# Dynamics of net-baryon density correlations near the QCD critical point

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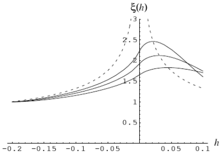
*From Correlation Functions to QCD Phenomenology* – Bad Honnef, Germany – April 4, 2018

# Dynamical effects are very important...

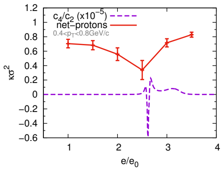
- **At the critical point:** $\xi \to \infty \Rightarrow$ fluctuations of the critical mode diverge!
- Higher moments more sensitive to $\xi$:

$$\langle \Delta\sigma^2 \rangle \propto \xi^2 \,, \quad \langle \Delta\sigma^3 \rangle \propto \xi^{9/2} \,, \quad \langle \Delta\sigma^4 \rangle_c \propto \xi^7 \,.$$

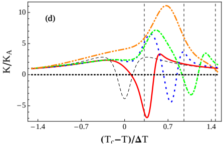
- Relaxation time $\tau_{\rm rel} \propto \xi^z$ diverges $\Rightarrow$ critical slowing down!




B. Berdnikov, K. Rajagopal PRD61 (2000)




C. Herold, MN, Y. Yan and C. Kobdaj PRC93 (2016) no.2




S. Mukherjee, R. Venugopalan, Y. Yin PRC92 (2015)

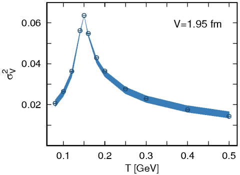
$\Rightarrow$ Matter created in a heavy-ion collision is small, short-lived and highly dynamical!

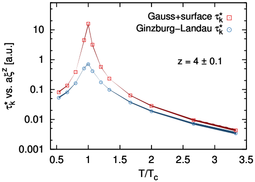
$\Rightarrow$ Study fluctuations within diffusive dynamics of net-baryon number!

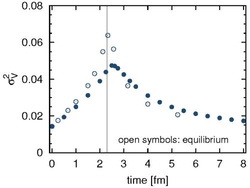
# Goal of this talk

Present a numerical implementation of the diffusive dynamics of net-baryon density fluctuations near the QCD critical point:

- thoroughly tested against equilibrium expectations
- discuss effects of resolution, system-size and net-baryon charge conservation




⇒ deviations in scaling behavior of fluctuation observables from infinite-volume expectations! Stephanov, Phys.Rev.Lett.102 (2009)

⇒ dynamical critical scaling of model B! Hohenberg, Halperin, Rev.Mod.Phys.49 (1977)

⇒ nonequilibrium and retardation effects in the dynamics!

M. Nahrgang, MB, T. Schäfer, S. Bass, in preparation

# Diffusive dynamics of the net-baryon density

$$\partial_\mu N_B^\mu = 0 \quad \text{net-baryon charge conservation}$$

The diffusive dynamics follows the minimized free energy $\mathcal{F}$:

$$\partial_t n_B(t,x) = \kappa \nabla^2 \left( \frac{\delta \mathcal{F}[n_B]}{\delta n_B} \right)$$

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$\Rightarrow$ respects the fluctuation-dissipation theorem:

$$P_{\text{eq}}[n_B] = \tfrac{1}{\mathcal{Z}} \exp\left(-\mathcal{F}[n_B]/T\right)$$

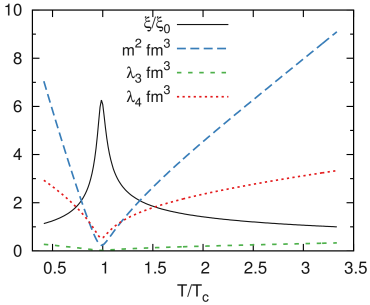
# Couplings motivated by 3-dimensional Ising model

$$\mathcal{F}[n_B] = T\int \mathrm{d}^3 r \left( \frac{m^2}{2n_c^2}\Delta n_B^2 + \frac{K}{2n_c^2}(\nabla \Delta n_B)^2 + \frac{\lambda_3}{3n_c^3}\Delta n_B^3 + \frac{\lambda_4}{4n_c^4}\Delta n_B^4 + \frac{\lambda_6}{6n_c^6}\Delta n_B^6 \right)$$

The couplings depend on temperature via $\xi(T)$:

$$m^2 = 1/(\xi_0 \xi^2)$$
$$K = \tilde{K}/\xi_0$$
$$\lambda_3 = n_c\,\tilde{\lambda}_3\,(\xi/\xi_0)^{-3/2}$$
$$\lambda_4 = n_c\,\tilde{\lambda}_4\,(\xi/\xi_0)^{-1}$$
$$\lambda_6 = n_c\,\tilde{\lambda}_6$$

M. Tsypin PRL73 (1994); PRB55 (1997)




in this Fig.: $\tilde{\lambda}_3 = 1$, $\tilde{\lambda}_4 = 10$

parameter choice: $\Delta n_B = n_B - n_c$

$\xi_0 \sim 0.5$ fm, $T_c = 0.15$ GeV, $n_c = 1/3$ fm$^{-3}$

$K = 1/\xi_0$ (surface tension)

$\tilde{\lambda}_3$, $\tilde{\lambda}_4$, $\tilde{\lambda}_6$ (universal, but mapping to QCD)

# Different physical scenarios

The diffusion equation:

$$\partial_t n_B = \frac{D}{n_c}\left(m^2 - K\nabla^2\right)\nabla^2 n_B$$
$$+D\nabla^2\left(\frac{\lambda_3}{n_c^2}\Delta n_B^2 + \frac{\lambda_4}{n_c^3}\Delta n_B^3 + \frac{\lambda_6}{n_c^5}\Delta n_B^5\right) + \sqrt{2Dn_c}\nabla\zeta$$

# Different physical scenarios

The diffusion equation:

**Gauss+surface**

**Ginzburg-Landau**

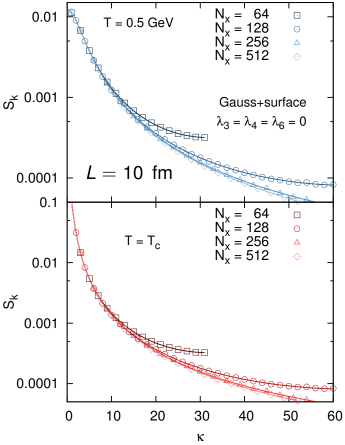
**noise**

$$\partial_t n_B = \frac{D}{n_c}\left(m^2 - K\nabla^2\right)\nabla^2 n_B + D\nabla^2\left(\frac{\lambda_3}{n_c^2}\Delta n_B^2 + \frac{\lambda_4}{n_c^3}\Delta n_B^3 + \frac{\lambda_6}{n_c^5}\Delta n_B^5\right) + \sqrt{2Dn_c}\nabla\zeta$$

- consider $3+1$d system with propagation in 1 spatial dimension
- diffusion coefficient $D = \kappa T / n_c$
  - for equilibrium calculations $D = 1$ fm
  - for dynamical calculations $T$-dependent
- equilibrium: let system equilibrate for long times
- dynamics: equilibrate first at high $T$, then evolve
- $\langle N_B \rangle$ perfectly conserved in the numerics!

# Equilibrium fluctuations

# Gaussian model - static structure factor




static structure factor in continuum ($\xi^2 = K/m^2$):

$$S(k) = \frac{n_c^2}{m^2} \frac{1}{1 + \xi^2 k^2}$$

discretized form ($k = 2\pi\kappa/L$):

(semi-implicit predictor-corrector scheme)

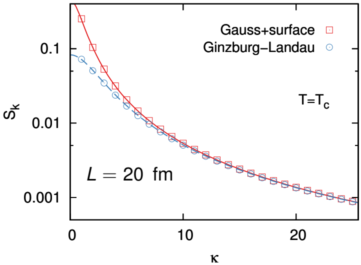
$$S_k = \frac{n_c^2}{m^2} \frac{1}{1 + \frac{2K}{m^2 \Delta x^2} \left(1 - \cos(k\Delta x)\right)}$$

$\Rightarrow$ perfectly reproduced!

$\Rightarrow S_k \to S(k)$ for $\Delta x = L/N_x \to 0$!

non-zero surface tension suppresses short-wavelength fluctuations!

# Ginzburg-Landau model - static structure factor

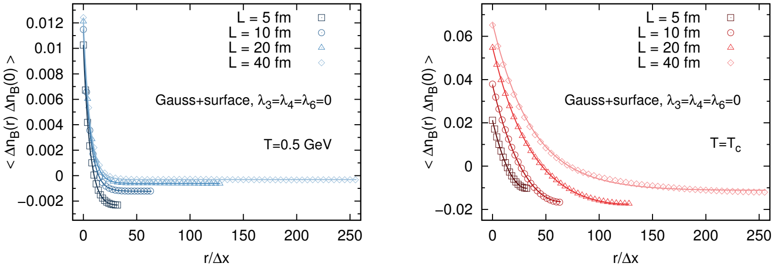



- nonlinear interactions reduce $S_k$ for long-wavelength fluctuations!

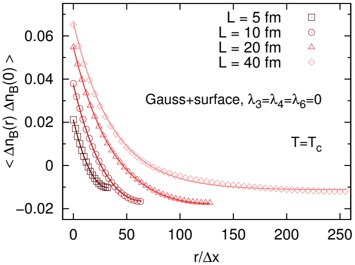
$\Rightarrow$ at this level, results of the Ginzburg-Landau model can effectively be described by a Gauss+surface model with modified $m^2$ ($K$ essentially unaffected)!

# Gaussian model - equal-time correlation function

- numerical correlation length $\xi > \Delta x$!
- spatial correlations broaden near $T_c$!



# Gaussian model - equal-time correlation function

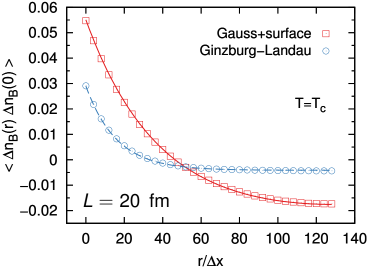



impact of exact $\langle N_B \rangle$-conservation: local $n_B$-fluctuations need to be balanced within $L$

$\int_L \mathrm{d}r \, \langle \Delta n_B(r) \Delta n_B(0) \rangle = 0 \;\rightarrow$ negative correction term expected

$\Rightarrow$ perfectly reproduced by the numerics!

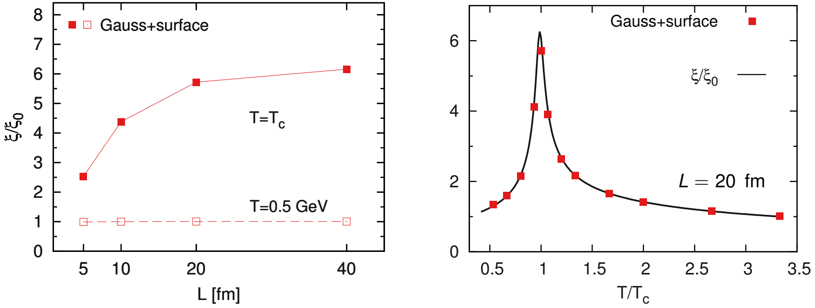
# Ginzburg-Landau model - correlation function




- spatial correlations are significantly smaller!
- less pronounced finite-size effects!

# Correlation length and finite-size effects

For finite size *L*, the numerical correlation length is strongly limited due to $\langle N_B \rangle$-conservation.




- very good realization of the relation $\xi(T) \propto 1/m(T)$!

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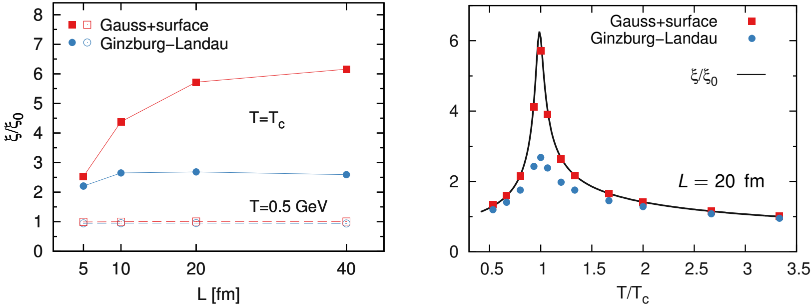
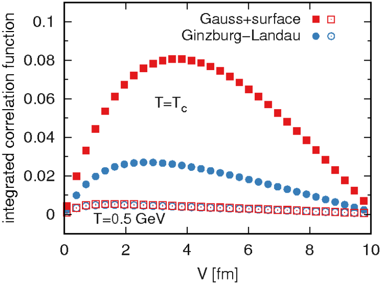



- very good realization of the relation $\xi(T) \propto 1/m(T)$!
- nonlinear mode couplings reduce the numerically realized correlation length in accordance with the effective $m^2$!

# Volume-integrated variance

- infinite-volume expectation:
  $\sigma_V^2 = \frac{1}{V^2} \int dx \int dy \, \langle \Delta n_B(x) \Delta n_B(y) \rangle \propto \xi^2$

M. Stephanov (et al.) PRL81 (1998); PRD60 (1999); PRL102 (2009)




$\Rightarrow$ exact charge conservation has a significant impact

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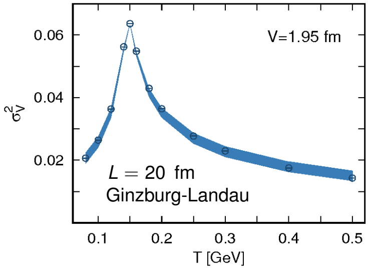


- including finite-size and $\langle N_B \rangle$-conservation we find for $V \simeq 2$ fm:

$$\sigma_V^2 \propto \xi^n \text{ with } n \simeq 1.4 \pm 0.1\,!$$

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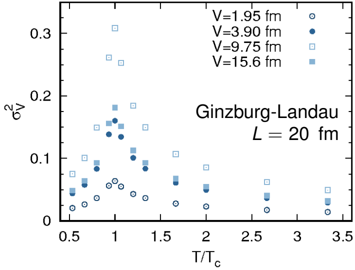


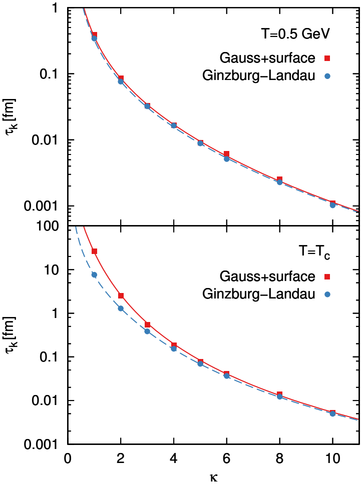

- increasing $V \rightarrow$ increase in $\sigma_V^2$ ($n$ gets closer to 2) before impact of charge conservation visible!

# Dynamics of fluctuations

# Dynamic structure factor and relaxation time

dynamic structure factor for Gaussian model in continuum:

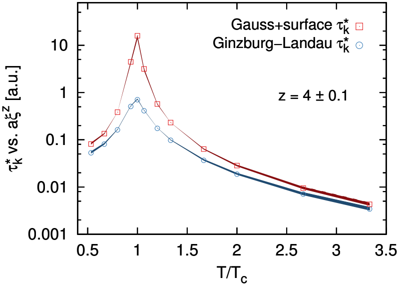
$$S(k,t) = S(k)\exp\left(-t/\tau_k\right) \quad \text{with} \quad \tau_k^{-1} = \frac{Dm^2}{n_c}\left(1+\frac{K}{m^2}k^2\right)k^2$$




- analytic results reproduced by numerics!
- long-wavelength modes relax slowly!
- $\tau_k$ significantly enhanced near $T_c$!
- nonlinear interactions reduce $\tau_k$ for modes with small $k$!

# Dynamical critical scaling

analyze $\xi$-dependence of relaxation time for modes with $k^* = 1/\xi$:




- for both models: $\tau_k^* = a\,\xi^z$
- best fit gives:

$$z = 4 \pm 0.1$$

$$a = \frac{n_c \xi_0}{D(1+\tilde{K})}$$

$\Rightarrow$ Simulations reproduce scaling of model B!

Hohenberg, Halperin, Rev.Mod.Phys.49 (1977)

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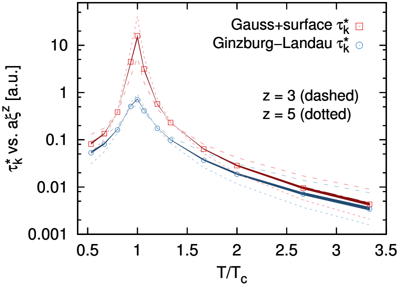


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- $z = 3,\ 5$ ruled out!

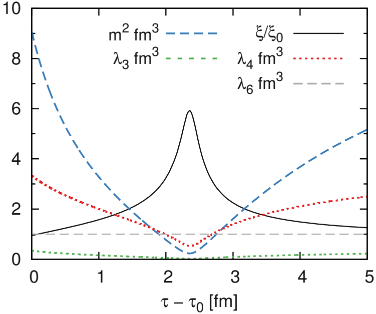
$\Rightarrow$ Simulations reproduce scaling of model B!

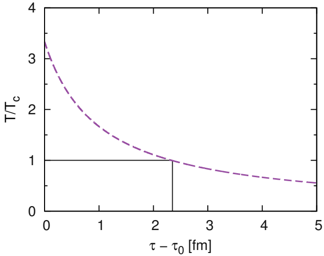
Hohenberg, Halperin, Rev.Mod.Phys.49 (1977)

# Time-dependent temperature

Hubble-like $T$-dynamics:

$$T(\tau) = T_0 \left(\frac{\tau_0}{\tau}\right)^{dc_s^2}$$

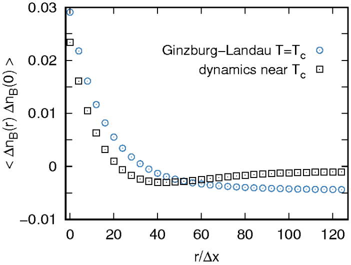


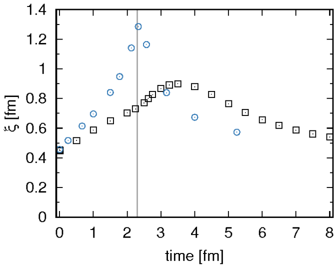




- choose $c_s^2 = 1/3$
- equilibrate system at $T_0 = 0.5$ GeV, $D_0 = 1$ fm
- $D(T) = D_0 T / T_0$
- $T_c$ reached at $\tau - \tau_0 = 2.33$ fm

# Correlation function and length

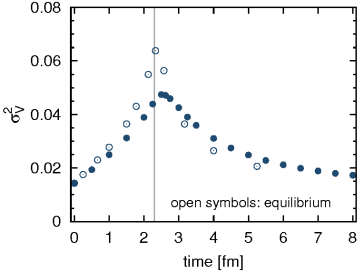
equal-time correlation function/length during the dynamics:




- clear deviations from equilibrium expectations
- rapid modification of $T \rightarrow$ retardation effect
- equilibrium magnitude of $\xi$ cannot develop $\rightarrow$ nonequilibrium

# Dynamics of the integrated variance




- shift of extremum (retardation effect)

  B. Berdnikov, K. Rajagopal PRD61 (2000), S. Mukherjee et al., PRC92(3) (2015)

- maximal value in dynamics smaller than equilibrium value (nonequilibrium effect)
- expected behavior with varying *D*

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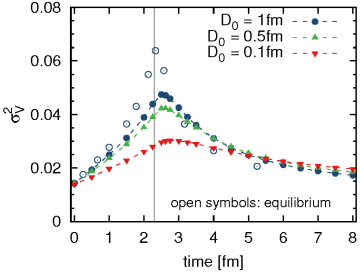


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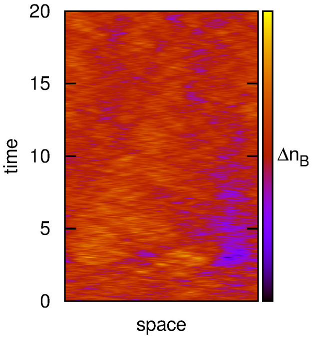
# Conclusions

- successful test of numerical implementation vs. analytic expectations!
- significant effect of net-baryon charge conservation!
- nonlinear interactions significantly reduce $\xi$
- finite size affects scaling behavior of fluctuation observables
- dynamics: critical scaling of model B, nonequilibrium and retardation effects!




Thanks to:

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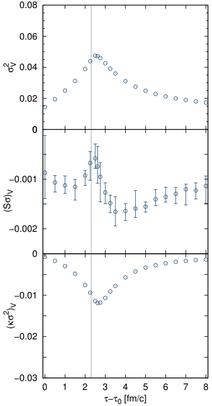



**talk by M. Nahrgang**

Thanks to:

# Fluid dynamical fluctuations in heavy-ion collisions

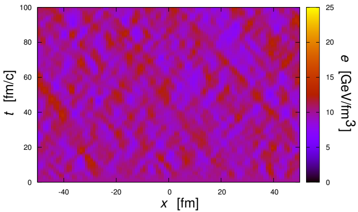
The average particle spectra are successfully described using conventional fluid dynamics. $\rightarrow$ What about fluctuations?

$$\partial_\mu T^{\mu\nu} = \partial_\mu \left( T^{\mu\nu}_{\rm eq} + \Delta T^{\mu\nu}_{\rm visc} + \Xi^{\mu\nu} \right) = 0$$

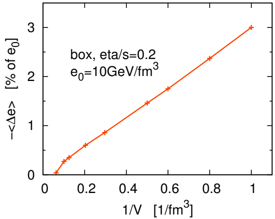
$$\partial_\mu N^{\mu} = \partial_\mu \left( N^{\mu}_{\rm eq} + \Delta N^{\mu}_{\rm visc} + I^{\mu} \right) = 0$$

- important consequence: nonlinearities cause cutoff dependent corrections to EoS, $\eta$ ... (e.g. $\langle \Delta e \rangle \sim -\Lambda^3$) P. Kovtun et al., JHEP1407 (2014)

box, $\Delta x = 1$ fm, $e_0 = 10$ GeV/fm$^3$, $\eta/s = 0.2$




based on vHLLE code



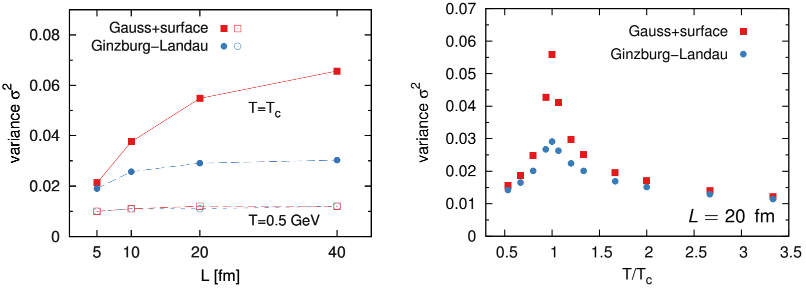

$\Rightarrow$ Implementing fluid dynamical fluctuations is important, but requires a sustained and systematic effort!

M. Nahrgang et al., CPOD2016

I. Karpenko et al., Comput.Phys.Commun.185 (2014)

# $T$-dependence of local Gaussian fluctuations

In finite systems: variance is reduced compared to TD expectations.

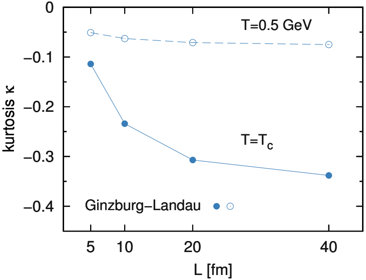


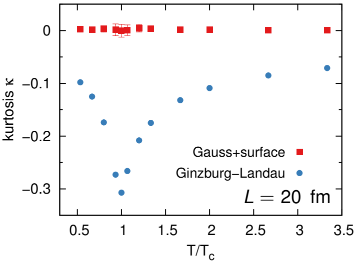

- nonlinear couplings in the Ginzburg-Landau model reduce the variance!

⇒ Could explain why no sign of criticality is observed in second-order moments at NA49 and RHIC BES phase I.

# $T$-dependence of local non-Gaussian fluctuations

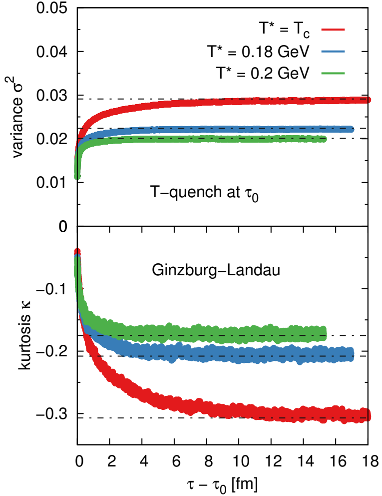
Kurtosis vanishes in the absence of nonlinear interactions!




- negative kurtosis observed for Ginzburg-Landau model with a pronounced signal near $T_c$!

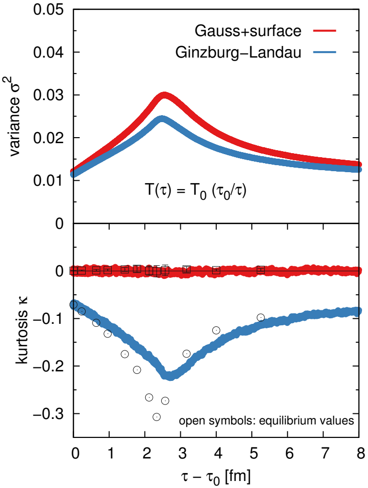
# Temperature quench and equilibration




- temperature quench: at $\tau_0$ temperature drops from $T_0 = 0.5$ GeV to $T^*$
- fast initial relaxation
- local variance approaches equilibrium value faster than local kurtosis
- long relaxation times near $T_c$

B. Berdnikov, K. Rajagopal PRD61 (2000)

# Dynamics of local (non-)Gaussian fluctuations



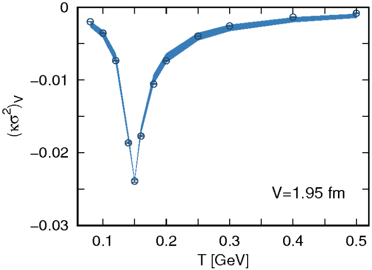

- shift of extrema for variance and kurtosis (retardation effect)

  B. Berdnikov, K. Rajagopal PRD61 (2000), S. Mukherjee et al., PRC92(3) (2015)

- |extremal values| in dynamical simulations smaller than equilibrium values (nonequilibrium effect)
- no dynamical creation of non-Gaussianity in Gauss+surface model
- expected behavior with varying *D*

# Volume-integrated kurtosis

- infinite-volume expectation: $(\kappa\sigma^2)_V \propto \xi^5$

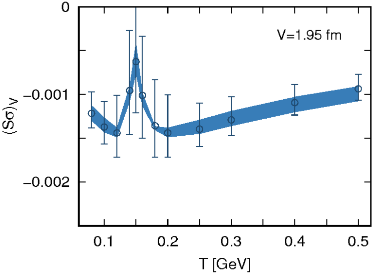



- including finite-size and $\langle N_B \rangle$-conservation we find for $V \simeq 2$ fm:

$$(\kappa\sigma^2)_V \propto \xi^n \text{ with } n \simeq 2.9 \pm 0.2\,!$$

# Volume-integrated skewness

- infinite-volume expectation: $(S\sigma)_V \propto \xi^{2.5}$




- including finite-size and $\langle N_B \rangle$-conservation we find for $V \simeq 2$ fm: $(S\sigma)_V = a\xi^n - b\xi^m$ with $n \simeq 1.47 \pm 0.05$ and $m \simeq 2.40 \pm 0.05\,!$