Effective dissipation

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Effective dissipation

- Dissipation is **generation** of entropy
- von Neumann definition

$$S = -\mathrm{Tr}\rho \ln \rho$$

- Entropy measures information we have about a state
  - maximal information for pure state with $S = 0$
  - minimal information for thermal state $S = \text{max.}\big|_{E,\vec{p},N}$
- Unitary evolution **conserves** entropy!
- What information is really accessible and relevant?

Entanglement entropy

- Consider splitting of system into two parts $A + B$
- Reduced density matrix

$$\rho_A = \mathsf{Tr}_B\ \rho$$

- Entanglement entropy between A and B

$$S_A = -\mathsf{Tr}_A\ \rho_A \ln \rho_A$$

- Spatial splitting: entanglement entropy of ground state
- C-theorem & A-theorem

Dissipation and effective field theory

- What are the RG equations for the dissipative terms?
- Is there universality in the effective dissipative sector?
- What dissipative terms are relevant for dynamics close to (quantum) phase transitions?

Close-to-equilibrium situations

- out-of-equilibrium situations
- close-to-equilibrium: description by field expectation values and thermodynamic fields
- more complete description by following more fields explicitly
- example: Viscous fluid dynamics plus additional fields
- usually discussed in terms of
    - phenomenological constitutive relations
    - as a limit of kinetic theory
    - in AdS/CFT
- want non-perturbative formulation in terms of QFT concepts
- Analytic continuation as an alternative to Schwinger-Keldysh
- direct generalization of equilibrium formalism

Local equilibrium states

- Dissipation: energy and momentum get transferred to a heat bath
- Even if one starts with pure state $T = 0$ initially, dissipation will generate nonzero temperature
- Close-to-equilibrium situations: dissipation is local
- Convenient to use general coordinates with metric

$$g_{\mu\nu}(x)$$

- Need approximate **local** equilibrium description with temperature $T(x)$ and fluid velocity $u^\mu(x)$, will appear in combination

$$\beta^\mu(x) = \frac{u^\mu(x)}{T(x)}$$

- **Global** thermal equilibrium corresponds to β^μ Killing vector

$$\nabla_\mu \beta_\nu(x) + \nabla_\nu \beta_\mu(x) = 0$$

Local equilibrium

- Use similarity between local density matrix and translation operator

$$e^{\beta^\mu(x)\mathscr{P}_\mu} \qquad \longleftrightarrow \qquad e^{i\Delta x^\mu \mathscr{P}_\mu}$$

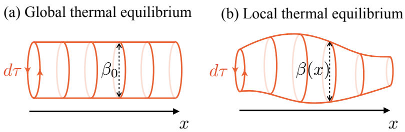
to represent partition function as functional integral with periodicity in imaginary direction such that

$$\phi(x^\mu - i\beta^\mu(x)) = \pm\phi(x^\mu)$$

- Partition function $Z[J]$, Schwinger functional $W[J]$ in Euclidean domain

$$Z[J] = e^{W_E[J]} = \int D\phi\, e^{-S_E[\phi] + \int_x J\phi}$$

- First defined on **Euclidean manifold** $\Sigma \times M$ at constant time
- Approximate local equilibrium at all times: Hypersurface Σ can be shifted



Effective action

- Defined in euclidean domain by Legendre transform

$$\Gamma_E[\Phi] = \int_x J_a(x)\Phi_a(x) - W_E[J]$$

with expectation values

$$\Phi_a(x) = \frac{1}{\sqrt{g}(x)} \frac{\delta}{\delta J_a(x)} W_E[J]$$

- Euclidean field equation

$$\frac{\delta}{\delta \Phi_a(x)} \Gamma_E[\Phi] = \sqrt{g}(x)\, J_a(x)$$

resembles classical equation of motion for $J = 0$.

- Need analytic continuation to obtain a viable equation of motion

Two-point functions

- Consider homogeneous background fields and global equilibrium

$$\beta^\mu = \left(\frac{1}{T}, 0, 0, 0\right)$$

- Propagator and inverse propagator

$$\frac{\delta^2}{\delta J_a(-p)\delta J_b(q)} W_E[J] = G_{ab}(i\omega_n, \mathbf{p})\ \delta(p-q)$$
$$\frac{\delta^2}{\delta \Phi_a(-p)\delta \Phi_b(q)} \Gamma_E[\Phi] = P_{ab}(i\omega_n, \mathbf{p})\ \delta(p-q)$$

- From definition of effective action

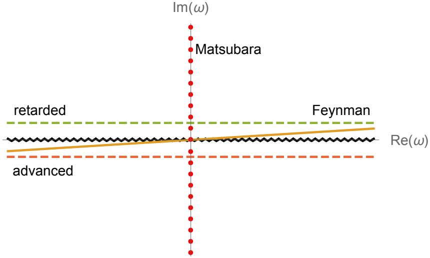
$$\sum_b G_{ab}(p) P_{bc}(p) = \delta_{ac}$$

Spectral representation

- Källen-Lehmann spectral representation

$$G_{ab}\left(\omega, \mathbf{p}\right) = \int_{-\infty}^{\infty} dz\, \frac{\rho_{ab}(z^2 - \mathbf{p}^2, z)}{z - \omega}$$

  with $\rho_{ab} \in \mathbb{R}$
- correlation functions can be analytically continued in $\omega = -u^\mu p_\mu$
- branch cut or poles on real frequency axis $\omega \in \mathbb{R}$ but nowhere else
- different propagators follow by evaluation of G_{ab} in different regions



$$\Delta^M_{ab}(p) = G_{ab}\left(i\omega_n, \mathbf{p}\right)$$
$$\Delta^R_{ab}(p) = G_{ab}\left(p^0 + i\epsilon, \mathbf{p}\right)$$
$$\Delta^A_{ab}(p) = G_{ab}\left(p^0 - i\epsilon, \mathbf{p}\right)$$
$$\Delta^F_{ab}(p) = G_{ab}\left(p^0 + i\epsilon\, \text{sign}\left(p^0\right), \mathbf{p}\right)$$

Inverse propagator

- spectral representation for G_{ab} implies that *inverse* propagator $P_{ab}(\omega, \mathbf{p})$
  - can have zero-crossings for $\omega = p^0 \in \mathbb{R}$
  - has in general branch-cut for $\omega = p^0 \in \mathbb{R}$
- so far reference frame with $u^\mu = (1, 0, 0, 0)$
- more general: analytic continuation with respect to

$$\omega = -u^\mu p_\mu$$

- use **decomposition**

$$P_{ab}(p) = P_{1,ab}(p) - i s_\mathsf{I}(-u^\mu p_\mu)\, P_{2,ab}(p)$$

with **sign function**

$$s_\mathsf{I}(\omega) = \mathsf{sign}(\mathsf{Im}\ \omega)$$

- both functions $P_{1,ab}(p)$ and $P_{2,ab}(p)$ are regular (no discontinuities)

Sign operator in position space

[Floerchinger, JHEP 1609 (2016) 099]

- In position space, **sign function** becomes **operator**

$$s_{\mathsf{I}}\left(-u^\mu p_\mu\right) = \text{sign}\left(\text{Im}(-u^\mu p_\mu)\right)$$
$$\rightarrow \text{sign}\left(\text{Im}\left(iu^\mu \tfrac{\partial}{\partial x^\mu}\right)\right) = \text{sign}\left(\text{Re}\left(u^\mu \tfrac{\partial}{\partial x^\mu}\right)\right) = s_{\mathsf{R}}\left(u^\mu \tfrac{\partial}{\partial x^\mu}\right)$$

- Geometric representation in terms of Lie derivative

$$s_{\mathsf{R}}(\mathcal{L}_u) \qquad \text{or} \qquad s_{\mathsf{R}}(\mathcal{L}_\beta)$$

- **Sign operator** appears also in analytically continued quantum effective action $\Gamma[\Phi]$

Analytically continued 1 PI effective action

[Floerchinger, JHEP 1609 (2016) 099]

- Analytically continued quantum effective action defined by analytic continuation of correlation functions
- Quadratic part

$$\Gamma_2[\Phi] = \frac{1}{2} \int_{x,y} \Phi_a(x) \left[P_{1,ab}(x-y) + P_{2,ab}(x-y) s_\mathsf{R} \left(u^\mu \frac{\partial}{\partial y^\mu} \right) \right] \Phi_b(y)$$

- Higher orders correlation functions less understood: no spectral representation
- Use inverse Hubbard-Stratonovich trick: terms quadratic in auxiliary field can be integrated out
- Allows to understand analytic structures of higher order terms

Equations of motion

- Can one obtain **causal** and **real** renormalized equations of motion from the 1 PI effective action?
- naively: time-ordered action / Feynman $i\epsilon$ prescription:

$$\frac{\delta}{\delta \Phi_a(x)} \Gamma_{\text{time ordered}}[\Phi] = \sqrt{g}\, J_a(x)$$

- This does not lead to causal and real equations of motion !
  [e.g. Calzetta & Hu: *Non-equilibrium Quantum Field Theory* (2008)]

Retarded functional derivative

[Floerchinger, JHEP 1609 (2016) 099]

- **Real** and **causal dissipative field equations** follow from analytically continued effective action

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi_a(x)}\bigg|_{\text{ret}} = \sqrt{g}J(x)$$

- to calculate retarded variational derivative determine

$$\delta\Gamma[\Phi]$$

  by varying the fields $\delta\Phi(x)$ including dissipative terms
- set signs according to

$$s_{\mathsf{R}}(u^\mu\partial_\mu)\;\delta\Phi(x) \to -\delta\Phi(x), \qquad\qquad \delta\Phi(x)\;s_{\mathsf{R}}(u^\mu\partial_\mu) \to +\delta\Phi(x)$$

- proceed as usual
- opposite choice of sign: field equations for backward time evolution
- Leads to causal equations of motion

Scalar field with $O(N)$ symmetry

- Consider effective action (with $\rho = \frac{1}{2}\varphi_j\varphi_j$)

$$\Gamma[\varphi, g_{\mu\nu}, \beta^\mu] = \int d^dx\sqrt{g}\Big\{\frac{1}{2}Z(\rho,T)g^{\mu\nu}\partial_\mu\varphi_j\partial_\nu\varphi_j + U(\rho,T) + \frac{1}{2}C(\rho,T)\;[\varphi_j, s_{\mathrm{R}}(u^\mu\partial_\mu)]\;\beta^\nu\partial_\nu\varphi_j\Big\}$$

- Variation at fixed metric $g_{\mu\nu}$ and β^μ gives

$$\begin{aligned}\delta\Gamma = \int d^dx\sqrt{g}\Big\{ & Z(\rho,T)g^{\mu\nu}\partial_\mu\delta\varphi_j\partial_\nu\varphi_j + \frac{1}{2}Z'(\rho,T)\varphi_m\delta\varphi_m\; g^{\mu\nu}\partial_\mu\varphi_j\partial_\nu\varphi_j \\ & + U'(\rho,T)\varphi_m\delta\varphi_m \\ & + \frac{1}{2}C(\rho,T)\;[\delta\varphi_j, s_{\mathrm{R}}(u^\mu\partial_\mu)]\,\beta^\nu\partial_\nu\varphi_j \\ & + \frac{1}{2}C(\rho,T)\;[\varphi_j, s_{\mathrm{R}}(u^\mu\partial_\mu)]\,\beta^\nu\partial_\nu\delta\varphi_j \\ & + \frac{1}{2}C'(\rho,T)\varphi_m\delta\varphi_m\;[\varphi_j, s_{\mathrm{R}}(u^\mu\partial_\mu)]\,\beta^\nu\partial_\nu\varphi_j\Big\}\end{aligned}$$

- set now $\delta\varphi_j\; s_{\mathrm{R}}(u^\mu\partial_\mu) \to \delta\varphi_j$ and $s_{\mathrm{R}}(u^\mu\partial_\mu)\,\delta\varphi_j \to -\delta\varphi_j$

Scalar field with $O(N)$ symmetry

- Field equation becomes

$$-\nabla_\mu \left[Z(\rho,T)\partial^\mu\varphi_j\right] + \frac{1}{2}Z'(\rho,T)\varphi_j\partial_\mu\varphi_m\partial^\mu\varphi_m$$
$$+U'(\rho,T)\varphi_j + C(\rho,T)\beta^\mu\partial_\mu\varphi_j = 0$$

- Generalized Klein-Gordon equation with additional damping term

Where do energy & momentum go?

- Modified variational principle leads to equations of motion with dissipation.
- But what happens to the dissipated energy and momentum?
- And other conserved quantum numbers?
- What about entropy production?

Energy-momentum tensor expectation value

- Analogous to field equation, obtain by retarded variation

$$\left.\frac{\delta\Gamma[\Phi, g_{\mu\nu}, \beta^\mu]}{\delta g_{\mu\nu}(x)}\right|_{\text{ret}} = -\frac{1}{2}\sqrt{g}\,\langle T^{\mu\nu}(x)\rangle$$

- Leads to Einstein's field equation when $\Gamma[\Phi, g_{\mu\nu}, \beta^\mu]$ contains Einstein-Hilbert term
- Useful to decompose

$$\Gamma[\Phi, g_{\mu\nu}, \beta^\mu] = \Gamma_R[\Phi, g_{\mu\nu}, \beta^\mu] + \Gamma_D[\Phi, g_{\mu\nu}, \beta^\mu]$$

  where reduced action Γ_R contains no dissipative / discontinuous terms and Γ_D only dissipative terms
- Energy-momentum tensor has two parts

$$\langle T^{\mu\nu}\rangle = (\bar{T}_R)^{\mu\nu} + (\bar{T}_D)^{\mu\nu}$$

General covariance

- Infinitesimal general coordinate transformations as a "gauge transformation" of the metric

$$\delta g^G_{\mu\nu}(x) = g_{\mu\lambda}(x)\frac{\partial \epsilon^\lambda(x)}{\partial x^\nu} + g_{\nu\lambda}(x)\frac{\partial \epsilon^\lambda(x)}{\partial x^\mu} + \frac{\partial g_{\mu\nu}(x)}{\partial x^\lambda}\epsilon^\lambda(x)$$

- Temperature / fluid velocity field transforms as vector

$$\delta \beta^\mu_G(x) = -\beta^\nu(x)\frac{\partial \epsilon^\mu(x)}{\partial x^\nu} + \frac{\partial \beta^\mu(x)}{\partial x^\nu}\epsilon^\nu(x)$$

- Also fields Φ_a transform in some representation, e. g. as scalars

$$\delta \Phi^G_a(x) = \epsilon^\lambda(x)\frac{\partial}{\partial x^\lambda}\Phi_a(x)$$

- Reduced action is invariant

$$\Gamma_R[\Phi + \delta\Phi^G, g_{\mu\nu} + \delta g^G_{\mu\nu}, \beta^\mu + \beta^\mu_G] = \Gamma_R[\Phi, g_{\mu\nu}, \beta^\mu]$$

Situation without dissipation

- Consider first situation **without dissipation** $\Gamma[\Phi, g_{\mu\nu}, \beta^\mu] = \Gamma_R[\Phi, g_{\mu\nu}]$
- Field equation implies (for $J = 0$)

$$\frac{\delta}{\delta\Phi_a(x)}\Gamma_R[\Phi, g_{\mu\nu}] = 0$$

- Gauge variation of the metric

$$\delta\Gamma_R = \int d^dx\sqrt{g}\,\epsilon^\lambda(x)\nabla_\mu\langle T^\mu{}_\lambda(x)\rangle$$

- General covariance $\delta\Gamma_R = 0$ and field equations imply covariant energy-momentum conservation

$$\nabla_\mu\,\langle T^\mu{}_\lambda(x)\rangle = 0$$

Situation with dissipation

[Floerchinger, JHEP 1609 (2016) 099]

- Consider now situation **with dissipation**. General covariance of Γ_R:

$$\delta\Gamma_R = \int d^dx \left\{ \frac{\delta\Gamma_R}{\delta\Phi_a}\delta\Phi_a^G + \sqrt{g}\,\epsilon^\lambda \nabla_\mu (\bar{T}_R)^\mu{}_\lambda + \frac{\delta\Gamma_R}{\delta\beta^\mu}\delta\beta^\mu_G \right\} = 0$$

- Reduced action **not stationary** with respect to field variations

$$\frac{\delta\Gamma_R}{\delta\Phi_a(x)} = -\frac{\delta\Gamma_D}{\delta\Phi_a(x)}\bigg|_{\text{ret}} =: -\sqrt{g}(x)\, M_a(x)$$

- Reduced energy-momentum tensor **not conserved**

$$\nabla_\mu (\bar{T}_R)^\mu{}_\lambda(x) = -\nabla_\mu (\bar{T}_D)^\mu{}_\lambda(x)$$

- Dependence on $\beta^\mu(x)$ **cannot be dropped**

$$\frac{\delta\Gamma_R}{\delta\beta^\mu(x)} =: \sqrt{g}(x)\, K_\mu(x)$$

- General covariance implies **four additional differential equations** that determine β^μ

$$M_a\partial_\lambda\Phi_a + \nabla_\mu (\bar{T}_D)^\mu{}_\lambda = \nabla_\mu \left[\beta^\mu K_\lambda\right] + K_\mu \nabla_\lambda \beta^\mu$$

Entropy production

[Floerchinger, JHEP 1609 (2016) 099]

- Contraction of previous equation with β^λ gives

$$M_a \beta^\lambda \partial_\lambda \Phi_a + \beta^\lambda \nabla_\mu (\bar{T}_D)^\mu{}_\lambda = \nabla_\mu \left[\beta^\mu \beta^\lambda K_\lambda \right]$$

- Consider special case

$$\sqrt{g}\, K_\mu(x) = \frac{\delta \Gamma_R}{\delta \beta^\mu(x)} = \frac{\delta}{\delta \beta^\mu(x)} \int d^d x \sqrt{g}\, U(T)$$

with grand canonical potential density $U(T) = -p(T)$ and temperature

$$T = \frac{1}{\sqrt{-g_{\mu\nu} \beta^\mu \beta^\nu}}$$

- Using $s = \partial p / \partial T$ gives entropy current

$$\beta^\mu \beta^\lambda K_\lambda = s^\mu = s u^\mu$$

- Local form of **second law of thermodynamics**

$$\nabla_\mu s^\mu = M_a \beta^\lambda \partial_\lambda \Phi_a + \beta^\lambda \nabla_\mu (\bar{T}_D)^\mu{}_\lambda \geq 0$$

Could dissipation affect the cosmological expansion ?

Backreaction: General idea

- for $0+1$ dimensional, non-linear dynamics

$$\dot{\varphi} = f(\varphi) = f_0 + f_1\,\varphi + \tfrac{1}{2} f_2\,\varphi^2 + \ldots$$

- one has for expectation values $\bar{\varphi} = \langle \varphi \rangle$

$$\dot{\bar{\varphi}} = f_0 + f_1\,\bar{\varphi} + \tfrac{1}{2} f_2\,\bar{\varphi}^2 + \tfrac{1}{2} f_2\,\langle (\varphi - \bar{\varphi})^2 \rangle + \ldots$$

- evolution equation for expectation value $\bar{\varphi}$ depends on two-point correlation function or spectrum $P_2 = \langle (\varphi - \bar{\varphi})^2 \rangle$
- evolution equation for spectrum depends on bispectrum and so on
- more complicated for higher dimensional theories
- more complicated for gauge theories such as gravity

Backreaction in gravity

- Einstein's equations are non-linear.
- Important question [G. F. R. Ellis (1984)]: If Einstein's field equations describe small scales, including inhomogeneities, do they also hold on large scales?
- Is there a sizable backreaction from inhomogeneities to the cosmological expansion?
- Difficult question, has been studied by many people [Ellis & Stoeger (1987); Mukhanov, Abramo & Brandenberger (1997); Unruh (1998); Buchert (2000); Geshnzjani & Brandenberger (2002); Schwarz (2002); Wetterich (2003); Räsänen (2004); Kolb, Matarrese & Riotto (2006); Brown, Behrend, Malik (2009); Gasperini, Marozzi & Veneziano (2009); Clarkson & Umeh (2011); Green & Wald (2011); ...]
- Recent reviews: [Buchert & Räsänen, Ann. Rev. Nucl. Part. Sci. 62, 57 (2012); Green & Wald, Class. Quant. Grav. 31, 234003 (2014)]
- No general consensus but most people believe now that **gravitational backreaction is rather small**.
- In the following we look at a new **backreaction on the matter side** of Einstein's equations.

Fluid equation for energy density

First order viscous fluid dynamics

$$u^\mu \partial_\mu \epsilon + (\epsilon + p)\nabla_\mu u^\mu - \zeta \Theta^2 - 2\eta \sigma^{\mu\nu}\sigma_{\mu\nu} = 0$$

For $\vec{v}^2 \ll c^2$ and Newtonian potentials $\Phi, \Psi \ll 1$

$$\dot{\epsilon} + \vec{v}\cdot\vec{\nabla}\epsilon + (\epsilon + p)\left(3\tfrac{\dot{a}}{a} + \vec{\nabla}\cdot\vec{v}\right)$$
$$= \tfrac{\zeta}{a}\left[3\tfrac{\dot{a}}{a} + \vec{\nabla}\cdot\vec{v}\right]^2 + \tfrac{\eta}{a}\left[\partial_i v_j \partial_i v_j + \partial_i v_j \partial_j v_i - \tfrac{2}{3}(\vec{\nabla}\cdot\vec{v})^2\right]$$

Fluid dynamic backreaction in Cosmology

[Floerchinger, Tetradis & Wiedemann, PRL 114, 091301 (2015)]

Expectation value of energy density $\bar{\epsilon} = \langle \epsilon \rangle$

$$\frac{1}{a}\dot{\bar{\epsilon}} + 3H\left(\bar{\epsilon} + \bar{p} - 3\bar{\zeta}H\right) = D$$

with dissipative backreaction term

$$D = \frac{1}{a^2}\langle \eta \left[\partial_i v_j \partial_i v_j + \partial_i v_j \partial_j v_i - \frac{2}{3}\partial_i v_i \partial_j v_j\right]\rangle + \frac{1}{a^2}\langle \zeta [\vec{\nabla}\cdot\vec{v}]^2\rangle + \frac{1}{a}\langle \vec{v}\cdot\vec{\nabla}\left(p - 6\zeta H\right)\rangle$$

- D vanishes for unperturbed homogeneous and isotropic universe
- D has contribution from shear & bulk viscous dissipation and thermodynamic work done by contraction against pressure gradients
- dissipative terms in D are positive semi-definite
- for spatially constant viscosities and scalar perturbations only

$$D = \frac{\bar{\zeta} + \frac{4}{3}\bar{\eta}}{a^2}\int d^3q \; P_{\theta\theta}(q)$$

Dissipation of perturbations

[Floerchinger, Tetradis & Wiedemann, PRL 114, 091301 (2015)]

- Dissipative backreaction does not need negative effective pressure

$$\frac{1}{a}\dot{\bar{\epsilon}} + 3H\left(\bar{\epsilon} + \bar{p}_{\text{eff}}\right) = D$$

- D is an integral over perturbations, could become large at late times.
- Can it potentially accelerate the universe?
- Need additional equation for scale parameter a
- Use trace of Einstein's equations $R = 8\pi G_{\text{N}} T^{\mu}{}_{\mu}$

$$\frac{1}{a}\dot{H} + 2H^2 = \frac{4\pi G_{\text{N}}}{3}\left(\bar{\epsilon} - 3\bar{p}_{\text{eff}}\right)$$

  does not depend on unknown quantities like $\langle(\epsilon + p_{\text{eff}})u^{\mu}u^{\nu}\rangle$
- To close the equations one needs equation of state $\bar{p}_{\text{eff}} = \bar{p}_{\text{eff}}(\bar{\epsilon})$ and dissipation parameter D

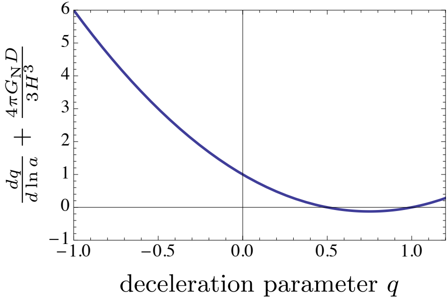
Deceleration parameter

[Floerchinger, Tetradis & Wiedemann, PRL 114, 091301 (2015)]

- assume now vanishing effective pressure $\bar{p}_{\text{eff}} = 0$
- obtain for deceleration parameter $q = -1 - \frac{\dot{H}}{aH^2}$

$$-\frac{dq}{d\ln a} + 2(q-1)\left(q - \frac{1}{2}\right) = \frac{4\pi G_{\mathrm{N}} D}{3H^3}$$

- for $D = 0$ attractive fixed point at $q_* = \frac{1}{2}$ (deceleration)
- for $D > 0$ fixed point shifted towards $q_* < 0$ (acceleration)



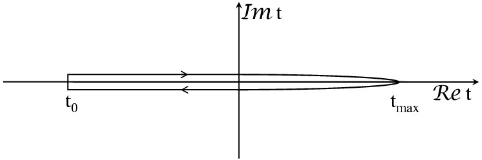
Conclusions

- Effective dissipation can arise in quantum field theories due to effective loss of information.
- Equations of motion for close-to-equilibrium theories can be obtained from analytic continuation.
- General covariance and energy-momentum conservation lead to equations for fluid velocity and entropy production.
- Local form of second law of thermodynamics is implemented on the level of the effective action $\Gamma[\Phi]$.
- Interesting applications in cosmology and condensed matter physics.

Backup

Double time path formalism

- formalism for general, far-from-equilibrium situations: Schwinger-Keldysh double time path
- can be formulated with two fields $\Phi = \frac{1}{2}(\phi_+ + \phi_-)$, $\chi = \phi_+ - \phi_-$
- in principle for arbitrary initial density matrices, in praxis mainly Gaussian initial states
- allows to treat also dissipation
- useful also to treat initial state fluctuations or forced noise in classical statistical theories
- difficult to recover thermal equilibrium, in particular non-perturbatively
- formalism algebraically somewhat involved



Causality

[Floerchinger, JHEP 1609 (2016) 099]

- consider derivative of field equation (in flat space with $\sqrt{g} = 1$)

$$\frac{\delta}{\delta \Phi_b(y)} \frac{\delta \Gamma}{\delta \Phi_a(x)} \bigg|_{\text{ret}} = \frac{\delta}{\delta \Phi_b(y)} J_a(x)$$

- inverting this equation gives retarded Green's function

$$\frac{\delta}{\delta J_b(y)} \Phi_a(x) = \Delta_{ab}^R(x, y)$$

- only non-zero for x future or null to y
- **Causality**: Field expectation value $\Phi_a(x)$ can only be influenced by the source $J_b(y)$ in or on the past light cone ✓

Energy-momentum tensor for scalar field

- Analytic action

$$\Gamma[\varphi, g_{\mu\nu}, \beta^\mu] = \int d^d x \sqrt{g} \Big\{ \frac{1}{2} Z(\rho, T) g^{\mu\nu} \partial_\mu \varphi_j \partial_\nu \varphi_j + U(\rho, T) + \frac{1}{2} C(\rho, T)\ [\varphi_j, s_{\mathsf{R}}(u^\mu \partial_\mu)]\ \beta^\nu \partial_\nu \varphi_j \Big\}$$

- Energy-momentum tensor

$$\langle T^{\mu\nu}(x) \rangle = Z(\rho, T) \partial^\mu \varphi_j \partial^\nu \varphi_j - \left(g^{\mu\nu} + u^\mu u^\nu T \frac{\partial}{\partial T} \right) \left\{ \frac{1}{2} Z(\rho, T) g^{\mu\nu} \partial_\mu \varphi_j \partial_\nu \varphi_j + U(\rho, T) \right\}$$

- Generalizes $T^{\mu\nu}$ for scalar field and $T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + g^{\mu\nu} p$ for ideal fluid with pressure $p = -U$ and enthalpy density $\epsilon + p = sT = -T \frac{\partial}{\partial T} U$.
- General covariance and covariant conservation law imply

$$\nabla_\mu \langle T^{\mu\nu}(x) \rangle = 0 \quad \Longrightarrow \quad \text{Differential eqs. for } \beta^\mu(x)$$

Entropy production for scalar field

- Entropy current

$$s^\mu = \beta^\mu \beta^\lambda K_\lambda = -\beta^\mu T \frac{\partial}{\partial T} \left\{ \frac{1}{2} Z(\rho, T) g^{\alpha\beta} \partial_\alpha \varphi_j \partial_\beta \varphi_j + U(\rho, T) \right\}$$

- Generalized entropy density

$$s_G = -\frac{\partial}{\partial T} \left\{ \frac{1}{2} Z(\rho, T) g^{\alpha\beta} \partial_\alpha \varphi_j \partial_\beta \varphi_j + U(\rho, T) \right\}$$

- Entropy generation positive semi-definite for $C(\rho, T) \geq 0$

$$\nabla_\mu s^\mu = C(\rho, T) \left(\beta^\mu \partial_\mu \varphi_j \right) \left(\beta^\nu \partial_\nu \varphi_j \right) \geq 0$$

- For fluid at rest $u^\mu = (1, 0, 0, 0)$

$$\nabla_\mu s^\mu = \dot{s}_G = \frac{C(\rho, T)}{T^2} \dot{\varphi}_j \dot{\varphi}_j$$

entropy increases when φ_j oscillates. Example: Reheating after inflation.

Damped harmonic oscillator 1

- Equation of motion

$$m\ddot{x} + c\dot{x} + kx = 0$$

or

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0$$

with $\omega_0 = \sqrt{k/m}$ and $\zeta = c/\sqrt{4mk}$

- What is action for damped oscillator? This does *not* work:

$$\int \frac{d\omega}{2\pi}\ \frac{m}{2} x^*(\omega)\left[\omega^2 + 2i\,\omega\,\zeta\omega_0 - \omega_0^2\right] x(\omega)$$

- Consider inverse propagator

$$\omega^2 + 2i\,s_{\mathsf{I}}(\omega)\,\omega\,\zeta\omega_0 - \omega_0^2$$

with

$$s_{\mathsf{I}}(\omega) = \mathsf{sign}\left(\mathsf{Im}\,\omega\right)$$

zero crossings (poles in the eff. propagator) are broadened to branch cut

Damped harmonic oscillator 2

- Take for effective action

$$\Gamma[x] = \int \frac{d\omega}{2\pi}\ \frac{m}{2} x^*(\omega) \left[-\omega^2 - 2i\, s_{\mathsf{I}}(\omega)\, \omega\, \zeta\omega_0 + \omega_0^2\right] x(\omega)$$
$$= \int dt \left\{-\frac{1}{2} m\dot{x}^2 + \frac{1}{2} c\, x\, s_{\mathsf{R}}(\partial_t)\dot{x} + \frac{1}{2} kx^2\right\}$$

where the second line uses

$$s_{\mathsf{I}}(\omega) = \mathsf{sign}(\mathsf{Im}\,\omega) \to \mathsf{sign}(\mathsf{Im}\, i\partial_t) = \mathsf{sign}(\mathsf{Re}\,\partial_t) = s_{\mathsf{R}}(\partial_t)$$

- Variation gives up to boundary terms

$$\delta\Gamma = \int dt \left\{m\ddot{x}\,\delta x + \frac{1}{2} c\,\delta x\, s_{\mathsf{R}}(\partial_t)\dot{x} - \frac{1}{2} c\,\dot{x}\, s_{\mathsf{R}}(\partial_t)\delta x + kx\,\delta x\right\}$$

Set now $s_{\mathsf{R}}(\partial_t)\delta x \to -\delta x$ and $\delta x\, s_{\mathsf{R}}(\partial_t) \to \delta x$. Defines $\frac{\delta\Gamma}{\delta x}\big|_{\mathsf{ret}}$.

- Equation of motion for forward time evolution

$$\frac{\delta\Gamma}{\delta x}\bigg|_{\mathsf{ret}} = m\ddot{x} + c\dot{x} + kx = 0$$

Ideal fluid

- Consider effective action

$$\Gamma[g_{\mu\nu}, \beta^\mu] = \Gamma_R[g_{\mu\nu}, \beta^\mu] = \int d^d x \sqrt{g}\, U(T)$$

with effective potential $U(T) = -p(T)$ and temperature

$$T = \frac{1}{\sqrt{-g_{\mu\nu}\beta^\mu \beta^\nu}}$$

- Variation of $g_{\mu\nu}$ at fixed β^μ leads to

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p g^{\mu\nu}$$

where $\epsilon + p = Ts = T\frac{\partial}{\partial T}p$ is the enthalpy density

- Describes ideal fluid. General covariance of covariant conservation $\nabla_\mu T^{\mu\nu} = 0$ leads to ideal fluid equations

$$u^\mu \partial_\mu \epsilon + (\epsilon + p)\nabla_\mu u^\mu = 0,$$
$$(\epsilon + p)u^\mu \nabla_\mu u^\nu + \Delta^{\nu\mu}\partial_\mu p = 0.$$

Viscous fluid

- Analytic action

$$\Gamma[g_{\mu\nu},\beta^\mu] = \int_x \left\{ U(T) + \frac{1}{4}\left[g_{\mu\nu}, s_{\mathsf{R}}(\mathcal{L}_u)\right]\left(2\eta(T)\sigma^{\mu\nu} + \zeta(T)\Delta^{\mu\nu}\nabla_\rho u^\rho\right) \right\}$$

with projector

$$\Delta^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}$$

and

$$\sigma^{\mu\nu} = \left(\frac{1}{2}\Delta^{\mu\alpha}\Delta^{\mu\beta} + \frac{1}{2}\Delta^{\mu\beta}\Delta^{\mu\alpha} - \frac{1}{d-1}\Delta^{\mu\nu}\Delta^{\alpha\beta}\right)\nabla_\alpha u_\beta$$

leads to

$$\langle T^{\mu\nu}\rangle = -\tfrac{2}{\sqrt{g}}\tfrac{\delta\Gamma[g_{\mu\nu},\beta^\mu]}{\delta g_{\mu\nu}}\Big|_{\mathsf{ret}} = (\epsilon+p)u^\mu u^\nu + pg^{\mu\nu} - 2\eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\nabla_\rho u^\rho$$

- Describes viscous fluid with shear viscosity $\eta(T)$ and bulk viscosity $\zeta(T)$
- Entropy production

$$\nabla_\mu s^\mu = \frac{1}{T}\left[2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta(\nabla_\rho u^\rho)^2\right]$$

Equations of motion from the Feynman action ?

- Consider damped harmonic oscillator as example. Time-ordered or Feynman action is obtained from analytic action by replacing $s_\mathsf{I}(\omega) \to \mathsf{sign}(\omega)$

$$\Gamma_{\text{time ordered}}[x] = \int \frac{d\omega}{2\pi}\,\frac{m}{2} x^*(\omega) \left[-\omega^2 - 2i|\omega|\,\zeta\omega_0 + \omega_0^2\right] x(\omega)$$

- Field equation $\frac{\delta}{\delta x(t)}\Gamma_{\text{time ordered}}[x] = J(t)$ would give

$$\left[-\omega^2 - 2i|\omega|\,\zeta\omega_0 + \omega_0^2\right] x(\omega) = J(\omega)$$

- Violates reality constraint $x^*(\omega) = x(-\omega)$ for $J^*(\omega) = J(-\omega)$
- Solution not causal

$$x(t) = \int_{t'} \Delta_F(t-t')J(t')$$

  because Feynman propagator $\Delta_F(t-t')$ not causal.
- In contrast, retarded variation of analytic action leads to real and causal equation of motion

Tree-like structures

- Discontinuous terms in analytic action could be of the form

$$\Gamma_{\mathsf{Disc}}[\Phi] = \int d^dx\sqrt{g}\ \{f[\Phi](x)\ s_{\mathsf{R}}\big(u^\mu(x)\tfrac{\partial}{\partial x^\mu}\big)\ g[\Phi](x)\}$$

- More general, tree-like structure are possible such as

$$\Gamma_{\mathsf{Disc}}[\Phi] = \int_{x,y}\ \Big\{f[\Phi](x)\ s_{\mathsf{R}}\big(u^\mu(x)\tfrac{\partial}{\partial x^\mu}\big)\ g[\Phi](x,y)\ s_{\mathsf{R}}\big(u^\mu(y)\tfrac{\partial}{\partial y^\mu}\big)\ h[\Phi](y)\Big\}$$

or

$$\Gamma_{\mathsf{Disc}}[\Phi] = \int_{x,y,z}\ \Big\{f[\Phi](x)\ s_{\mathsf{R}}\big(u^\mu(x)\tfrac{\partial}{\partial x^\mu}\big)\ g[\Phi](x,y,z)\ s_{\mathsf{R}}\big(u^\mu(y)\tfrac{\partial}{\partial y^\mu}\big)\ h[\Phi](y)$$
$$\times\ s_{\mathsf{R}}\big(u^\mu(z)\tfrac{\partial}{\partial z^\mu}\big)\ j[\Phi](z)\Big\}$$

- For retarded variation calculate $\delta\Gamma$ and set $s_{\mathsf{R}}(u^\mu\partial_\mu) \to -1$ if derivative operator points towards node that is varied and $s_{\mathsf{R}}(u^\mu\partial_\mu) \to 1$ if derivative operator points in opposite direction

Analytic continuation of FRG equations

[Floerchinger, JHEP 1205 (2012) 021]

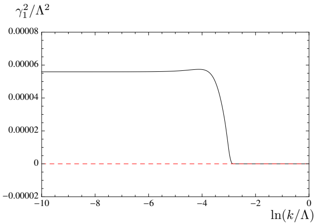
- Consider a point $p_0^2 - \vec{p}^2 = m^2$ where $P_1(m^2) = 0$.
- One can expand around this point

$$P_1 = Z(-p_0^2 + \vec{p}^2 + m^2) + \cdots$$
$$P_2 = Z\gamma^2 + \cdots$$

- Leads to Breit-Wigner form of propagator (with $\gamma^2 = m\Gamma$)

$$G(p) = \frac{1}{Z} \frac{-p_0^2 + \vec{p}^2 + m^2 + i\, s(p_0)\, m\Gamma}{(-p_0^2 + \vec{p}^2 + m^2)^2 + m^2\Gamma^2}.$$

- A few flowing parameters describe efficiently the singular structure of the propagator.



Truncation for relativistic scalar $O(N)$ theory

$$\Gamma_k = \int_{t,\vec{x}} \left\{ \sum_{j=1}^{N} \frac{1}{2} \bar{\phi}_j \, \bar{P}_\phi(i\partial_t, -i\vec{\nabla}) \, \bar{\phi}_j \right.$$
$$\left. + \frac{1}{4} \bar{\rho} \, \bar{P}_\rho(i\partial_t, -i\vec{\nabla}) \, \bar{\rho} + \bar{U}_k(\bar{\rho}) \right\}$$

with $\bar{\rho} = \frac{1}{2} \sum_{j=1}^{N} \bar{\phi}_j^2$.

- Goldstone propagator massless, expanded around $p_0 - \vec{p}^2 = 0$

$$\bar{P}_\phi(p_0, \vec{p}) \approx \bar{Z}_\phi \, (-p_0^2 + \vec{p}^2)$$

- Radial mode is massive, expanded around $p_0^2 - \vec{p}^2 = m_1^2$

$$\bar{P}_\phi(p_0, \vec{p}) + \bar{\rho}_0 \bar{P}_\rho(p_0, \vec{p}) + \bar{U}_k' + 2\bar{\rho}\bar{U}_k''$$
$$\approx \bar{Z}_\phi Z_1 \left[(-p_0^2 + \vec{p}^2 + m_1^2) - is(p_0) \, \gamma_1^2 \right]$$

Flow of the effective potential

$$\partial_t U_k(\rho)\big|_{\bar\rho} = \frac{1}{2}\int_{p_0=i\omega_n,\vec p}\left\{\frac{(N-1)}{\vec p^2 - p_0^2 + U' + \frac{1}{\bar Z_\phi}R_k}\right.$$
$$\left. + \frac{1}{Z_1\left[(\vec p^2 - p_0^2) - i\,s(p_0)\gamma_1^2\right] + U' + 2\rho U'' + \frac{1}{\bar Z_\phi}R_k}\right\}\frac{1}{\bar Z_\phi}\partial_t R_k.$$

- Summation over Matsubara frequencies $p_0 = i2\pi Tn$ can be done using contour integrals.
- Radial mode has non-zero decay width since it can decay into Goldstone excitations.
- Use Taylor expansion for numerical calculations

$$U_k(\rho) = U_k(\rho_{0,k}) + m_k^2(\rho-\rho_{0,k}) + \frac{1}{2}\lambda_k(\rho-\rho_{0,k})^2$$

Bulk viscosity

- Bulk viscous pressure is negative for expanding universe

$$\pi_{\mathsf{bulk}} = -\zeta\,\nabla_\mu u^\mu = -\zeta\,3H < 0$$

- Negative effective pressure

$$p_{\mathsf{eff}} = p + \pi_{\mathsf{bulk}} < 0$$

  would act similar to dark energy in Friedmann's equations
  [Murphy (1973), Padmanabhan & Chitre (1987), Fabris, Goncalves & de Sa Ribeiro (2006), Li & Barrow (2009), Velten & Schwarz (2011), Gagnon & Lesgourgues (2011), ...]

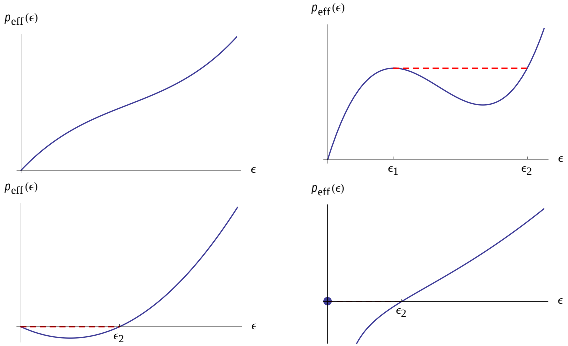
- **Is negative effective pressure physical?**
- In context of heavy ion physics: instability for $p_{\mathsf{eff}} < 0$ ("cavitation")
  [Torrieri & Mishustin (2008), Rajagopal & Tripuraneni (2010), Buchel, Camanho & Edelstein (2014), Habich & Romatschke (2015), Denicol, Gale & Jeon (2015)]
- What precisely happens at the instability?

Is negative effective pressure physical?

- Kinetic theory

$$p_{\text{eff}}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}^2}{3E_{\vec{p}}} f(x,\vec{p}) \geq 0$$

- Stability argument



If there is a vacuum with $\epsilon = p_{\text{eff}} = 0$, phases with $p_{\text{eff}} < 0$ cannot be mechanically stable. (But could be metastable.)