Effective Description of Dark Matter as a Viscous Fluid

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Distribution of dark and baryonic matter in the Universe

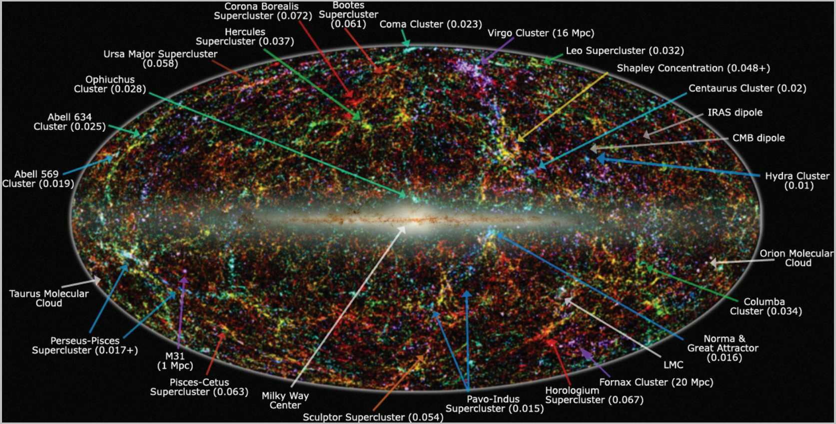

Figure: *2MASS Galaxy Catalog (more than 1.5 million galaxies).*

Inhomogeneities

- Inhomogeneities are treated as perturbations on top of an expanding homogeneous background.
- Under gravitational attraction, the matter overdensities grow and produce the observed large-scale structure.
- The distribution of matter at various redshifts reflects the detailed structure of the cosmological model.
- Define the density field $\delta = \delta\rho/\rho_0$ and its spectrum

$$\langle \delta(\mathbf{k})\delta(\mathbf{q})\rangle \equiv \delta_D(\mathbf{k}+\mathbf{q})P(\mathbf{k}).$$

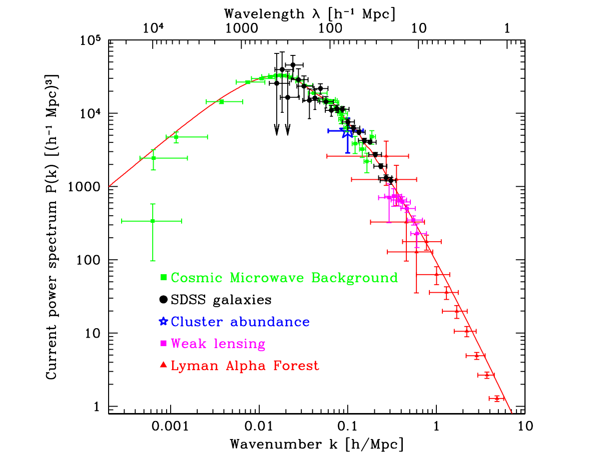

Figure: Matter power spectrum (Tegmark *et al.* 2003).

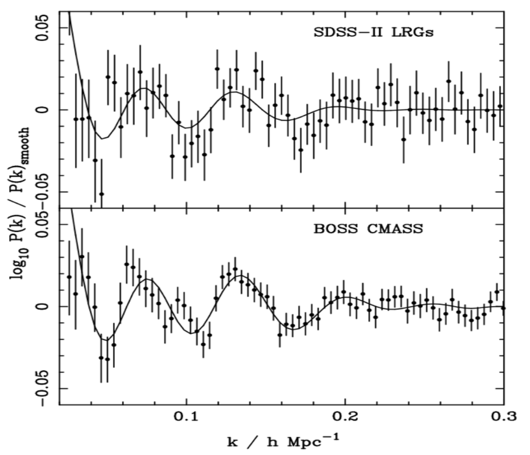

Figure: Matter power spectrum in the range of baryon acoustic oscillations.

A scale from the early Universe

- The characteristic scale of the baryon acoustic oscillations is approximately 150 Mpc (490 million light-years) today.
- It corresponds to the wavelength of sound waves (the sound horizon) in the baryon-photon plasma at the time of recombination ($z = a_{\rm today}/a - 1 = 1100$).
- It is also imprinted on the spectrum of the photons of the cosmic microwave background.
- Comparing the measured with the theoretically calculated spectra constrains the cosmological model.
- The aim is to achieve a 1% precision both for the measured and calculated spectra.
- Galaxy surveys: Euclid, DES, LSST, SDSS ...

Problems with standard perturbation theory

- In the linearized hydrodynamic equations each mode evolves independently. Higher-order corrections take into account mode-mode coupling.
- Calculation of the matter spectrum beyond the linear level. (Crocce, Scoccimarro 2005)
- Baryon acoustic oscillations ($k \simeq 0.05 - 0.2\, h/\text{Mpc}$): Mildly nonlinear regime of perturbation theory.
- Higher-order corrections dominate for $k \simeq 0.3 - 0.5\, h/\text{Mpc}$.
- The theory becomes strongly coupled for $k \gtrsim 1\, h/\text{Mpc}$.
- The deep UV region is out of the reach of perturbation theory.
- Way out: Introduce an effective low-energy description in terms of an imperfect fluid (Baumann, Nicolis, Senatore, Zaldarriaga 2010, Carrasco, Hertzberg,Senatore 2012, Pajer, Zaldarriaga 2013, Carrasco, Foreman, Green, Senatore 2014)

Lessons from the functional renormalization group

- Coarse-graining: Integrate out the modes with $k > k_m$ and replace them with effective couplings in the low-k theory.
- Wetterich equation for the coarse-grained effective action $\Gamma_k[\phi]$:

$$\frac{\partial \Gamma_k[\phi]}{\partial t} = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)}[\phi] + \hat{R}_k\right)^{-1} \frac{\partial \hat{R}_k}{\partial \ln k}\right].$$

- For a standard kinetic term and potential $U_k[\phi]$, with a sharp cutoff, the first step of an iterative solution gives

$$U_{k_m}(\phi) = V(\phi) + \frac{1}{2}\int_{k_m}^{\Lambda} \frac{d^d q}{(2\pi)^d} \ln\left(q^2 + V''(\phi)\right).$$

- The low-energy theory contains new couplings, not present in the tree-level action. It comes with a UV cutoff k_m.

- Why is this intuition relevant for the problem of classical cosmological perturbations?
- The primordial Universe is a stochastic medium.
- The fluctuating fields (density, velocity) at early times are Gaussian random variables with an almost scale-invariant spectrum.
- The generation of this spectrum is usually attributed to inflation.
- The coarse graining can be implemented formally on the initial condition for the spectrum at recombination.

The question

- Question: Is dark matter at large scales (in the BAO range) best described as a perfect fluid?
- I shall argue that there is a better description in the context of the effective theory.
- Going beyond the perfect-fluid approximation, the description must include effective (shear and bulk) viscosity and nonzero speed of sound.
- Formulate the perturbative approach for viscous dark matter.

Scales

- $k_\Lambda \sim 1 - 3\ h$/Mpc (length $\sim 3 - 10$ Mpc):
  The fluid description becomes feasible.
- Scales $k > k_\Lambda$ correspond to virialized structures, which are essentially decoupled.
- $k_m \sim 0.5 - 1\ h$/Mpc (length $\sim 10 - 20$ Mpc):
  The fluid parameters have a simple form. The description includes effective viscosity and speed of sound, arising through coarse-graining.
- The viscosity results from the integration of the modes $k > k_m$. The form of the power spectrum $\sim k^{-3}$ implies that the effective viscosity is dominated by $k \simeq k_m$.
- k_m acts as an UV cutoff for perturbative corrections in the large-scale theory.
- Good convergence, in contrast to standard perturbation theory.

- Dark matter can be treated as a fluid because of its small velocity and the finite age of the Universe. Dark matter particles drift over a finite distance, much smaller than the Hubble radius.
- The phase space density $f(\mathbf{x},\mathbf{p},\tau)=f_0(p)[1+\delta_f(\mathbf{x},p,\hat{\mathbf{p}},\tau)]$ can be expanded in Legendre polynomials:

$$\delta_f(\mathbf{k},p,\hat{\mathbf{p}},\tau)=\sum_{n=0}^{\infty}(-i)^n(2n+1)\delta_f^{[n]}(\mathbf{k},p,\tau)P_n(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}).$$

The Vlasov equation leads to:

$$\frac{d\delta_f^{[n]}}{d\tau}=kv_p\left[\frac{n+1}{2n+1}\delta_f^{[n+1]}-\frac{n}{2n+1}\delta_f^{[n-1]}\right],\qquad n\geq 2,$$

with $v_p=p/am$ the particle velocity.

- The time τ available for the higher $\delta_f^{[n]}$ to grow is $\sim 1/\mathcal{H}$.
- A fluid description is possible for $kv_p/\mathcal{H}\lesssim 1$.
- Estimate the particle velocity from the fluid velocity v at small length scales.

- At the comoving scale k, the linear evolution indicates that

$$(\theta/\mathcal{H})^2 \sim k^3 P^L(k),$$

with $\theta = \vec{k}\vec{v}$ and $P^L(k)$ the linear power spectrum.

- The linear power spectrum scales roughly as k^{-3} above $\sim k_m$. $k^3 P^L(k)$ is rougly constant, with a value of order 1 today. Its time dependence is given by D_L^2, with D_L the linear growth factor.
- If the maximal particle velocity is identified with the fluid velocity at the scale k_m, we have

$$v_p \sim \frac{\mathcal{H}}{k_m} D_L.$$

- The dimensionless factor characterizing the growth of higher $\delta_f^{[n]}$ is $kv_p/\mathcal{H} \sim D_L k/k_m$.
- Scales with $k \gg k_m/D_L$ require the use of the whole Boltzmann hierarchy.
- In practice, the validity of the fluid description extends beyond k_m.

- The shear viscosity is estimated as $\eta/(\rho+p) \sim l_{\rm free} v_p$, with $l_{\rm free}$ the mean free path.
- For the effective viscosity we can estimate $l_{\rm free} \sim v_p/H$, with $H = \mathcal{H}/a$. In this way we obtain

$$\nu_{\rm eff}\mathcal{H} = \frac{\eta_{\rm eff}}{(\rho+p)a}\mathcal{H} \sim l_{\rm free} v_p H \sim \frac{\mathcal{H}^2}{k_m^2} D_L^2.$$

- There is also a nonzero speed of sound. The linearized treatment of the Boltzmann hierarchy gives

$$\nu_{\rm eff}\mathcal{H} = \frac{3}{5} c_s^2$$

on the growing mode.

Plan of the talk

- Basic formalism
- Determination of the effective viscosity
- Calculation of the spectrum
- Conclusions

- S. Floerchinger, N. T., U. Wiedemann
  arXiv:1411.3280[gr-qc], Phys. Rev. Lett. 114: 9, 091301 (2015)
- D. Blas, S. Floerchinger, M. Garny, N. T., U. Wiedemann
  arXiv:1507.06665[astro-ph.CO], JCAP 1511, 049 (2015)
- S. Floerchinger, M. Garny, N. T., U. Wiedemann
  arXiv:1607.03453[astro-ph.CO], JCAP 1701 no.01, 048 (2017)

Covariant hydrodynamic description of a viscous fluid

- Work within the first-order formalism.
- Energy-momentum tensor:

$$T^{\mu\nu} = \rho u^\mu u^\nu + (p + \pi_b)\Delta^{\mu\nu} + \pi^{\mu\nu}.$$

ρ: energy density
p: pressure in the fluid rest frame
π_b: bulk viscosity
$\pi^{\mu\nu}$: shear viscosity, satisfying: $u_\mu \pi^{\mu\nu} = \pi^\mu_{\ \mu} = 0$
$\Delta^{\mu\nu}$ projector orthogonal to the fluid velocity: $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$

- New elements:
Bulk viscosity: $\pi_b = -\zeta \nabla_\rho u^\rho$
Shear viscosity tensor:

$$\pi^{\mu\nu} = -2\eta\sigma^{\mu\nu} = -2\eta \left(\frac{1}{2} \left(\Delta^{\mu\rho} \nabla_\rho u^\nu + \Delta^{\nu\rho} \nabla_\rho u^\mu \right) - \frac{1}{3} \Delta^{\mu\nu} (\nabla_\rho u^\rho) \right).$$

Dynamical equations

- Einstein equations:

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu},$$

- Conservation of the energy momentum tensor ($\nabla_\nu T^{\mu\nu} = 0$):

$$\begin{aligned} u^\mu \nabla_\mu \rho + (\rho + p)\nabla_\mu u^\mu - \zeta\left(\nabla_\mu u^\mu\right)^2 - 2\eta\sigma^{\mu\nu}\sigma_{\mu\nu} &= 0 \\ (\rho + p + \pi_b)u^\mu \nabla_\mu u^\rho + \Delta^{\rho\mu}\nabla_\mu(p + \pi_b) + \Delta^\rho{}_\nu \nabla_\mu \pi^{\mu\nu} &= 0. \end{aligned}$$

Subhorizon scales

- Ansatz for the metric:

$$ds^2 = a^2(\tau)\left[-\left(1+2\Psi(\tau,\mathbf{x})\right)d\tau^2 + \left(1-2\Phi(\tau,\mathbf{x})\right)d\mathbf{x}\,d\mathbf{x}\right].$$

- The potentials Φ and Ψ are weak. Their difference is governed by the shear viscosity. We can take $\Phi \simeq \Psi \ll 1$.
- The four-velocity $u^\mu = dx^\mu/\sqrt{-ds^2}$ can be expressed through the coordinate velocity $v^i = dx^i/d\tau$ and the potentials Φ and Ψ:

$$u^\mu = \frac{1}{a\sqrt{1+2\Psi-(1-2\Phi)\vec{v}^2}}(1,\vec{v}).$$

- Neglect vorticity and consider the density $\delta = \frac{\delta\rho}{\rho_0}$ and velocity $\theta = \vec{\nabla}\vec{v}$ fields. Combine them in a doublet $\varphi_{1,2} = (\delta, -\theta/\mathcal{H})$, where $\mathcal{H} = \dot{a}/a$ is the Hubble parameter.
- The spectrum is

$$\langle\varphi_a(\mathbf{k},\tau)\varphi_b(\mathbf{q},\tau)\rangle \equiv \delta_D(\mathbf{k}+\mathbf{q})P_{ab}(\mathbf{k},\tau).$$

- $P(\mathbf{k},\tau) \equiv P_{11}(\mathbf{k},\tau)$.

Convergence of perturbation theory (no viscosity)

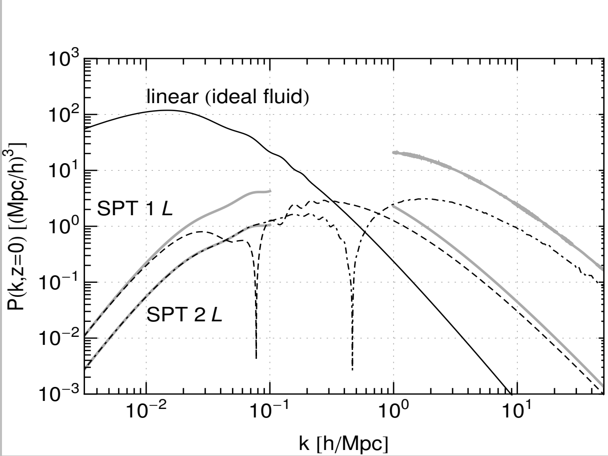

Figure: Linear power spectrum and the one- and two-loop corrections in standard perturbation theory (SPT) at $z = 0$.

- Higher-order (loop) corrections dominate for $k \gtrsim 0.3 - 0.5\ h/\text{Mpc}$.
- The theory becomes strongly coupled for $k \simeq 1\ h/\text{Mpc}$.
- The deep UV region is out of the reach of perturbation theory.
- Higher-order corrections are increasingly more UV sensitive.
- For small k, the one-loop depends on the dimensionful scale

$$\sigma_d^2(\eta) = \frac{4\pi}{3} \int_0^\infty dq\, P^L(q,\eta) = \frac{4\pi}{3} D_L^2(\eta) \int_0^\infty dq\, P^L(q,0),$$

  with $\eta = \ln a = -\ln(1+z)$ and $D_L(\eta)$ the linear growth factor: $\delta(\mathbf{k},\eta) = D_L(\eta)\delta(\mathbf{k},0)$ on the growing mode.
- For the spectrum, the complete expression is (Blas, Garny, Konstandin 2013)

$$P_{ab}^{1-\text{loop}}(k,\eta) = -\begin{pmatrix} \frac{61}{105} & \frac{25}{21} \\ \frac{25}{21} & \frac{9}{5} \end{pmatrix} k^2 \sigma_d^2 P^L(k,\eta).$$

Effective description (in terms of pressure and viscosity)

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arXiv:1507.06665[astro-ph.CO], JCAP 1511, 049 (2015)

- Introduce an effective low-energy description in terms of an imperfect fluid (Baumann, Nicolis, Senatore, Zaldarriaga 2010, Carrasco, Hertzberg,Senatore 2012, Pajer, Zaldarriaga 2013, Carrasco, Foreman, Green, Senatore 2014)
- Integrate out the modes with $k \gtrsim k_m$ and replace them with effective couplings (viscosity, pressure), determined in terms of

$$\sigma^2_{dk}(\eta) = \frac{4\pi}{3}\int_{k_m}^{\infty} dq\, P^L(q,\eta) = \frac{4\pi}{3} D_L^2(\eta)\int_{k_m}^{\infty} dq\, P^L(q,0).$$

- For a spectrum $\sim 1/k^3$, the integral is dominated by the region near k_m. The deep UV does not contribute significantly.

Parameters of an effective viscous theory

$$\rho(\mathbf{x},\tau) = \rho_0(\tau) + \delta\rho(\mathbf{x},\tau) \qquad p(\mathbf{x},\tau) = 0 + \delta p(\mathbf{x},\tau)$$

$$\delta = \frac{\delta\rho}{\rho_0} \qquad \theta = \vec{\nabla}\vec{v}$$

$$\mathcal{H} = \frac{\dot{a}}{a} \qquad H = \frac{1}{a}\mathcal{H}$$

$$c_s^2(\tau) = \frac{\delta p}{\delta\rho} = \alpha_s(\tau)\frac{\mathcal{H}^2}{k_m^2} \qquad \nu(\tau)\mathcal{H} = \frac{\eta\mathcal{H}}{\rho_0 a} = \frac{3}{4}\alpha_\nu(\tau)\frac{\mathcal{H}^2}{k_m^2}.$$

with $\alpha_s, \alpha_\nu = \mathcal{O}(1)$ today.

- We rely on a hierarchy supported by linear perturbation theory for subhorizon perturbations.
  1. We treat δ and $\theta/\mathcal{H}$ as quantities of order 1, $\vec{v}$ as a quantity of order $\mathcal{H}/k$ and Ψ, Φ as quantities of order $\mathcal{H}^2/k^2$.
  2. We assume that a time derivative is equivalent to a factor of $\mathcal{H}$, while a spatial derivative to a factor of k.
  3. We assume that $c_s^2, \nu\mathcal{H}$ are of order $\mathcal{H}^2/k_m^2$.
- Keeping the dominant terms, we obtain

$$
\begin{aligned}
\dot{\delta} + \vec{\nabla}\vec{v} + (\vec{v}\vec{\nabla})\delta + \delta\vec{\nabla}\vec{v} &= 0 \\
\dot{\vec{v}} + \mathcal{H}\vec{v} + (\vec{v}\vec{\nabla})\vec{v} + \vec{\nabla}\Phi + c_s^2(1-\delta)\vec{\nabla}\delta & \\
-\nu(1-\delta)\left(\nabla^2\vec{v} + \frac{1}{3}\vec{\nabla}(\vec{\nabla}\vec{v})\right) &= 0. \\
\nabla^2\Phi &= \frac{3}{2}\Omega_m\mathcal{H}^2\delta.
\end{aligned}
$$

Use Fourier-transformed quantities to obtain

$$\begin{aligned}
\dot{\delta}_{\mathbf{k}} &+ \theta_{\mathbf{k}} + \int d^3\mathbf{p}\, d^3\mathbf{q}\, \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\, \alpha_1(\mathbf{p},\mathbf{q})\, \delta_{\mathbf{p}}\, \theta_{\mathbf{q}} = 0 \\
\dot{\theta}_{\mathbf{k}} &+ \left(\mathcal{H} + \frac{4}{3}\nu k^2\right) \theta_{\mathbf{k}} + \left(\frac{3}{2}\Omega_m \mathcal{H}^2 - c_s^2 k^2\right) \delta_{\mathbf{k}} \\
&+ \int d^3\mathbf{p}\, d^3\mathbf{q}\, \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \\
&\qquad \Big(\beta_1(\mathbf{p},\mathbf{q})\, \delta_{\mathbf{p}}\, \delta_{\mathbf{q}} + \beta_2(\mathbf{p},\mathbf{q})\, \theta_{\mathbf{p}}\, \theta_{\mathbf{q}} + \beta_3(\mathbf{p},\mathbf{q})\, \delta_{\mathbf{p}}\, \theta_{\mathbf{q}}\Big) = 0,
\end{aligned}$$

with

$$\begin{aligned}
\alpha_1(\mathbf{p},\mathbf{q}) &= \frac{(\mathbf{p}+\mathbf{q})\mathbf{q}}{q^2} \\
\beta_1(\mathbf{p},\mathbf{q}) &= c_s^2(\mathbf{p}+\mathbf{q})\mathbf{q} \\
\beta_2(\mathbf{p},\mathbf{q}) &= \frac{(\mathbf{p}+\mathbf{q})^2\mathbf{p}\cdot\mathbf{q}}{2p^2q^2} \\
\beta_3(\mathbf{p},\mathbf{q}) &= -\frac{4}{3}\nu(\mathbf{p}+\mathbf{q})\mathbf{q}.
\end{aligned}$$

Define the doublet

$$\begin{pmatrix} \varphi_1(\mathbf{k},\eta) \\ \varphi_2(\mathbf{k},\eta) \end{pmatrix} = \begin{pmatrix} \delta_{\mathbf{k}}(\tau) \\ -\dfrac{\theta_{\mathbf{k}}(\tau)}{\mathcal{H}} \end{pmatrix},$$

where $\eta = \ln a(\tau)$. The evolution equations take the form

$$\partial_\eta \phi_a(\mathbf{k}) = -\Omega_{ab}(\mathbf{k},\eta)\varphi_b(\mathbf{k}) + \int d^3p\, d^3q\, \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\gamma_{abc}(\mathbf{p},\mathbf{q},\eta)\,\varphi_b(\mathbf{p})\,\varphi_c(\mathbf{q}),$$

where

$$\Omega(\mathbf{k},\eta) = \begin{pmatrix} 0 & -1 \\ -\frac{3}{2}\Omega_m + \alpha_s \frac{k^2}{k_m^2} & 1 + \frac{\mathcal{H}'}{\mathcal{H}} + \alpha_\nu \frac{k^2}{k_m^2} \end{pmatrix}.$$

and a prime denotes a derivative with respect to η. The nonzero elements of γ_{abc} are expressed in terms of $\alpha_1(\mathbf{p},\mathbf{q})$, $\beta_1(\mathbf{p},\mathbf{q})$, $\beta_2(\mathbf{p},\mathbf{q})$, $\beta_3(\mathbf{p},\mathbf{q})$.

The evolution of the spectrum

- Define the spectra, bispectra and trispectra as

$$
\begin{aligned}
\langle \varphi_a(\mathbf{k},\eta)\varphi_b(\mathbf{q},\eta)\rangle &\equiv \delta_D(\mathbf{k}+\mathbf{q})P_{ab}(\mathbf{k},\eta)\\
\langle \varphi_a(\mathbf{k},\eta)\varphi_b(\mathbf{q},\eta)\varphi_c(\mathbf{p},\eta)\rangle &\equiv \delta_D(\mathbf{k}+\mathbf{q}+\mathbf{p})B_{abc}(\mathbf{k},\mathbf{q},\mathbf{p},\eta)\\
\langle \varphi_a(\mathbf{k},\eta)\varphi_b(\mathbf{q},\eta)\varphi_c(\mathbf{p},\eta)\varphi_d(\mathbf{r},\eta)\rangle &\equiv \delta_D(\mathbf{k}+\mathbf{q})\delta_D(\mathbf{p}+\mathbf{r})P_{ab}(\mathbf{k},\eta)P_{cd}(\mathbf{p},\eta)\\
&+\delta_D(\mathbf{k}+\mathbf{p})\delta_D(\mathbf{q}+\mathbf{r})P_{ac}(\mathbf{k},\eta)P_{bd}(\mathbf{q},\eta)\\
&+\delta_D(\mathbf{k}+\mathbf{r})\delta_D(\mathbf{q}+\mathbf{p})P_{ad}(\mathbf{k},\eta)P_{bc}(\mathbf{q},\eta)\\
&+\delta_D(\mathbf{k}+\mathbf{p}+\mathbf{q}+\mathbf{r})Q_{abcd}(\mathbf{k},\mathbf{p},\mathbf{q},\mathbf{r},\eta).
\end{aligned}
$$

- Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy.
- Essential (rather crude) approximation: Neglect the effect of the trispectrum on the evolution of the bispectrum (Pietroni 2008).
- In this way we obtain

$$
\begin{aligned}
\partial_\eta P_{ab}(\mathbf{k},\eta) = & -\Omega_{ac}P_{cb}(\mathbf{k},\eta) - \Omega_{bc}P_{ac}(\mathbf{k},\eta) \\
& + \int d^3q \big[\gamma_{acd}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k})B_{bcd}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k}) \\
& + \gamma_{bcd}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k})B_{acd}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k})\big], \\
\partial_\eta B_{abc}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k}) = & -\Omega_{ad}B_{dbc}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k}) - \Omega_{bd}B_{adc}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k}) \\
& -\Omega_{cd}B_{abd}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k}) \\
& + 2\int d^3q \big[\gamma_{ade}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k})P_{db}(\mathbf{q},\eta)P_{ec}(\mathbf{k}-\mathbf{q},\eta) \\
& + \gamma_{bde}(-\mathbf{q},\mathbf{q}-\mathbf{k},\mathbf{k})P_{dc}(\mathbf{k}-\mathbf{q},\eta)P_{ea}(\mathbf{k},\eta) \\
& + \gamma_{cde}(\mathbf{q}-\mathbf{k},\mathbf{k},-\mathbf{q})P_{da}(\mathbf{k},\eta)P_{eb}(\mathbf{q},\eta)\big].
\end{aligned}
$$

Alternative approach

- Expand the fields in powers of the initial perturbations at $\eta = \eta_0$,

$$\phi_a(\mathbf{k},\eta) = \sum_n \int d^3q_1 \cdots d^3q_n\,(2\pi)^3\delta^{(3)}(\mathbf{k}-\sum_i \mathbf{q}_i) \times F_{n,a}(\mathbf{q}_1,\ldots,\mathbf{q}_n,\eta)\delta_{\mathbf{q}_1}(\eta_0)\cdots\delta_{\mathbf{q}_n}(\eta_0)\,.$$

- From the equation of motion we can get evolution equations for the kernels $F_{n,a}$

$$(\partial_\eta\delta_{ab}+\Omega_{ab}(\mathbf{k},\eta))F_{n,b}(\mathbf{q}_1,\ldots,\mathbf{q}_n,\eta) = \sum_{m=1}^{n}\gamma_{abc}(\mathbf{q}_1+\cdots+\mathbf{q}_m,\mathbf{q}_{m+1}+\cdots+\mathbf{q}_n) \times F_{m,b}(\mathbf{q}_1,\ldots,\mathbf{q}_m,\eta)F_{n-m,c}(\mathbf{q}_{m+1},\ldots,\mathbf{q}_n,\eta)\,.$$

- When neglecting the pressure and viscosity terms, the solution is known analytically for an Einstein-de Sitter Universe.
- For non-zero pressure and viscosity, the time-dependence does not factorize. We solve the differential equations numerically.

Matching the perfect-fluid and viscous theories

- For the matching one could use the propagator

$$G_{ab}(\mathbf{k},\tilde{\eta},\tilde{\eta}')\delta^{(3)}(\mathbf{k}-\mathbf{k}') = \left\langle \frac{\delta\phi_a(\mathbf{k},\tilde{\eta})}{\delta\phi_b(\mathbf{k}',\tilde{\eta}')} \right\rangle ,$$

- Define appropriate fields for the background to be effectively Einstein-de Sitter to a very good approximation: $D_L(\tilde{\eta}) = \exp(\tilde{\eta})$.
- The one-loop propagator of the perfect-fluid theory is

$$G_{ab}(\mathbf{k},\tilde{\eta},\tilde{\eta}') = g_{ab}(\tilde{\eta}-\tilde{\eta}') - k^2 e^{\tilde{\eta}-\tilde{\eta}'}\sigma_d^2(\tilde{\eta}) \begin{pmatrix} \frac{61}{350} & \frac{61}{525} \\ \frac{27}{50} & \frac{9}{25} \end{pmatrix} ,$$

where the linear propagor for the growing mode is

$$g_{ab}(\tilde{\eta}-\tilde{\eta}') = \frac{e^{\tilde{\eta}-\tilde{\eta}'}}{5} \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} .$$

- The contribution from $k > k_m$ can be isolated by taking

$$\sigma_d^2(\tilde{\eta}) \longrightarrow \sigma_{dk}^2(\tilde{\eta}) \equiv \frac{4\pi}{3}\exp(2\tilde{\eta}) \int_{k_m}^{\infty} dq\, P^L(q,0) .$$

- Compute the propagator of the viscous theory at linear order, for kinematic viscosity and sound velocity of the form

$$\nu\mathcal{H} = \frac{\eta}{\rho_0\, a}\mathcal{H} = \beta_\nu\, e^{2\tilde{\eta}}\,\frac{\mathcal{H}^2}{k_m^2}\,, \qquad c_s^2 = \frac{\delta p}{\delta\rho} = \frac{3}{4}\,\beta_s\, e^{2\tilde{\eta}}\,\frac{\mathcal{H}^2}{k_m^2}\,.$$

- The propagator contains a contribution

$$\delta g_{ab}(\mathbf{k},\tilde{\eta}) = -\frac{k^2}{k_m^2}(\beta_\nu+\beta_s)\,\frac{e^{3\tilde{\eta}}}{45}\begin{pmatrix}3 & 2\\ 9 & 6\end{pmatrix}$$

  in addition to the perfect-fluid linear contribution (for $\tilde{\eta} \gg \tilde{\eta}'$).
- Identify the linear contribution $\sim k^2 c_s^2$ and $\sim k^2\nu\mathcal{H}$ with the one-loop correction $\sim k^2\sigma_{dk}^2$ of the perfect-fluid propagator.
- This can be achieved with 1% accuracy, and gives

$$\beta_s + \beta_\nu = \frac{27}{10}\, k_m^2\sigma_{dk}^2(0).$$

- Solve for the nonlinear spectrum in the effective viscous theory with an UV cutoff k_m in the momentum integrations.

Particular features

- There are no free parameters in this approach.
- The results are independent of the ratio β_ν/β_s to a good approximation. They depend mainly on the value of $\beta_\nu + \beta_s$.
- They are also insensitive to the (mode-mode) couplings proportional to the viscosity or the shound velocity. The (mode-mode) couplings of the perfect-fluid theory are the dominant ones.

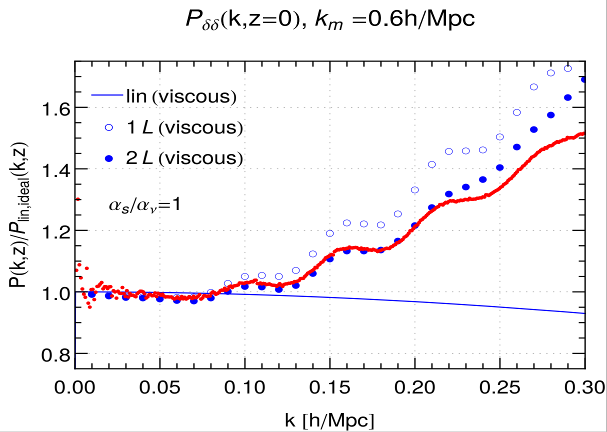

Figure: Power spectrum obtained in the viscous theory for redshift $z = 0$, normalized to $P_{lin,ideal}$. The open (filled) circles show the one-loop (two-loop) result in the viscous theory. The solid blue line is the linear spectrum in the viscous theory, and the red points show results of the Horizon N-body simulation.

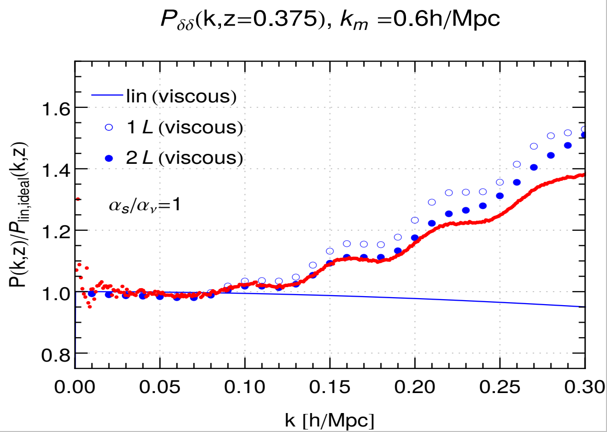

Figure: One-loop spectrum in the effective theory at $z = 0.375$.

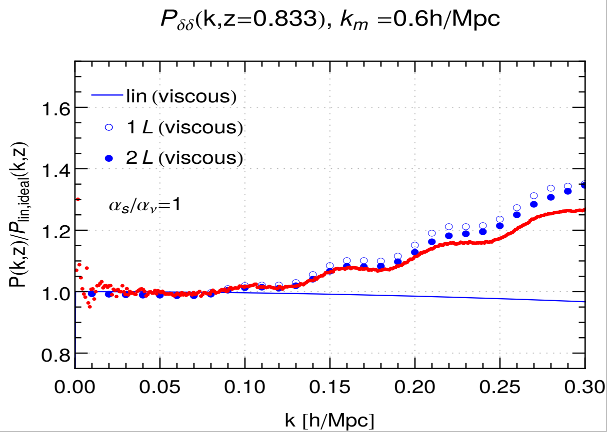

Figure: One-loop spectrum in the effective theory at $z = 0.833$.

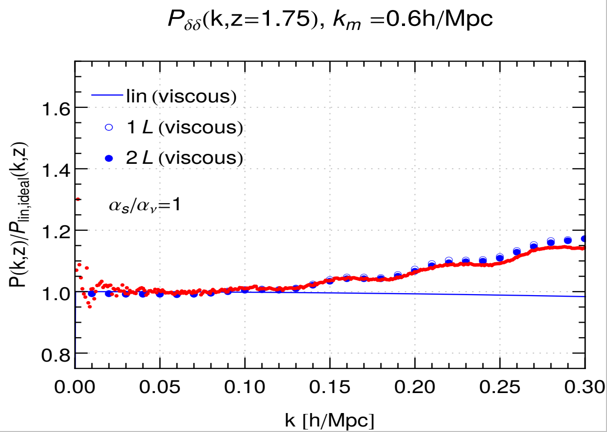

Figure: One-loop spectrum in the effective theory at $z = 1.75$.

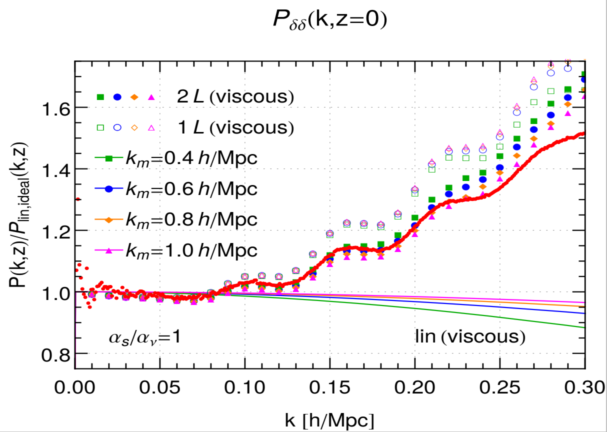

Figure: One-loop spectrum in the effective theory at $z = 0$ for various values of k_m.

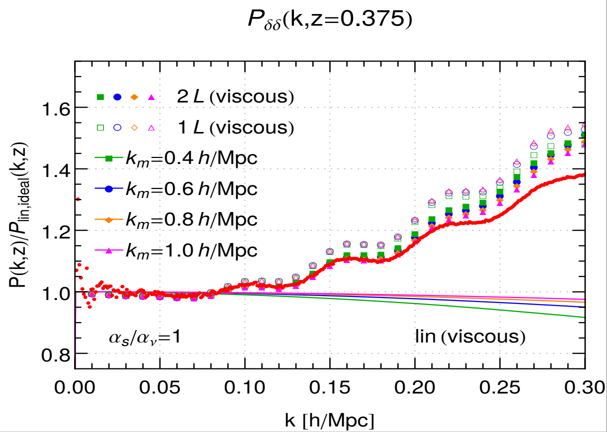

Figure: One-loop spectrum in the effective theory at $z = 0.375$ for various values of k_m.

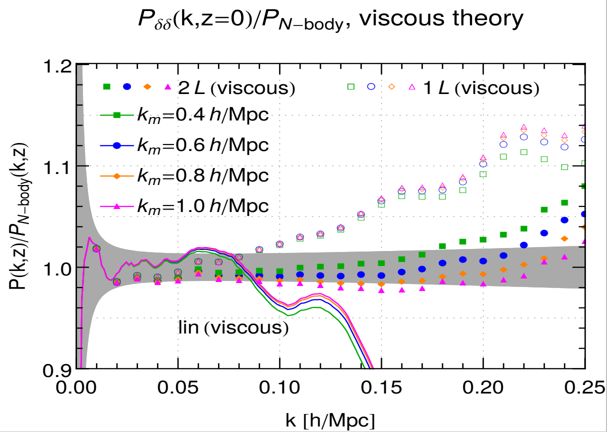

Figure: Comparison of results for the power spectrum obtained within the viscous theory normalized to the N-body result at $z = 0$. The grey band corresponds to an estimate for the error of the N-body result.

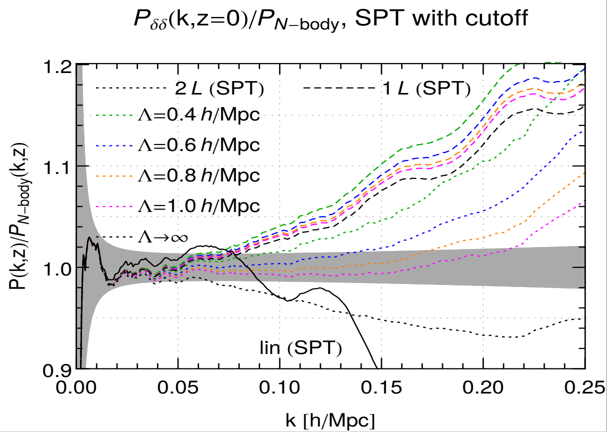

Figure: One- and two-loop results in standard perturbation theory (shown as dashed and dotted lines, respectively), computed with various values of an ad-hoc cutoff Λ (coloured lines), as well as in the limit $\Lambda \to \infty$ (black lines).

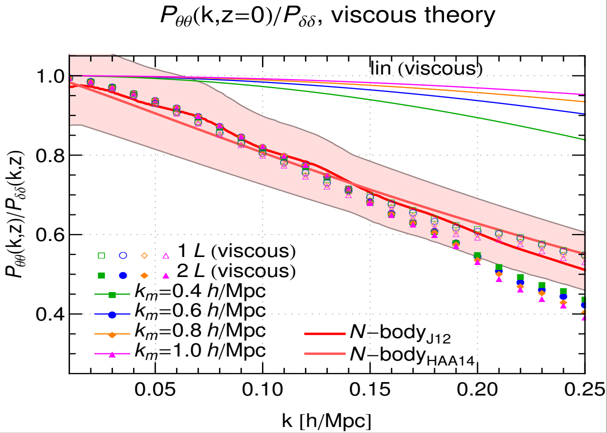

Figure: The velocity-velocity spectrum obtained within the viscous theory, compared to results from N-body simulations at $z = 0$. The pink band corresponds to a 10% error for the N-body results.

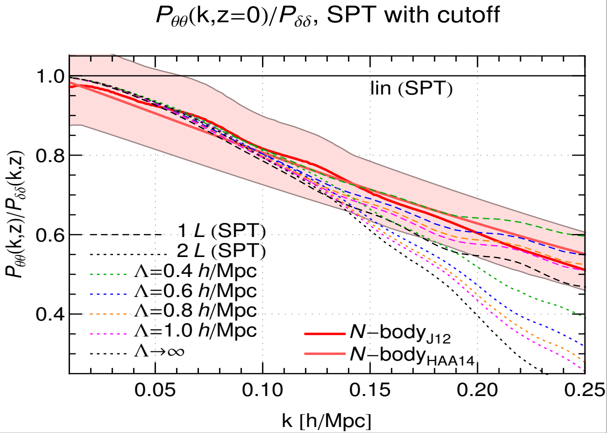

Figure: One- and two-loop results in standard perturbation theory (shown as dashed and dotted lines, respectively), computed with various values of an ad-hoc cutoff Λ (coloured lines), as well as in the limit $\Lambda \to \infty$ (black lines).

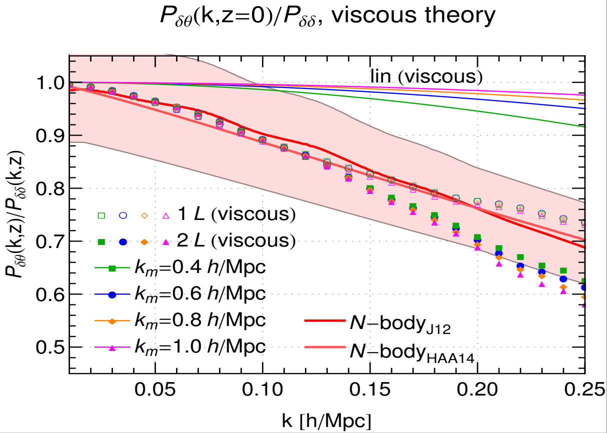

Figure: The density-velocity spectrum obtained within the viscous theory, compared to results from N-body simulations at $z = 0$. The pink band corresponds to a 10% error for the N-body results.

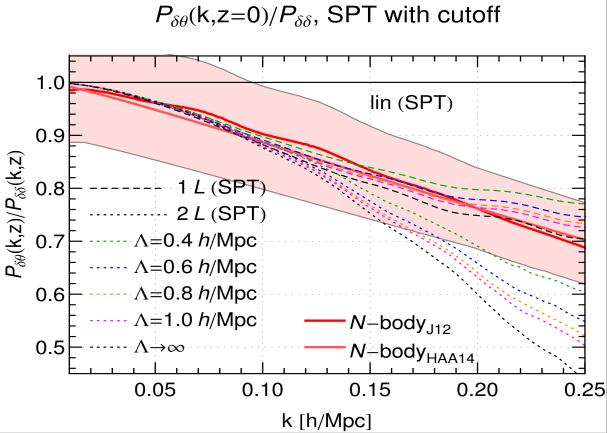

Figure: One- and two-loop results in standard perturbation theory (shown as dashed and dotted lines, respectively), computed with various values of an ad-hoc cutoff Λ (coloured lines), as well as in the limit $\Lambda \to \infty$ (black lines).

- We guessed
  $\nu(\tau)\mathcal{H} = \frac{3}{4}\alpha_\nu(\tau)\frac{\mathcal{H}^2}{k_m^2}, \quad c_s^2(\tau) = \alpha_s(\tau)\frac{\mathcal{H}^2}{k_m^2}$, with $\alpha_\nu, \ \alpha_s \lesssim 1$.
- At one loop we found
  $\nu\mathcal{H} = \frac{3}{4}\,\beta_\nu\, e^{2\eta}\,\frac{\mathcal{H}^2}{k_m^2}\,, \quad c_s^2 = \beta_s\, e^{2\eta}\,\frac{\mathcal{H}^2}{k_m^2}$, with $\beta_s + \beta_\nu = \frac{27}{10}\,k_m^2\sigma_{dk}^2(0)$.
- We generalize the framework by considering
  $\nu\mathcal{H} = \frac{3}{4}\lambda_\nu(k)\,e^{\kappa(k)\eta}\mathcal{H}^2, \quad c_s^2 = \lambda_s(k)\,e^{\kappa(k)\eta}\mathcal{H}^2$.
- The scale dependence of the parameters can be determined through renormalization-group methods. This requires a functional representation of the problem.

Functional representation

S. Floerchinger, M. Garny, N. T., U. Wiedemann
arXiv:1607.03453[astro-ph.CO], JCAP 1701 no.01, 048 (2017)

- We follow and extend Matarrese, Pietroni 2007.
- We are interested in solving

$$\partial_\eta \phi_a(\mathbf{k}) = -\Omega_{ab}(\mathbf{k},\eta)\phi_b(\mathbf{k}) + \int d^3p\, d^3q\, \delta^{(3)}(\mathbf{k}-\mathbf{p}-\mathbf{q})\gamma_{abc}(\mathbf{p},\mathbf{q},\eta)\,\phi_b(\mathbf{p})\,\phi_c(\mathbf{q})$$

with stochastic initial conditions determined by the primordial power spectrum $P^0_{ab}(k)$.

- This can be achieved by computing the generating functional

$$Z[J,K;P^0] = \int \mathcal{D}\phi\mathcal{D}\chi \exp\Big\{ -\frac{1}{2}\chi_a(0)P^0_{ab}\chi_b(0) + i\int d\eta\,[\chi_a(\delta_{ab}\partial_\eta + \Omega_{ab})\phi_b - \gamma_{abc}\chi_a\phi_b\phi_c + J_a\phi_a + K_b\chi_b]\Big\}$$

- One can now define the generating functional of connected Green's functions

$$W[J,K;P^0] = -i\log Z[J,K;P^0].$$

- The full power spectrum P_{ab} and the propagator G_{ab} can be obtained through second functional derivatives of W,

$$\begin{aligned}
\left.\frac{\delta^2 W}{\delta J_a(-\mathbf{k},\eta)\,\delta J_b(\mathbf{k}',\eta')}\right|_{J,\,K=0} &= i\delta(\mathbf{k}-\mathbf{k}')\,P_{ab}(\mathbf{k},\eta,\eta')\,, \\
\left.\frac{\delta^2 W}{\delta J_a(-\mathbf{k},\eta)\,\delta K_b(\mathbf{k}',\eta')}\right|_{J,\,K=0} &= -\delta(\mathbf{k}-\mathbf{k}')G^R_{ab}(\mathbf{k},\eta,\eta')\,, \\
\left.\frac{\delta^2 W}{\delta K_a(-\mathbf{k},\eta)\,\delta K_b(\mathbf{k}',\eta')}\right|_{J,\,K=0} &= 0\,.
\end{aligned}$$

- The effective action is the Legendre transform

$$\Gamma[\phi,\chi;P^0] = \int d\eta d^3\mathbf{k}\,\{J_a\phi_a + K_b\chi_b\} - W[J,K;P^0],$$

where $\phi_a(\mathbf{k},\eta) = \delta W/\delta J_a(\mathbf{k},\eta)$, $\quad \chi_b(\mathbf{k},\eta) = \delta W/\delta K_b(\mathbf{k},\eta)$.

- The inverse retarded propagator satisfies

$$\int d\eta'\; D^R_{ab}(\mathbf{k},\eta,\eta')G^R_{bc}(\mathbf{k},\eta',\eta'') = \delta_{ac}\,\delta(\eta-\eta'')\,.$$

- It can be computed from the effective action

$$\left.\frac{\delta^2\Gamma}{\delta\phi_a(-\mathbf{k},\eta)\,\delta\phi_b(\mathbf{k}',\eta')}\right|_{J,\,K=0} = 0\,,$$

$$\left.\frac{\delta^2\Gamma}{\delta\chi_a(-\mathbf{k},\eta)\,\delta\phi_b(\mathbf{k}',\eta')}\right|_{J,\,K=0} = -\delta(\mathbf{k}-\mathbf{k}')D^R_{ab}(\mathbf{k},\eta,\eta')\,,$$

$$\left.\frac{\delta^2\Gamma}{\delta\chi_a(-\mathbf{k},\eta)\,\delta\chi_b(\mathbf{k}',\eta')}\right|_{J,\,K=0} = -i\delta(\mathbf{k}-\mathbf{k}')H_{ab}(\mathbf{k},\eta,\eta')$$

- "Renormalized" field equations

$$\frac{\delta}{\delta\phi_a(\mathbf{x},\eta)}\Gamma[\phi,\chi] = J_a(\mathbf{x},\eta),$$
$$\frac{\delta}{\delta\chi_a(\mathbf{x},\eta)}\Gamma[\phi,\chi] = K_a(\mathbf{x},\eta),$$

- For vanishing source fields $J = K = 0$, we have $\chi = 0$ and the first equation is trivially satisfied.

Renormalization-group improvement

- Modify the initial power spectrum so that it includes only modes with wavevectors $|\mathbf{q}|$ larger than the coarse-graining scale k:

$$P^0_k(\mathbf{q}) = P^0(\mathbf{q})\,\Theta(|\mathbf{q}| - k).$$

- The coarse-grained effective action satisfies the Wetterich equation

$$\partial_k \Gamma_k[\phi,\chi] = \frac{1}{2}\,\mathrm{Tr}\left\{\left(\Gamma^{(2)}_k[\phi,\chi] - i\left(P^0_k - P^0\right)\right)^{-1}\partial_k P^0_k\right\}.$$

- Use an ansatz of the form

$$\Gamma_k[\phi,\chi] = \int d\eta \Bigg[\int d^3q\, \chi_a(-\mathbf{q},\eta) \left(\delta_{ab}\partial_\eta + \hat{\Omega}_{ab}(\mathbf{q},\eta)\right) \phi_b(\mathbf{q},\eta) - \int d^3k\, d^3p\, d^3q\, \delta^{(3)}(\mathbf{k}-\mathbf{p}-\mathbf{q})\gamma_{abc}(\mathbf{k},\mathbf{p},\mathbf{q})\chi_a(-\mathbf{k},\eta)\phi_b(\mathbf{p},\eta) - \frac{i}{2}\int d^3q\, \chi_a(\mathbf{q},\eta) H_{ab,k}(\mathbf{q},\eta,\eta')\chi_b(\mathbf{q},\eta') + \ldots \Bigg],$$

where

$$\hat{\Omega}(\mathbf{q},\eta) = \begin{pmatrix} 0 & -1 \\ -\frac{3}{2}\Omega_m + \lambda_s(k)\, e^{\kappa(k)\eta}\, \mathbf{q}^2 & 1 + \frac{\mathcal{H}'}{\mathcal{H}} + \lambda_\nu(k)\, e^{\kappa(k)\eta}\, \mathbf{q}^2 \end{pmatrix}.$$

- Derive differential equations for the k-dependence of $\lambda_\nu(k)$, $\lambda_s(k)$, $\kappa(k)$.

Comments

- A prescription is needed in order to project the general form of the inverse retarded propagator

$$D^R_{ab}(\mathbf{q},\eta,\eta') = \delta_{ab}\delta'(\eta-\eta') + \Omega_{ab}(\mathbf{q},\eta)\delta(\eta-\eta') + \Sigma^R_{ab}(\mathbf{q},\eta,\eta')$$

  to the form

$$D^R_{ab}(\mathbf{q},\eta,\Delta\eta) = \left(\delta_{ab}\partial_\eta - \hat{\Omega}_{ab}(\mathbf{q},\eta)\right)\,\delta(\eta-\eta')\,.$$

- The projection is performed through a Laplace transform.
- At the first order of an iterative solution of the exact RG equation, one finds

$$\lambda_s(k) = \frac{31}{70}\sigma^2_{dk}, \qquad \lambda_\nu(k) = \frac{78}{35}\sigma^2_{dk}, \qquad \kappa(k) = 2.$$

  This validates our intuitive matching through the propagator.
- At the next level, the k-dependence of $\lambda_\nu(k)$, $\lambda_s(k)$, $\kappa(k)$ can be derived.

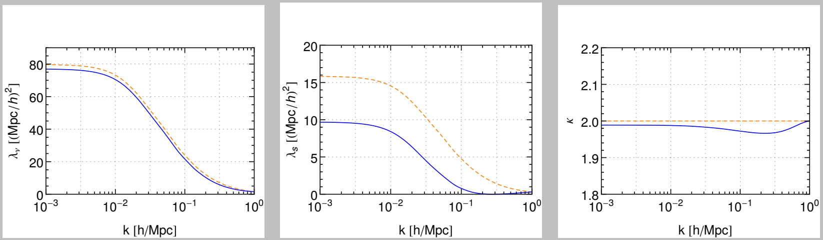

Figure: RG evolution of $\lambda_\nu(k)$, $\lambda_s(k)$ and $\kappa(k)$. We have initialized the flow at $k = \Lambda = 1\ h/$ Mpc with the one-loop values. The solid lines correspond to the solution of the full flow equations, while the dashed lines correspond to the solution of the one-loop approximation.

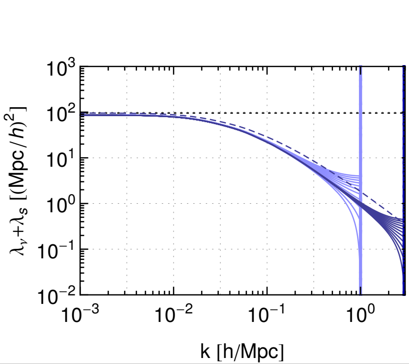

Figure: RG evolution of the sum $\lambda_\nu(k) + \lambda_s(k)$. The various lines correspond to the RG evolution obtained when imposing initial values at $\Lambda = 1\ h/\text{Mpc}$ (light blue) or $\Lambda = 3\ h/\text{Mpc}$ (dark blue), respectively. The dashed line shows the perturbative one-loop estimate for comparison.

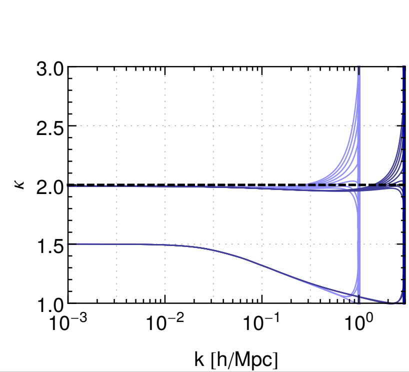

Figure: RG evolution of the power law index $\kappa(k)$ characterizing the time-dependence of the effective sound velocity and viscosity. The various blue lines show the RG evolution when initializing the RG flow at $\Lambda = 1\, h/\mathrm{Mpc}$ (light blue) or $\Lambda = 3\, h/\mathrm{Mpc}$ (dark blue), respectively.

Conclusions

- The nature of dark matter is still unknown. It is reasonable to consider possibilities beyond an ideal, pressureless fluid.
- Standard perturbation theory cannot describe reliably the short-distance cosmological perturbations.
- It is possible to "integrate out" the short-distance modes in order to obtain an effective description of the long-distance modes. One must allow for nonzero speed of sound and viscosity, whose form and time-dependence can be computed through the FRG.
- The nonlinear spectrum computed through the effective theory is in good agreement with results from N-body simulations.
- Perturbation theory seems to converge quickly for the effective theory if the UV cutoff is taken in the region $0.4 - 1\ h/$ Mpc.