Statistical physics of polymerized membranes

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Functional Renormalization - from quantum gravity and dark energy to ultracold atoms and condensed matter

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membranes: D-dimensional extended objects embedded in a d-dimensional space subject to quantum and/or thermal fluctuations

fluctuating membranes / random surfaces occur in several domains:
**chemical physics / biology**: 

(Aronovitz - Lubensky, Helfrich, David - Guitter, Le Doussal - Radzihovsky, Nelson - Peliti, '70's - 90's)

⇒ structures made of *amphiphile molecules* (ex: phospholipid)
  - one hydrophilic head
  - hydrophobic tails

⇒ bilayers:

![Phospholipid Bilayer Diagram](image)
condensed matter physics: graphene, silicene, phosphorene …

uni-layers of atoms located on a honeycomb lattice

striking properties:

- high electronic mobility, transmittance, conductivity,…
- mechanical properties: both extremely **strong** and **soft** material:

  ➞ example of genuine **2D fluctuating membrane**
Properties of fluid membranes

- very weak interaction between molecules
  \[\implies\] free diffusion inside the membrane plane
  \[\implies\] no shear modulus

- very small compressibility and elasticity

\[\implies\] main contribution to the energy: \textit{bending energy}
Energy:

- point of the surface described by the embedding:

\[ \mathbf{r}: \sigma = (\sigma^1, \sigma^2) \rightarrow \mathbf{r}(\sigma^1, \sigma^2) \in \mathbb{R}^d \]

- \((\sigma^1, \sigma^2) \equiv \) local coordinates on the membrane
- tangent vectors \( \mathbf{e}_a = \frac{\partial \mathbf{r}}{\partial \sigma^a} = \partial_a \mathbf{r} \quad a = 1, 2 \)
- a unit norm vector normal to \((\mathbf{e}_1, \mathbf{e}_2)\):

\[ \hat{\mathbf{n}} = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|} \]
curvature tensor $K$: $K_{ab} = -\mathbf{n} \cdot \partial_b \mathbf{e}_a = \mathbf{e}_a \cdot \partial_b \mathbf{n}$

$K_{ab}$ can be locally diagonalized with eigenvalues $K_1$ and $K_2$

- mean or *extrinsic* curvature:

\[ H = \frac{1}{2}(K_1 + K_2) = \frac{1}{2} \text{Tr} K \]

- Gaussian or *intrinsic* curvature: $K = K_1 K_2 = \det K_a^b$

  \[ \Rightarrow \text{no role in fixed topology (Gauss-Bonnet theorem)} \]

\[ \rightarrow \text{bending energy:} \]

\[ F = \frac{\kappa}{2} \int d^2\sigma \sqrt{g} H^2 \]

- $g_{\mu\nu} = \partial_\mu \mathbf{r} \cdot \partial_\nu \mathbf{r} \equiv \text{metric induced by the embedding } \mathbf{r}(\sigma)$

- $\sqrt{g}$ ensures reparametrization invariance of $F$
a remark: with $\partial_a \hat{n} = K_{ab} e^b$ one has:

$$F = \frac{\kappa}{2} \int d^2 \sigma \ (\partial_a \hat{n})^2 \quad \text{or} \quad F = -\frac{\kappa'}{2} \sum_{\langle i,j \rangle} \hat{n}_i \cdot \hat{n}_j$$

where $\hat{n}_i$ is a unit normal vector on the plaquette $i$

very close to a $O(N)$ nonlinear $\sigma$-model / Heisenberg spin system:

- with (rigidity) coupling constant $\kappa$
- with "spins" living on a fluctuating surface
- with $d$ playing the role of the number of components $N$
- Low temperature: Monge parametrization
  \[ x = \sigma_1, \ y = \sigma_2 \text{ and } z = h(x, y) \text{ with } h \text{ height, capillary mode} \]

- \[ \mathbf{r}(x, y) = (x, y, h(x, y)) \]

- \[ \hat{\mathbf{n}}(x, y) = \frac{(-\partial_x h, -\partial_y h, 1)}{\sqrt{1 + |\nabla h|^2}} \]

- \[ \hat{\mathbf{n}}(x, y) \cdot \mathbf{e}_z = \cos \theta(x, y) = \frac{1}{\sqrt{1 + |\nabla h|^2}} \]
Free energy:

\[ F \simeq \frac{\kappa}{2} \int d^2x (\Delta h)^2 + O(h^4) \]

flat phase ? \( \implies \) fluctuations of \( \theta(x,y) \) ?

\[ \langle \theta(x,y)^2 \rangle = k_B T \int d^2q \frac{1}{\kappa q^2} \simeq \frac{k_B T}{\kappa} \ln \left( \frac{L}{a} \right) \to \infty \]

\( \implies \) no long-range order between the normals
At next order in $\hbar$, $\kappa$ is renormalized and decreased at long distances:

$$\kappa_R(q) = \kappa - \frac{3k_B T}{2\pi} \left( \frac{d}{2} \right) \ln \left( \frac{1}{qa} \right)$$

$\implies$ divergence of $\langle \theta(x,y)^2 \rangle$: worse

$\implies$ strong analogy with 2D-NL$\sigma$ model:

- correlations: $\langle \hat{n}(r).\hat{n}(0) \rangle \sim e^{-r/\xi}$
- correlation length – mass gap: $\xi \simeq a e^{4\pi\kappa/3k_B T d}$
- $d/2 \implies N - 2$
- nothing really new ...
Polymerized membranes

ex:
- organic: red blood cell, ...
- inorganic: graphene, phosphorene, ...

made of molecules linked by $V(|r_i - r_j|)$

$\implies$ free energy built from both bending and elastic energy
Free energy and low-temperature fluctuations in polymerized membranes

- reference configuration: $r_0(x, y) = (x, y, z = 0)$
- fluctuations: $r(x, y) = r_0 + u_x e_1 + u_y e_2 + h \hat{n}$
stress tensor: \( u_{ab} = \frac{1}{2} (\partial_a r \cdot \partial_b r - \partial_a r_0 \cdot \partial_b r_0) = \frac{1}{2} (\partial_a r \cdot \partial_b r - \delta_{ab}) \)

\[ \implies u_{ab} = \frac{1}{2} \left[ \partial_a u_b + \partial_b u_a + \partial_a u \cdot \partial_b u + \partial_a h \partial_b h \right] \]

- \( u_\nu \) describes the longitudinal – phonon-like – degrees of freedom
- \( h \) describes height, capillary – degrees of freedom

free energy:

\[ F \approx \int d^2 x \left[ \frac{\kappa}{2} (\Delta h)^2 + \mu(u_{ab})^2 + \frac{\lambda}{2} (u_{ab})^2 \right] \]

\( \kappa \equiv \) bending rigidity \( \quad \lambda, \mu \equiv \) elastic coupling constants

- non-trivial coupling between longitudinal - in plane - and height fluctuations \( \implies \text{frustration} \) of height fluctuations
Gaussian approximation on phonon fields:

\[ u_{ab} \simeq \frac{1}{2} [ \partial_a u_b + \partial_b u_a + \partial_a h \partial_b h ] \]

integrate over \( u \):

\[ F_{\text{eff}} = \frac{\kappa}{2} \int d^2 x \ (\Delta h)^2 + \frac{\mathcal{K}}{8} \int d^2 x \ (P_{ab}^T \partial_a h \partial_b h)^2 \]

- \( P_{ab}^T = \delta_{ab} - \partial_a \partial_b / \nabla^2 \)
- \( \kappa \) bending, rigidity coupling constant
- \( \mathcal{K} = 4\mu(\lambda + \mu)/(2\mu + \lambda) \): Young elasticity modulus
Self-consistent screening approximation (SCSA)

\( \kappa_{\text{eff}}(\mathbf{q}) = \kappa + k_B T K \int d^2k \left( \frac{[\hat{q}_a P_{ab}^T \hat{q}_b]^2}{\kappa_{\text{eff}}(\mathbf{q} + \mathbf{k})|\mathbf{q} + \mathbf{k}|^4} \right) \)

\[ \Rightarrow \kappa_{\text{eff}}(\mathbf{q}) \sim \frac{\sqrt{k_B T K}}{q} \text{ rigidity increased by fluctuations!} \]

Normal fluctuations:

\[ \langle \theta(x, y)^2 \rangle = k_B T \int d^2q \frac{1}{\kappa_{\text{eff}}(\mathbf{q})q^2} < \infty! \]

\[ \Rightarrow \text{Long-range order between normals even in } D = 2! \]
polymerized membranes $\Rightarrow$ possibility of spontaneous symmetry breaking in $D = 2$ and even in $D < 2$

$\Rightarrow$ low-temperature - flat - phase with non-trivial correlations in the I.R.

\[
\begin{align*}
G_{hh}(q) &\sim q^{-(4-\eta)} \\
G_{uu}(q) &\sim q^{-(6-D-2\eta)}
\end{align*}
\]

with $\eta \neq 0 \Rightarrow$ associated e.g. to stable sheet of graphene
Perturbative approach of the flat phase

(Aronovitz and Lubensky’88)

- Field theory of the flat phase:

$$ F \sim \int d^2x \left[ \frac{\kappa}{2} (\Delta h)^2 + \mu (u_{ab})^2 + \frac{\lambda}{2} (u_{aa})^2 \right] $$

$$ \Rightarrow \text{perturbative expansion in} \ \bar{\lambda} \equiv \lambda/\kappa^2 \text{ and } \bar{\mu} \equiv \mu/\kappa^2 \ \text{in} \ D_{uc} = 4 - \epsilon $$

- non-trivial fixed point governs the flat phase

- increasing rigidity $\kappa_{eff}(q) \sim q^{-\eta} \Rightarrow \text{orientational order}$

- decreasing elasticity $K_{eff}(q) \sim q^\eta \Rightarrow \text{positional disorder}$

$\sim \text{ripples formation}$
However:

- flat phase properties: very poorly determined in $D = 2$ because $D_{uc} = 4$
- SCSA or weak-coupling tedious beyond leading order due to:
  - derivative interaction
  - multiplicity of fields: $h$, $u$
  - propagator structure:

  **Capillary modes:**
  
  $$G_{\alpha\beta}(q^2) = \frac{\delta_{\alpha\beta}}{\kappa q^4}$$

  **Phonon modes:**
  
  $$G_{ij}(q^2) = G_1(q^2) \left[ \delta_{ij} - \frac{q_i q_j}{q^2} \right] + G_2(q^2) \frac{q_i q_j}{q^2}$$

  with:
  
  $$G_1(q^2) = \frac{1}{\kappa q^4 + \zeta^2 \mu q^2}$$
  
  $$G_2(q^2) = \frac{1}{\kappa q^4 + \zeta^2 (2\mu + \lambda) q^2}$$
FRG approach to polymerized membranes

(Kownacki and D.M.’08, Essafi, Kownacki and D.M.’14, Coquand and D.M.’16)

- Effective action: $\Gamma_k[\partial_\mu \mathbf{r}]$ expanded around the flat phase configuration:

$$\mathbf{r}(\mathbf{x}) = \zeta \sum_{\alpha=1}^{D} x_\alpha \mathbf{e}_\alpha$$

$$\Gamma_k[\partial_\mu \mathbf{r}] = \int d^D \mathbf{x} \frac{Z}{2} (\partial_\alpha \partial_\alpha \mathbf{r})^2 + u_1 (\partial_\alpha \mathbf{r} \cdot \partial_\beta \mathbf{r} - \zeta^2 \delta_{\alpha\beta})^2 + u_2 (\partial_\alpha \mathbf{r} \cdot \partial_\alpha \mathbf{r} - D \zeta^2)^2 + \ldots + \ldots + u_{10} (\partial_\alpha \mathbf{r} \cdot \partial_\beta \mathbf{r} - \zeta^2 \delta_{\alpha\beta}) (\partial_\beta \mathbf{r} \cdot \partial_\gamma \mathbf{r} - \zeta^2 \delta_{\beta\gamma}) \times (\partial_\gamma \mathbf{r} \cdot \partial_\delta \mathbf{r} - \zeta^2 \delta_{\gamma\delta}) (\partial_\delta \mathbf{r} \cdot \partial_\alpha \mathbf{r} - \zeta^2 \delta_{\delta\alpha})$$
Flat phase: \( \eta = 0.849 \) (SCSA: 0.821 \cite{LeDoussalRadzihovsky92})

MC computation with a interatomic potential for graphene: \( \eta = 0.850 \) ! \cite{LosKatsnelsonYazyevZakharchenkoFasolino09}

amazingly:

- no correction beyond the leading order in field: \( (\partial r)^4 \) ! \cite{EssafiKownackiDM14}
- almost no correction beyond the leading order in field-derivatives \( \partial^4 \) ! \cite{BraghinHasselmann10}

**key point**: graphene very well described by the ordered phase of a derivative-“\( \phi^4 \)-like” theory at leading order !
Extension to membranes in various physical situations:

- **anisotropic membranes** $\rightarrow$ tubular phase
  (Essafi, Kownacki and D.M.’11)
  - production of organic nanotubes
  - applications in bio- and nano-technology (drug delivery devices, electrochemical sensors, etc)
anisotropy between the $x$ and $y$ directions

$$\Gamma_k[r] = \int dx\ dy \left\{ \frac{Z_y}{2} (\partial_y^2 r)^2 + t_x (\partial_x r)^2 + \frac{u_y}{2} (\partial_y r \cdot \partial_y r - \zeta_y^2)^2 \right\}$$

transition between a crumpled phase with $\zeta_y = 0$ at high $T$ and a tubular phase with $\zeta_y \neq 0$ at low $T$

general phenomenon of anisotropic scaling: $q_\perp \propto q_y^2$

- Lifshitz critical behaviour: disordered+homogenous ordered+spatially modulated, phases meet together
- Horava-Lifshitz theory/gravity: breaks Lorentz invariance

$$S = \int dt\ d^D x \left\{ \frac{1}{2} (\partial_t \phi)^2 - \frac{1}{2} (\partial_i^z \phi)^2 + V(\phi) \right\}$$

$\implies$ improves UV behaviour
Upper critical dimension: $D = 5/2$ "very close" to $D = 2$

$\implies \epsilon = 5/2 - D$ in good position ?

- perturbatively: $\eta = -0.0015 < 0$ ! (rigidity: $\kappa \sim 1/q^n$)
  
  $\epsilon$-expansion: "unreliable" and "qualitatively wrong"
  
  (Radzihovsky and Toner'95)

- FRG approach:
  
  (Essafi, Kownacki and D.M.'11)

\[ \eta = 0.358(4) > 0 \] to be compared to MC data ...
- effects of **quantum fluctuations** on the flat phase of polymerized membrane / graphene (quantum fluctuation important up to $T \sim 1000$ K)

- **perturbative approach**: (Kats and Lebedev.’14, Amorim et al’14) quantum membranes at $T = 0$ asymptotically free in the UV! $\implies$ unstable wrt quantum fluctuations!
preliminary work: FRG flow from the effective action for quantum membranes (Coquand and D.M.’16)

\[ \Gamma [r] = \int_{0}^{\beta} d\tau \int d^{D}x \left\{ \frac{\rho}{2} (\partial_{\tau} r)^{2} + \frac{\kappa}{2} (\partial_{\gamma} \partial_{\gamma} r)^{2} + \frac{\mu k}{4} (\partial_{\gamma} r \cdot \partial_{\nu} r - \zeta_{k}^{2} \delta_{\gamma\nu})^{2} + \frac{\lambda k}{8} (\partial_{\gamma} r \cdot \partial_{\gamma} r - D \zeta_{k}^{2})^{2} \right\} \]

RG equations $\overline{\lambda}_{k}$, $\overline{\mu}_{k}$ and $\zeta_{k}$

- quantum membranes governed by a IR trivial fixed point $\implies$ stability of quantum membranes at $T=0$
- **cross-overs**:
  - quantum to classical regime
  - classical weak-coupling to classical strong-coupling regime
  $\implies$ improved with respect to SCSA approach

(see O. Coquand, ERG 2016 and Phys. Rev. E 94, 032125 (2016))
- effect of **disorder**

  origin: imperfect polymerization, protein, etc
  - isotropic defects $\longrightarrow$ elastic disorder
  - anisotropic defect $\longrightarrow$ curvature disorder
Hamiltonian:

\[
H[r] = \int d^D x \left\{ \frac{\kappa}{2} \left( \partial_\mu \partial_\mu r(x) - \frac{c(x)}{\kappa} \right)^2 + \lambda \left( \partial_\mu r(x).\partial_\nu r(x) - \zeta^2 \delta_{\mu \nu} (1 + 2 m(x)) \right)^2 + \mu \left( \partial_\mu r(x).\partial_\mu r(x) - \zeta^2 D (1 + 2 m(x)) \right)^2 \right\}
\]

with \( c(x) \) and \( m(x) \) Gaussian random fields

- average over (quenched) disorder using replica trick:

\[
F = \log Z = \lim_{n \to 0} \frac{Z^n - 1}{n}
\]
effective action with interacting replica:

$$\Gamma[r] = \int d^d x \sum_\alpha \left\{ \frac{\kappa}{2} \left( \partial_i \partial_i r^\alpha(x) \right)^2 + \frac{\lambda}{8} \left( \partial_i r^\alpha(x) \cdot \partial_i r^\alpha(x) - D\zeta^2 \right)^2 + \frac{\mu}{4} \left( \partial_i r^\alpha(x) \cdot \partial_j r^\alpha(x) - \zeta^2 \delta_{ij} \right)^2 \right\}$$

$$- \frac{\Delta_\kappa}{2} \sum_{\alpha, \beta} \partial_i \partial_i r^\alpha(x) \cdot \partial_j \partial_j r^\beta(x)$$

$$- \frac{\Delta_\lambda}{8} \sum_{\alpha, \beta} \left( \partial_i r^\alpha(x) \cdot \partial_i r^\alpha(x) - D\zeta^2 \right) \left( \partial_j r^\beta(x) \cdot \partial_j r^\beta(x) - D\zeta^2 \right)$$

$$- \frac{\Delta_\mu}{8} \sum_{\alpha, \beta} \left( \partial_i r^\alpha(x) \cdot \partial_j r^\alpha(x) - \zeta^2 \delta_{ij} \right) \left( \partial_i r^\beta(x) \cdot \partial_j r^\beta(x) - \zeta^2 \delta_{ij} \right)$$

with $\Delta_\kappa, \Delta_\lambda, \Delta_\mu$ disorder variances
SCSA: \textit{(Radzihovsky and Nelson’91)}

\[
\kappa_{\text{eff}}^D(q) = \kappa_{\text{eff}}(q) + \Delta \kappa \mathcal{K} \int d^2k \frac{[\hat{q}_a P^{T}_{ab} \hat{q}_b]^2}{\kappa_{\text{eff}}^2(q + k)|q + k|^4}
\]

\[
-(\Delta \lambda + \Delta \mu) \kappa^2 \int d^2k \frac{[\hat{q}_a P^{T}_{ab} \hat{q}_b]^2}{\kappa_{\text{eff}}(q + k)|q + k|^4}
\]

with $\kappa_{\text{eff}}$ renormalized only by thermal fluctuations

- weak coupling \textit{(Morse and Lubensky’92)}

$\implies$ stability of the ordered fixed point
SCSA: (Radzihovsky and Nelson'91)

\[
\kappa_{eff}^D(q) = \kappa_{eff}(q) + \Delta \kappa \int d^2 k \frac{[\hat{q}_a P_{ab}^T \hat{q}_b]^2}{\kappa_{eff}^2(q+k)|q+k|^4}
\]

\[
-(\Delta \lambda + \Delta \mu) \kappa^2 \int d^2 k \frac{[\hat{q}_a P_{ab}^T \hat{q}_b]^2}{\kappa_{eff}(q+k)|q+k|^4}
\]

with \(\kappa_{eff}\) renormalized only by thermal fluctuations

weak coupling (Morse and Lubensky'92)

\[\implies\] stability of the ordered fixed point

FRG approach (Coquand, Essafi, Kownacki, D.M.'17): new fixed point not seen within perturbation theory

new failure of perturbative approach ?
Conclusion

Perturbative approaches of membranes fail in several situations

- $D = 2$ far from the upper critical dimension $D_{uc}$
- $D_{uc}$ is fractional
- missing of fixed point ?
- ...

The FRG seems efficient in all these cases ... but
Prospects

- **self-avoidance** (David, Duplantier, Guitter, Le Doussal, Wiese)

\[ H = H_0 + \frac{b}{2} \int d^D x \, d^D y \, \delta(\mathbf{r}(\mathbf{x}) - \mathbf{r}(\mathbf{y})) \]

\[ \Rightarrow \text{disappearance of the -- high } T -- \text{ crumpled phase?} \]

**problem:** non-locality in \( D \)-space

- graphene-like systems: interaction between electronic and membranes degrees of freedom

\[ H = -i \int d^2 x \sqrt{g} \, \bar{\Psi} \gamma^a e^i_a (\partial_i + \Omega_i) \Psi \]

with: \( g_{ij} = \delta_{ij} + 2u_{ij} \)

(Coquand, Le Doussal, D.M., Radzihovsky)