l \Gamma^{(4)}_\kappa(q,l)\, G^2_\kappa(l)\, \mathcal{I}_\kappa(l,p)$$

and using this equation to eliminate $\mathcal{I}_\kappa(q,p)$

we are left with

$$\partial_\kappa \Sigma_\kappa(p) = -\frac{1}{2}\int_q \partial_\kappa R_\kappa(q)\, G^2_\kappa(q)\, \Gamma^{(4)}_\kappa(q,p)$$

The exact flow equation for the 2-point function



A possible truncation scheme (1)

Truncate the Luttinger-Ward functional (keeping selected skeletons)

Obtain the kernel $\mathcal{I}(q,p) = 4\frac{\delta^2\Phi}{\delta G(q)\delta G(p)}$

Then solve the coupled equations

$$\Gamma_\kappa^{(4)}(q,p) = \mathcal{I}_\kappa(q,p) - \frac{1}{2}\int_l \Gamma_\kappa^{(4)}(q,l)\, G_\kappa^2(l)\, \mathcal{I}_\kappa(l,p)$$

$$\partial_\kappa \Sigma_\kappa(p) = -\frac{1}{2}\int_q \partial_\kappa R_\kappa(q)\, G_\kappa^2(q)\, \Gamma_\kappa^{(4)}(q,p)$$

NB. i) The solution is independent of the choice of the "regulator"

ii) Not only a truncation of fRG, but an alternative to solving the 2PI equations

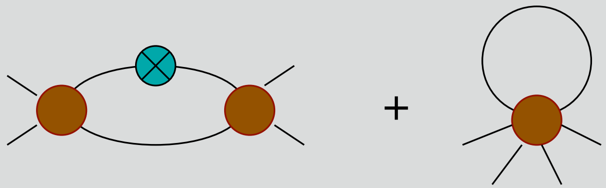
A possible truncation scheme (2)

Instead of solving the Bethe-Salpeter eqn., write a flow equation for the 4-point function

$$\begin{aligned}
\partial_\kappa \Gamma_\kappa^{(4)}(p,q) = \partial_\kappa \mathcal{I}_\kappa(p,q) \quad &- \quad \frac{1}{2}\int_l \Gamma_\kappa^{(4)}(p,l)\,\partial_\kappa G_\kappa^2(l)\,\Gamma_\kappa^{(4)}(l,q) \\
&- \quad \frac{1}{2}\int_l \partial_\kappa \mathcal{I}_\kappa(p,l)\,G_\kappa^2(l)\,\Gamma_\kappa^{(4)}(l,q) \\
&- \quad \frac{1}{2}\int_l \Gamma_\kappa^{(4)}(p,l)\,G_\kappa^2(l)\,\partial_\kappa \mathcal{I}_\kappa(l,q) \\
&+ \quad \frac{1}{4}\int_l\int_s \Gamma_\kappa^{(4)}(p,l)\,G_\kappa^2(l)\,\partial_\kappa \mathcal{I}_\kappa(l,s)\,G_\kappa^2(s)\,\Gamma_\kappa^{(4)}(s,q)
\end{aligned}$$

NB. This equation is NOT the "usual" flow equation for the 4-point function

$\partial_k \Gamma_k^{(4)}$ =



Renormalization issues

Not a priori obvious that the integrals are finite

$$\begin{aligned}\partial_\kappa \Gamma^{(4)}_\kappa(p,q) = \partial_\kappa \mathcal{I}_\kappa(p,q) \quad &- \quad \frac{1}{2}\int_l \Gamma^{(4)}_\kappa(p,l)\,\partial_\kappa G^2_\kappa(l)\,\Gamma^{(4)}_\kappa(l,q) \\ &- \quad \frac{1}{2}\int_l \partial_\kappa \mathcal{I}_\kappa(p,l)\,G^2_\kappa(l)\,\Gamma^{(4)}_\kappa(l,q) \\ &- \quad \frac{1}{2}\int_l \Gamma^{(4)}_\kappa(p,l)\,G^2_\kappa(l)\,\partial_\kappa \mathcal{I}_\kappa(l,q) \\ &+ \quad \frac{1}{4}\int_l\int_s \Gamma^{(4)}_\kappa(p,l)\,G^2_\kappa(l)\,\partial_\kappa \mathcal{I}_\kappa(l,s)\,G^2_\kappa(s)\,\Gamma^{(4)}_\kappa(s,q)\end{aligned}$$

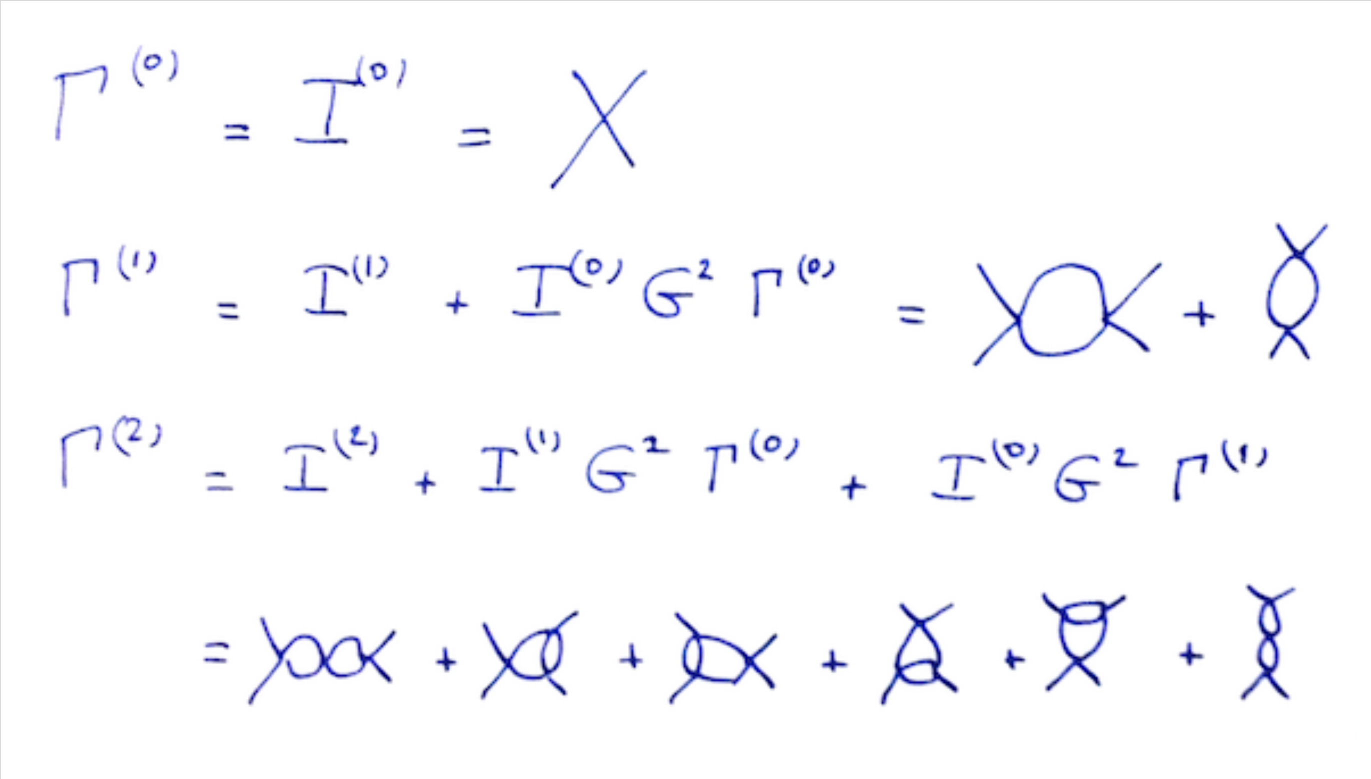
Standard lore in fRg: things become "simple" at the "cutoff scale"

$$\Gamma^{(2)}_\kappa(p) \equiv Z_\kappa p^2 + m^2_\kappa, \quad Z_\kappa \sim \ln\kappa, \quad m^2_\kappa \sim \kappa^2 \quad \Gamma^{(4)}_\kappa \sim \ln\kappa, \quad \Gamma^{(n>4)}_\kappa \sim \kappa^{4-n}$$

One expects of course similar features in the 2PI truncation…
…but working out the "details" turned out to be tricky

Divergences, and subdivergences.....

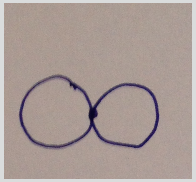
Consider the loop expansion of the 4-point function



In addition to counterterms needed to renormalise the kernel $\mathcal{I}$

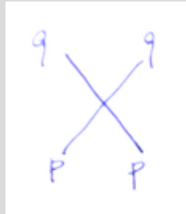
an infinite number of counterterms are needed to renormalise the BS equation….

A simple example (1)



$\Phi[G]$

$$\Sigma(p) = 2\frac{\delta\Phi}{\delta G(p)}$$



$\mathcal{I}_\kappa(q,p)$

Standard 2PI renormalization

Gap equation

$$\bar{\Sigma} = \delta m^2 + \frac{\lambda + \delta\lambda^{\rm BS}_{(0)}}{2}\int_q G(q) \qquad G(q) = \frac{1}{q^2 + m^2 + \bar{\Sigma}}$$

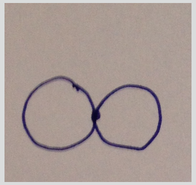
BS equation

$$\frac{1}{\Gamma^{\rm BS}_{(0)}} = \frac{1}{\lambda + \lambda^{\rm BS}_{(0)}} + \frac{1}{2}\int_q G^2(q)$$

Counterterms

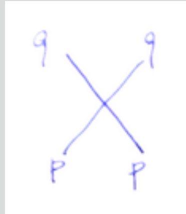
$$\delta\lambda^{\rm BS}_{(0)} = \lambda\frac{a\lambda}{1-a\lambda}, \qquad a \equiv \frac{1}{2}\int_q G^2(q) \qquad \frac{\delta m^2}{m^2} = \frac{a\lambda}{1-a\lambda} = \frac{\delta\lambda}{\lambda}$$

A simple example (2)



$$\Phi[G]$$

$$\Sigma(p) = 2\frac{\delta\Phi}{\delta G(p)}$$



$$\mathcal{J}_\kappa(q,p)$$

The two equations to be solved

$$\partial_\kappa m_\kappa^2 = -\tfrac{1}{2}\Gamma_\kappa \int_q (\partial_\kappa R_\kappa) G_\kappa^2(q) \qquad\qquad \partial_\kappa \Gamma_\kappa = -\frac{1}{2}\Gamma_\kappa^2 \int_q \partial_\kappa G_\kappa^2(q)$$

Solution

$$m_\kappa^2 = m^2 = m_\Lambda^2 + \frac{\Gamma_\Lambda}{2}\int_q \left\{ G_\kappa(q) - G_\Lambda(q) + (m_\kappa^2 - m_\Lambda^2) G_\Lambda^2(q) \right\}$$

Elimination of "subdivergences " is automatically taken care of by the coupled flow equations

Conclusions

- Two non perturbative methods were compared
- Approximation schemes exist where they completely match
- The comparison help to clarify some renormalisation issues in non perturbative schemes, such as 2PI
- Truncating the fRG flow equations with 2PI relations may be useful in some applications