# 2-, 3- and 4-point functions in 2, 3 and 4 dimensions

Markus Q. Huber

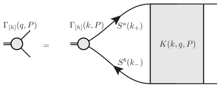
Institute of Theoretical Physics, Giessen University
Institute of Physics, University of Graz

666. WE-Heraeus-Seminar - From correlation functions to QCD phenomenology
Bad Honnef, Germany

April 3, 2018

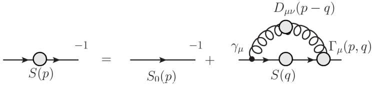
# Hadronic bound states

Bound state equations:




Ingredients:

- Interaction kernel $K$
- Quark propagator $S$



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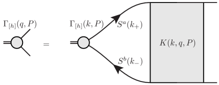


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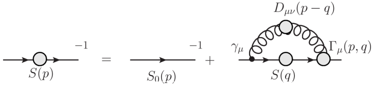


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  Piecing together the pieces

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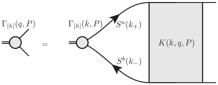


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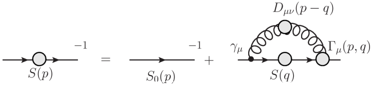


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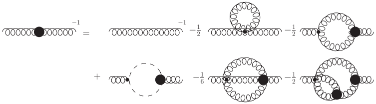
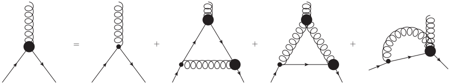


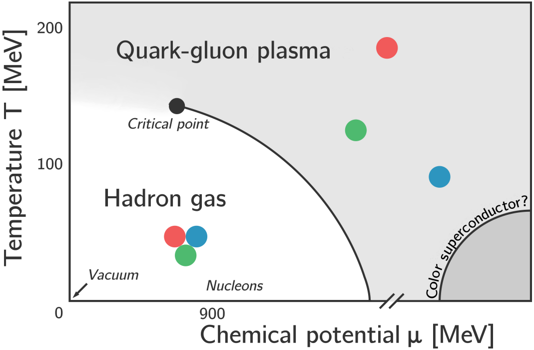


$\rightarrow$ Couples to infinity of equations.

# QCD phase diagram

Questions:

- Phases and transitions between them, critical point
- Experimental signatures




*Alexander Gorfer (quant.uni-graz.at), (CC-BY-SA 4.0)*

Theoretical challenges:

- Model description
- Mathematical, e.g., complex action for lattice QCD
- Complexity, e.g., truncations of function eqs.
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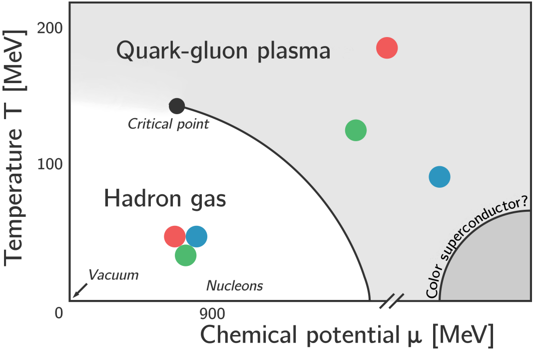




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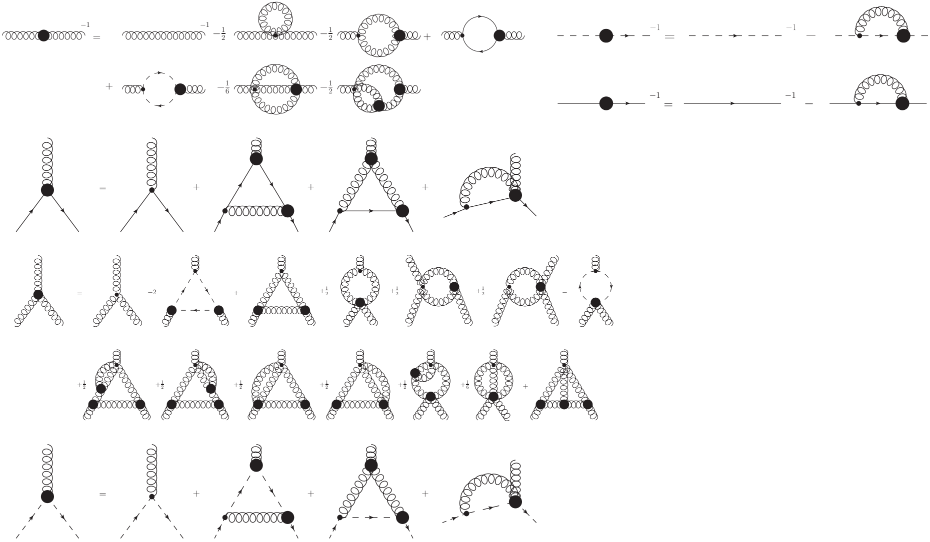
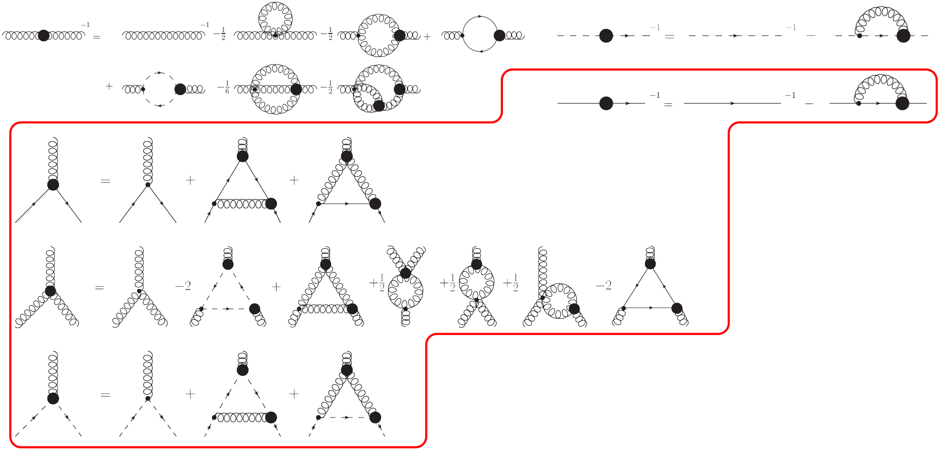
- How to realize resummation?
  higher loop contributions?
- Equivalence between different functional methods?
  FRG, DSEs, nPI, Hamiltonian approach

# Dyson-Schwinger equations



# Coupled systems of Dyson-Schwinger equations



quark propagator + 3-point functions: [Williams, Fischer, Heupel '15] $\rightarrow$ application to bound states

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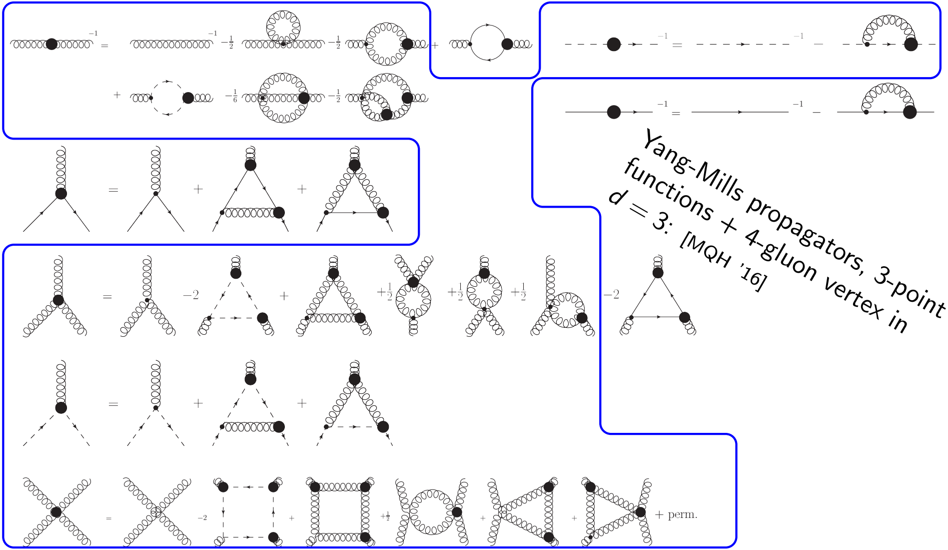
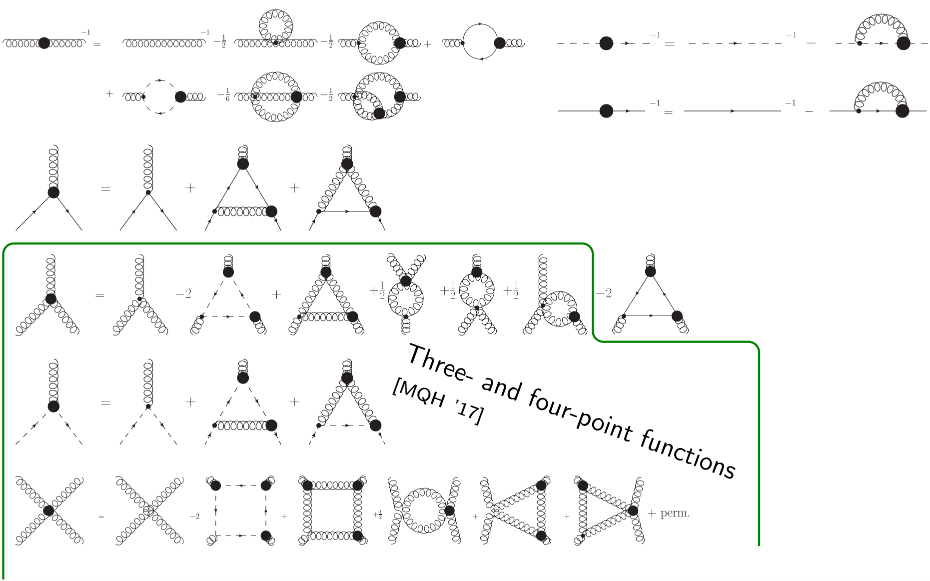



Yang-Mills propagators, 3-point functions + 4-gluon vertex in $d = 3$: [MQH '16]

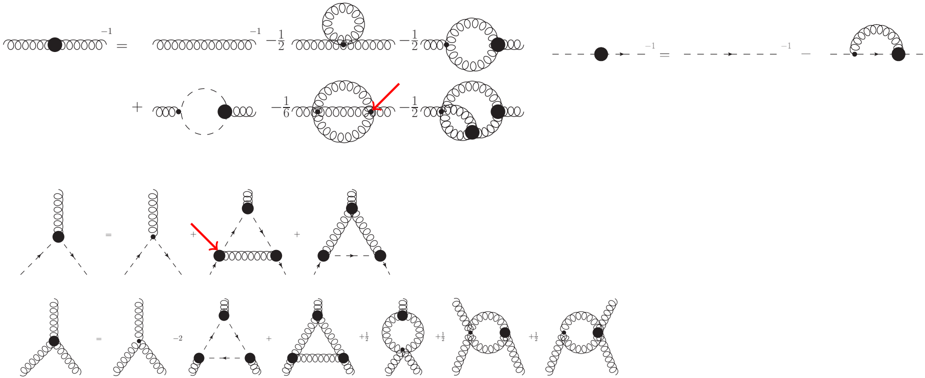
# Coupled systems of Dyson-Schwinger equations




Three- and four-point functions
[MQH '17]

# 3PI system of equations

Three-loop expansion of PI effective action [Berges '04]:



# UV behavior of the gluon propagator

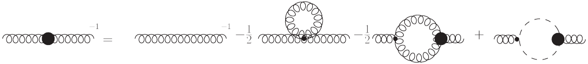
Resummed one-loop order: anomalous dimension $\gamma = -13/22$

$$\left(1 + \frac{\alpha(s)11N_c}{12\pi}\ln\frac{p^2}{s}\right)^{\gamma}$$

## One-loop anomalous dimension

Origin in resummation of higher order diagrams.

However, one-loop truncation discards some terms.



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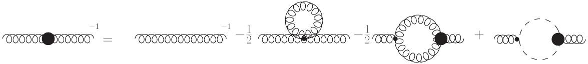
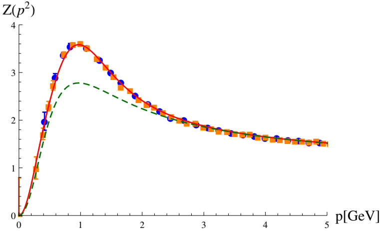
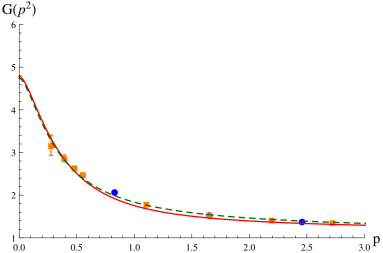


$\rightarrow$ Puts constraints on UV behavior of vertices [von Smekal, Hauck, Alkofer '97].
Way out: Include in models (for now).

# Propagators and ghost-gluon vertex with three-gluon vertex model

One-loop truncation of gluon propagator with an optimized effective model (contains zero crossing) [MQH, von Smekal '13]:

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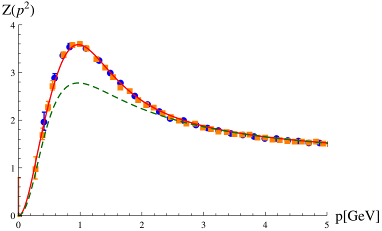


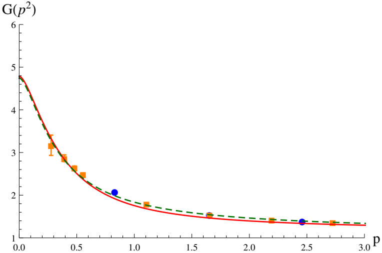


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$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \sum_j \bar{\psi}_j\left[i\gamma^\mu D_\mu - m_j\right]\psi_j$$

WOBEI $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\left[A_\mu, A_\nu\right]$

UND $D_\mu = \partial_\mu + igA_\mu$

# Resummed behavior

Minimal requirements to obtain one-loop resummed behavior:

- Squint diagram
- Correct anomalous dimensions of three-point functions
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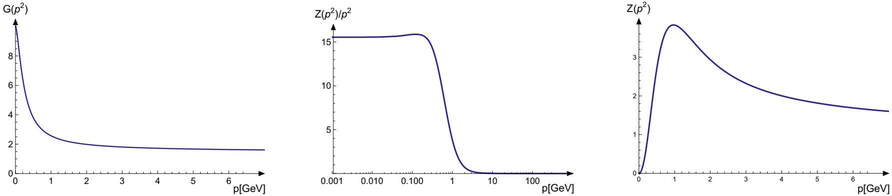
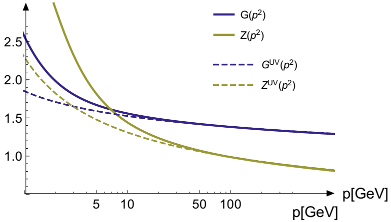



[propagator+ghost-gluon eqs. full, 3-gluon vertex model, bare 4-gluon vertex]

- Resummed behavior is recovered [MQH '17].



# Lower dimensional Yang-Mills theories as testing ground

Advantages:

- UV finite: no renormalization, no anomalous running
- Spurious divergences easier to handle
- UV behavior 'easier': $\propto \frac{g^2}{p^n}$ instead of resummed logarithm

$\rightarrow$ Many complications from $d = 4$ absent.
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Historically interesting because cheaper on the lattice $\rightarrow$ easier to reach the IR.

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- Only scaling solution exists

  [Cucchieri, Dudal, Vandersickel '12; MQH, Maas, von Smekal '12; Zwanziger '12]
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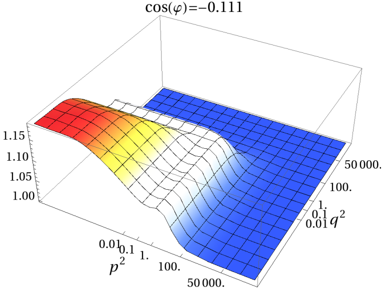
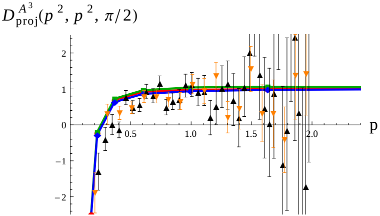
Ghost-gluon vertex:




[MQH, Maas, von Smekal '12]

Three-gluon vertex:




[Maas '07; MQH, Maas, von Smekal '12]

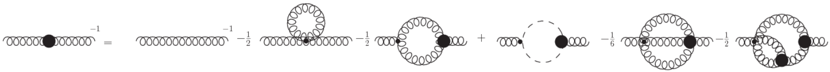
# Three dimensions

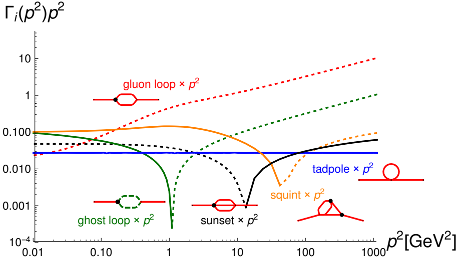
- Four-point functions numerically cheaper.
- Perturbation theory works.

Continuum results:

- Coupled propagator DSEs: [Maas, Wambach, Grüter, Alkofer '04]
- (R)GZ: [Dudal, Gracey, Sorella, Vandersickel, Verschelde '08]
- DSEs of PT-BFM: [Aguilar, Binosi, Papavassiliou '10]
- YM + mass term: [Tissier, Wschebor '10, '11]

# Gluon propagator: Single diagrams





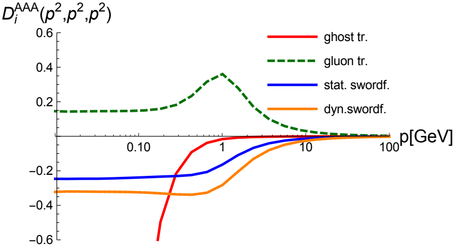

[MQH '16]

→ Clear hierarchies identified.

- UV: as expected perturbatively
- non-perturbative: squint important, sunset small (d=4: [Mader, Alkofer '13; Meyers, Swanson '14])

# Cancellations in gluonic vertices

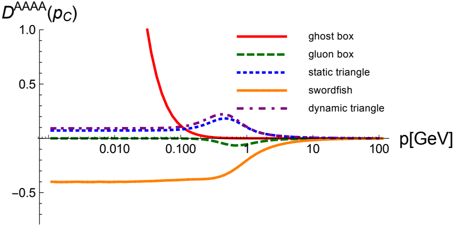
## Three-gluon vertex:




[MQH '16]

- Individual contributions large.
- Sum is small!

## Four-gluon vertex:



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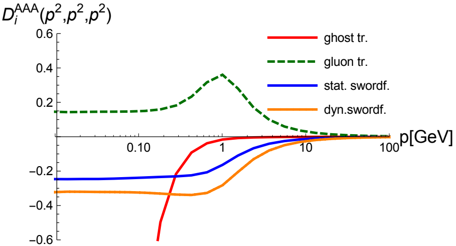


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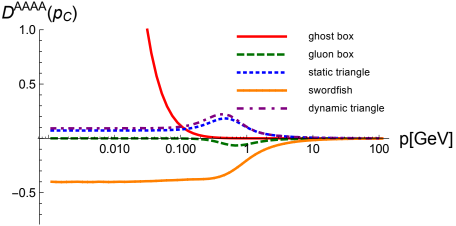


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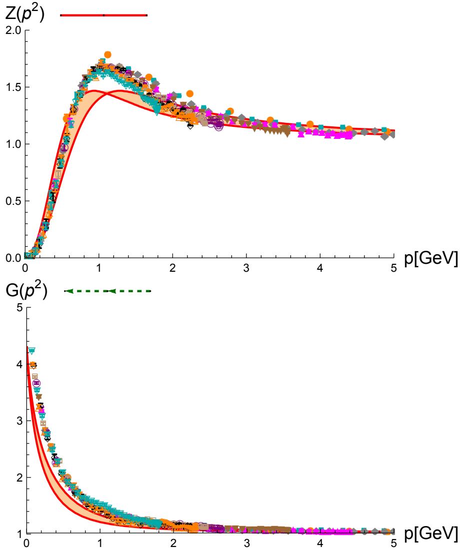
- Sum is small!

$\Downarrow$

Higher contributions:

- Higher vertices close to 'tree-level'? $\rightarrow$ Small.
- If pattern changes (higher vertices large): cancellations required.

# Results: Propagators

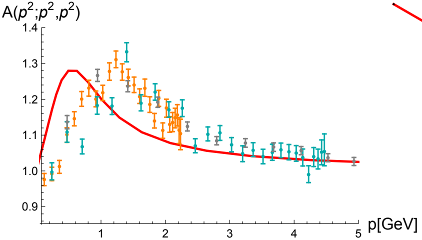



Bands from uncertainty in setting the physical scale.

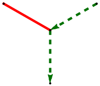
[MQH '16; lattice: Maas '14]

# Comparison of three-point functions with lattice results

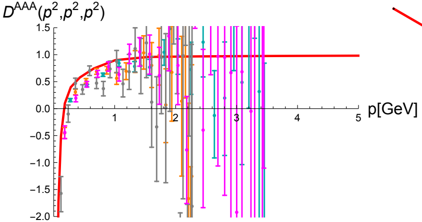
Dressings:



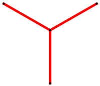

[MQH '16; lattice: Maas, unpublished]



- Deviation from tree-level 'small'
- Position of maximum shifted (as observed with other continuum methods for $SU(2)$, e.g., [Pelaez, Matthieu, Wschebor '13; Cyrol, Fister, Mitter, Pawlowski, Strodthoff '15; Corell '18]).



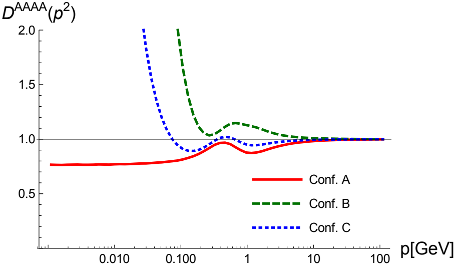

[MQH '16; lattice: Cucchieri, Maas, Mendes '08]



- Close to tree-level above 1 GeV
- Good agreement with lattice data.
- Linear IR divergence [Pelaez, Tissier, Wschebor '13; Aguilar et al. '13]

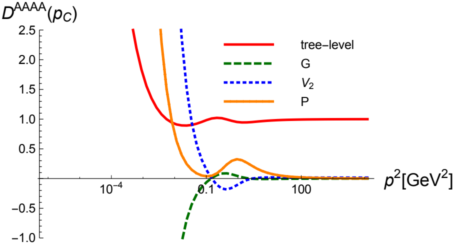
# Four-gluon vertex

## Different momentum configurations:




[MQH '16]

## Different projections:




## Four-gluon vertex:

- Close to tree-level down to 1 GeV

# Solution from the 3PI effective action

Different set of functional equations:
Equations of motion from 3PI effective action (at three-loop level)

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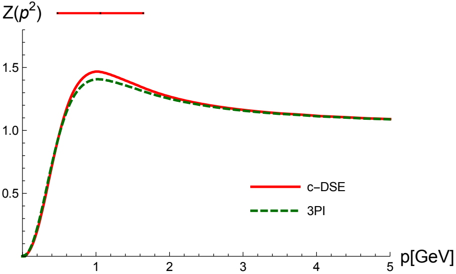
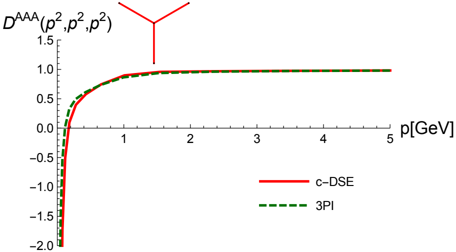


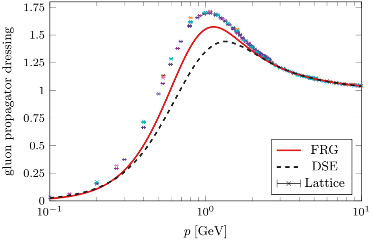



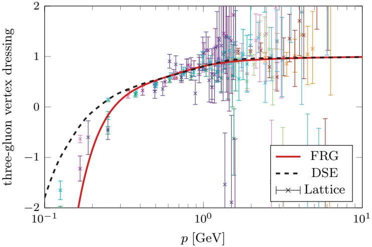

$\rightarrow$ Very similar results.

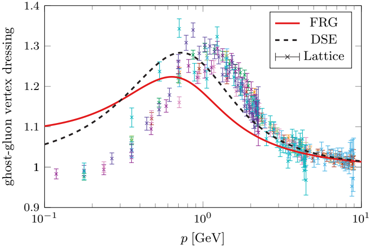
[MQH '16]

# Solutions from the FRG

FRG calculations by Corell, Cyrol, Mitter, Pawlowski, Strodthoff, arXiv:1803.10092




NB: Scaling (FRG) and decoupling (DSEs)

- FRG has 'additional' diagrams (tadpoles).
- Equivalence of truncations not trivial.

[Cucchieri, Maas, Mendes '08; MQH '16; Corell et al. '18; Maas, unpublished]

# Conclusions from three dimensions

- Hierarchy of correlation functions and diagrams
- Cancellations lead to small deviations from the perturbative behavior above 2 $GeV$.
- Some degree of stability (but no complete list of checks done) when
  - varying *system* of equations.
  - varying *equations* of system.
- Discrepancies with lattice results:
  - Nonperturbative gauge fixing?
  - Missing diagrams for vertices?
  - Incomplete tensor bases for some vertices?

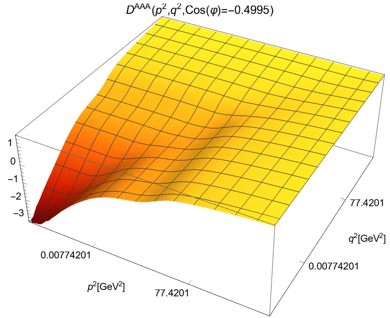
# Extending truncations

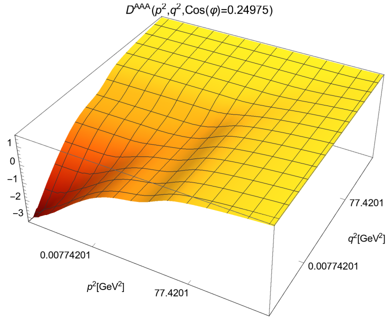
Various ways to extend truncations:

- Vertex tensors beyond tree-level
- Neglected diagrams
- Neglected correlation functions

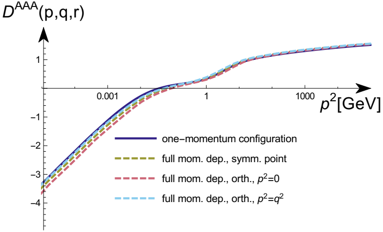
Extensions also test the previous truncations!

# Three-gluon vertex: Kinematic dependence



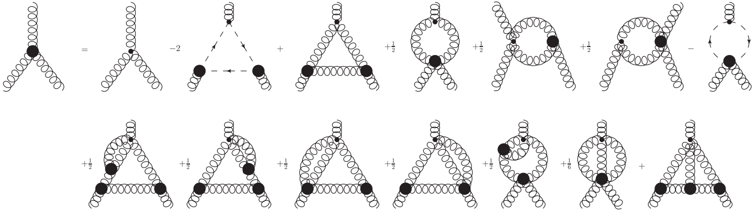




- Kinematic dependence weak.
- In the following: One-momentum approximation



# Three-gluon vertex DSE

Full DSE:



# Three-gluon vertex DSE

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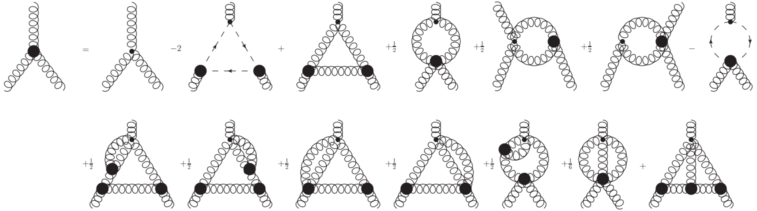
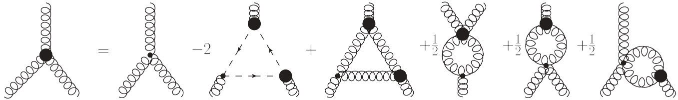


Perturbative one-loop truncation [Blum, MQH, Mitter von Smekal '14; Eichmann, Alkofer, Vujinovic '14]:



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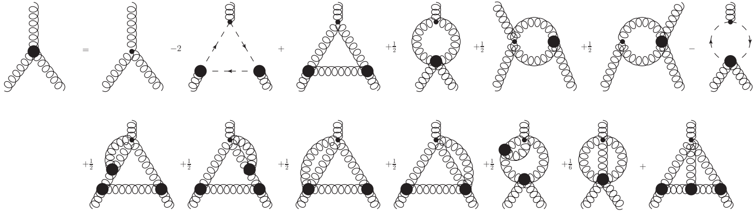

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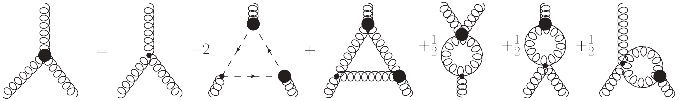
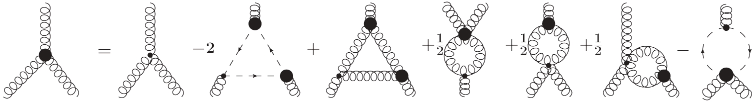
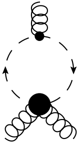


Non-perturbative one-loop truncation [MQH '17]:



# Influence of two-ghost-two-gluon vertex



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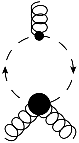
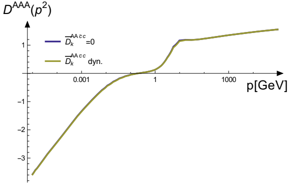


Coupled system of ghost-gluon, three-gluon and four-gluon vertices with and without two-ghost-two-gluon vertex [MQH '17]:



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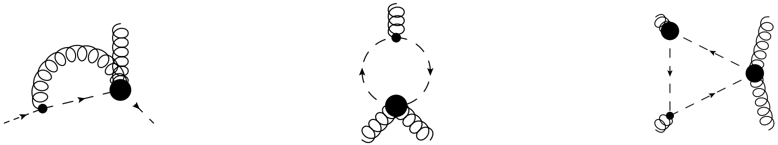

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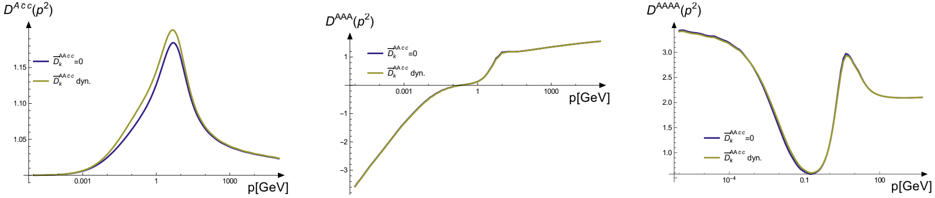
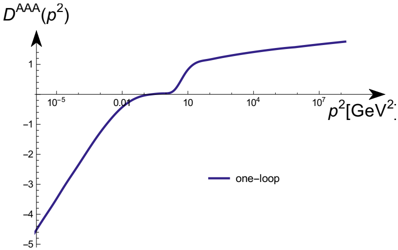



- Small influence on ghost-gluon vertex ($< 1.7\%$)
- Negligible influence on three- and four-gluon vertices.

# Three-gluon vertex results



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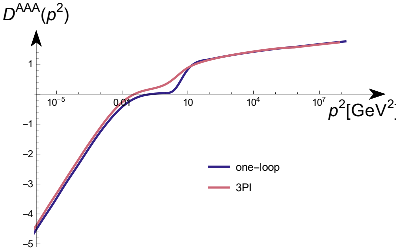

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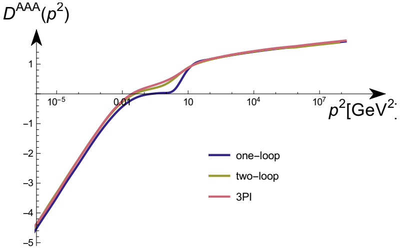


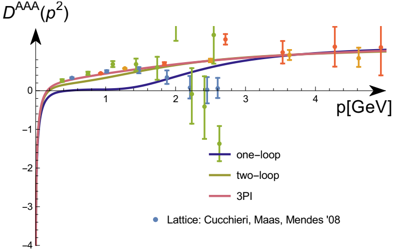

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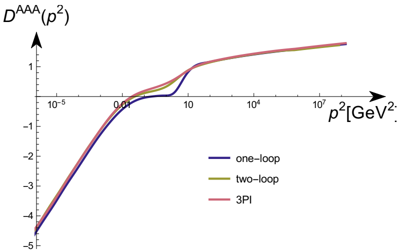
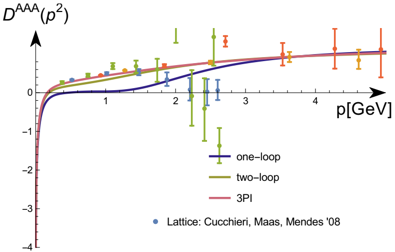
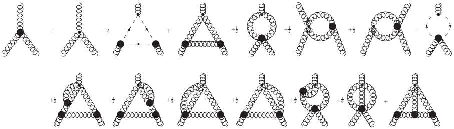



- Two-loop truncation: All diagrams except the one with a five-point function.



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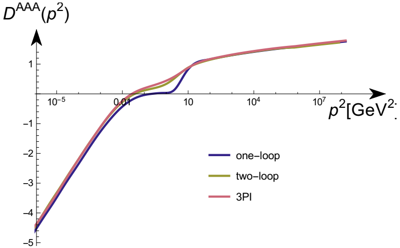
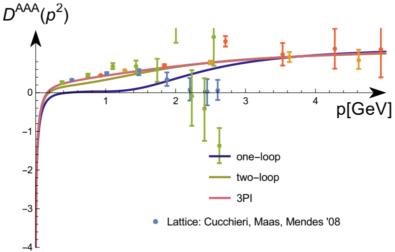
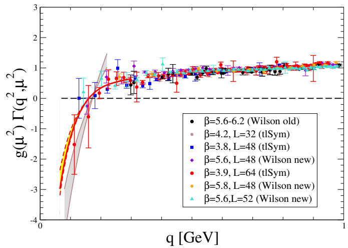



- Difference between two-loop DSE and 3PI smaller than lattice error.
- Resolves ambiguity in zero crossing due to RG improvement [Blum et al. '14; Eichmann et al. '14; Williams et al. '16]
- Zero crossing in agreement with other approaches, e.g., [Pelaez et al. '13; Aguilar et al. '13; Athenodorou et al. '16; Duarte et al. '16; Sternbeck et al. '17]




[Athenodorou et al. '16, '18]

# Four-point functions: Color space

15 possibilities:

$$\delta\,\delta: \quad 3\,\text{combinations}$$
$$f\,f: \quad 3\,\text{combinations}$$
$$d\,d: \quad 3\,\text{combinations}$$
$$d\,f: \quad 6\,\text{combinations}$$

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9/8/3 linearly independent in $SU(N/3/2)$, $N>3$ [Pascual, Tarrach '80].

$SU(3)$: $\{\sigma_1,\ldots,\sigma_8\}$ chosen with these symmetries:

| | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ | $\sigma_5$ | $\sigma_6$ | $\sigma_7$ | $\sigma_8$ |
|---|---|---|---|---|---|---|---|---|
| $a\leftrightarrow b$ | + | + | + | - | - | - | - | + |
| $c\leftrightarrow d$ | + | + | + | - | - | + | - | - |

$\{\sigma_1,\ldots,\sigma_5\}$ orthogonal to $\{\sigma_6,\sigma_7,\sigma_8\}$. $\rightarrow$ $\{\sigma_6,\sigma_7,\sigma_8\}$ decouple.

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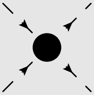
## Four-ghost vertex



$$\Gamma^{\bar{c}\bar{c}cc,abcd}(p,q,r,s)=\mathbf{g^4}\sum_{k=1}^{8}\sigma^{k,abcd}E_k^{\bar{c}\bar{c}cc}(p,q,r,s).$$

# The two-ghost-two-gluon vertex: Lorentz space

Non-primitively divergent correlation function → No guide from tree-level tensor. → Use full basis.

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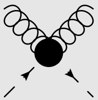


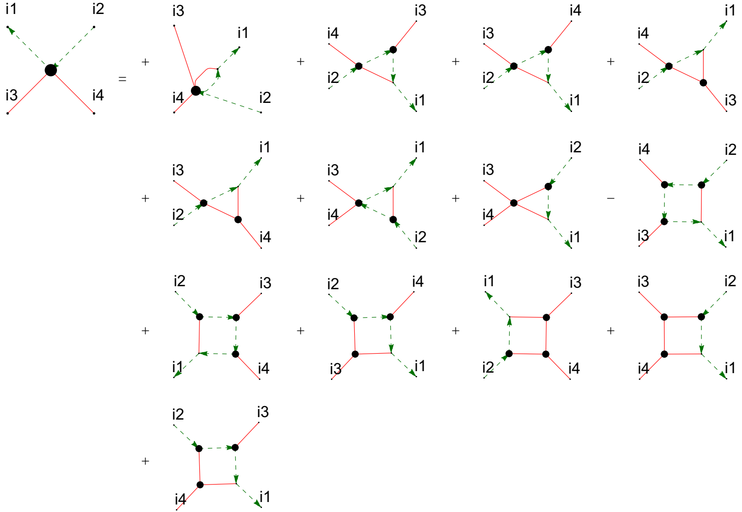
$$\Gamma^{AA\bar{c}c,abcd}_{\mu\nu}(p,q;r,s) = g^4 \sum_{k=1}^{40} \rho^{k,abcd}_{\mu\nu} D^{AA\bar{c}c}_{k(i,j)}(p,q;r,s)$$

with

$$\rho^{k,abcd}_{\mu\nu} = \sigma^{abcd}_i \tau^j_{\mu\nu}, \qquad k = k(i,j) = 5(i-1)+j$$

# The two-ghost-two-gluon vertex DSE

2 DSEs, choose the one with the ghost leg attached to the bare vertex
$\rightarrow$ Truncation discards only one diagram.



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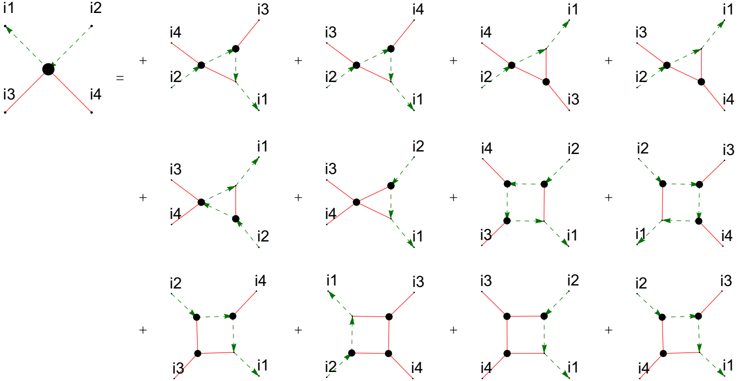
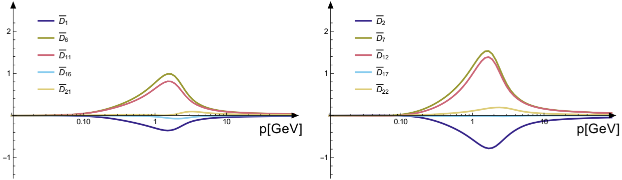
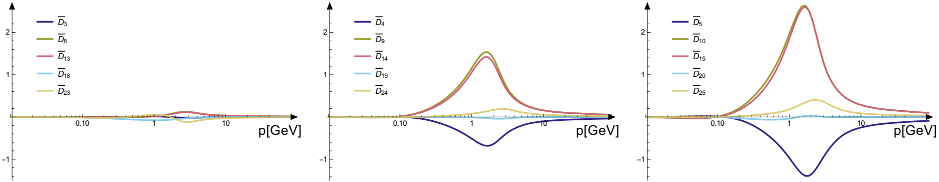


# Results for the two-ghost-two-gluon vertex

Kinematic approximation: one-momentum configuration

- Dimensionless dressing functions $\overline{D}_k$.
- Each plot one Lorentz tensor.



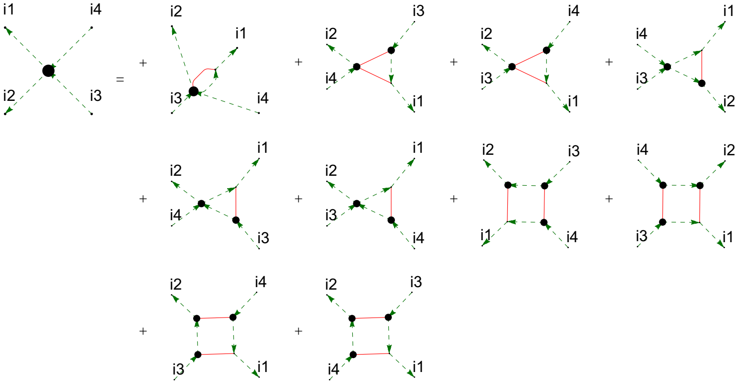




→ Two classes of dressings: 13 very small, 12 not small

→ No nonzero solution for $\{\sigma_6, \sigma_7, \sigma_8\}$ found. [MQH '17]

# The four-ghost vertex DSE

→ Truncation discards only one diagram.



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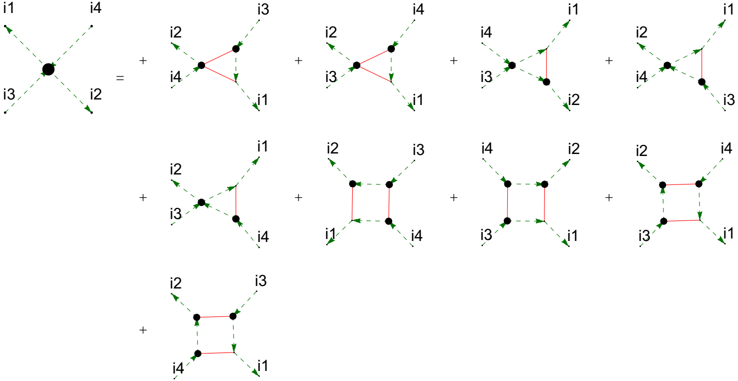

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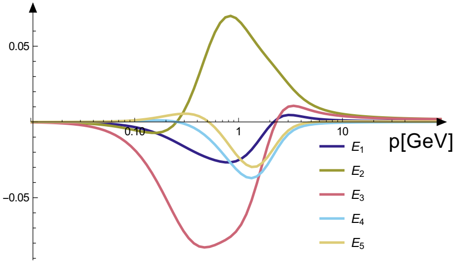


→ All dressings very small.

[MQH '17]

## $E_6$, $E_7$, $E_8$ ($\{\sigma_6, \sigma_7, \sigma_8\}$)

Decouple into a homogeneous, linear equation. → Trivial solution always exists.
Nontrivial one? → None found.
(Same applies to two-ghost-two-gluon vertex.)

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- Add quarks
- Finite temperature
- Bound states
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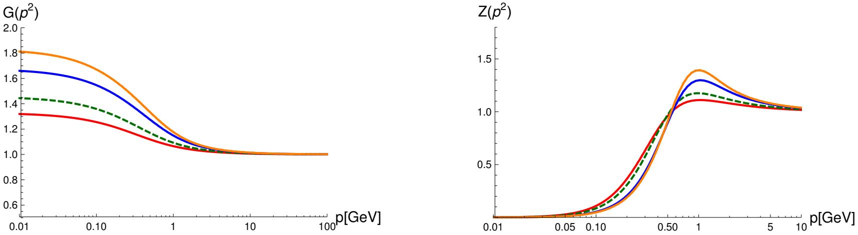
Thank you for your attention!

# Family of solutions in three dimensions

Cf. FRG results: Bare mass parameter from modified STIs [Cyrol, Fister, Mitter, Pawlowski, Strodthoff '15].

DSEs: Enforce family of solutions by fixing the gluon propagator at $p^2 = 0$.

Simple toy system with bare vertices [MQH, 1606.02068]:




$\Rightarrow$ Possibility of family of solutions.

NB: Effect overestimated here since vertices are fixed.