QCD 2- and 3-point Green's functions:
From lattice results to phenomenology

In collaboration with:

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Lattice two- and three-point Green's function

\[
\mathcal{G}^{abc}_{\alpha\mu\nu}(q, r, p) = \langle A^a_\alpha(q)A^b_\mu(r)A^c_\nu(p) \rangle = f^{abc}\mathcal{G}_{\alpha\mu\nu}(q, r, p),
\]

\[
\Delta^{ab}_{\mu\nu}(q) = \langle A^a_\mu(q)A^b_\nu(-q) \rangle = \delta^{ab}\Delta(p^2)P_{\mu\nu}(q),
\]

\[
\tilde{A}^a_\mu(q) = \frac{1}{2} \text{Tr} \sum_x A_\mu(x + \hat{\mu}/2) \exp[iq \cdot (x + \hat{\mu}/2)]\lambda^a
\]

\[
A_\mu(x + \hat{\mu}/2) = \frac{U_\mu(x) - U_\mu^\dagger(x)}{2iag_0} - \frac{1}{3} \text{Tr} \frac{U_\mu(x) - U_\mu^\dagger(x)}{2iag_0}
\]

Tree-level Symanzik gauge action

\[
S_g = \frac{\beta}{3} \sum_x \left\{ b_0 \sum_{\mu, \nu=1}^{4} \left[ 1 - \text{Re} \text{Tr} (U_{x,\mu,\nu}^{1 \times 1}) \right] + b_1 \sum_{\mu, \nu=1}^{4} \left[ 1 - \text{Re} \text{Tr} (U_{x,\mu,\nu}^{1 \times 2}) \right] \right\}
\]

The gauge fields are to be nonperturbatively obtained from lattice QCD simulations and applied then to get the gluon Green's functions.
The gluon propagator

\[ \Delta^{ab}_{\mu\nu}(q) = \langle A^a_\mu(q)A^b_\nu(-q) \rangle = \delta^{ab}\Delta(p^2)P_{\mu\nu}(q), \]

where \( P_{\mu\nu}(q) = \delta_{\mu\nu} - q_\mu q_\nu/q^2 \), implies directly that \( \mathcal{G} \) is totally transverse: \( q \cdot \mathcal{G} = r \cdot \mathcal{G} = p \cdot \mathcal{G} = 0. \)

Quenched lattice gluon propagators for different large volumes!
The gluon propagator

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Duarte, Oliveira, Silva
PRD94(2016)014502

Quenched lattice gluon propagators for different beta and similar volume!
The gluon propagator

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ArXiv:1704.02053 (PRD): Essentially, a scale setting problem!!
The gluon propagator

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- Effective gluon mass increases with the number of flavours

Unquenched lattice gluon propagators!
The gluon propagator

$$\Delta^{ab}_{\mu\nu}(q) = \langle A^a_{\mu}(q)A^b_{\nu}(-q) \rangle = \delta^{ab} \Delta(p^2)P_{\mu\nu}(q),$$

where $P_{\mu\nu}(q) = \delta_{\mu\nu} - q_{\mu}q_{\nu}/q^2$, implies directly that $G$ is totally transverse: $q \cdot G = r \cdot G = p \cdot G = 0.$

- Effective gluon mass increases with the number of flavours

Unquenched lattice gluon propagators!
The gluon propagator

\[ \Delta_{\mu
u}^{ab}(q) = \langle A_{\mu}^{a}(q) A_{\nu}^{b}(-q) \rangle = \delta^{ab} \Delta(p^2) P_{\mu\nu}(q), \]

where \( P_{\mu\nu}(q) = \delta_{\mu\nu} - q_{\mu} q_{\nu}/q^2 \), implies directly that \( G \) is totally transverse: \( q \cdot G = r \cdot G = p \cdot G = 0 \).

- Effective gluon mass increases with the number of flavours
  ... and decreases with the dynamical quark mass

Unquenched lattice gluon propagators!
The ghost propagator

\[
\begin{align*}
(F^{(2)})_{xy}^{ab} & \equiv \langle (M^{-1})_{xy}^{ab} \rangle, \quad M(U) = -\frac{1}{N} \nabla \cdot \vec{D}(U) \\
\vec{D}(U) \eta(x) &= \frac{1}{2} \left( U_{\mu}(x) \eta(x + \mu) - \eta(x) U_{\mu}(x) + \eta(x + \mu) U_{\mu}^\dagger - U_{\mu}^\dagger(x) \eta(x) \right)
\end{align*}
\]

Ayala et al.
PRD86(2012)074512

Unquenched lattice ghost propagators!
The vertex and the three-gluon Green's function

\[ \mathcal{G}_{\alpha \mu \nu}(q, r, p) = \langle A_{\alpha}(q) A_{\mu}(r) A_{\nu}(p) \rangle = f^{abc} \mathcal{G}_{\alpha \mu \nu}(q, r, p), \]

\[ \mathcal{G}_{\alpha \mu \nu}(q, r, p) = g \Gamma_{\alpha' \mu' \nu'}(q, r, p) \Delta_{\alpha'}(q) \Delta_{\mu'}(r) \Delta_{\nu'}(p), \]

\[ G_{\alpha \mu \nu}(q, r, p) = T^{\text{sym}}(q^2) \lambda_{\alpha \mu \nu}^{\text{tree}}(q, r, p) + S^{\text{sym}}(q^2) \lambda_{\alpha \mu \nu}^{S}(q, r, p) \]

\[ \Gamma_{\alpha \mu \nu}(q, r, p) = \Gamma_{\alpha \mu \nu}^{\text{sym}}(q^2) \lambda_{\alpha \mu \nu}^{\text{tree}}(q, r, p) + \Gamma_{\alpha \mu \nu}^{\text{sym}}(q^2) \lambda_{\alpha \mu \nu}^{S}(q, r, p) \]

\[ \Delta_{\mu \nu}(q) = \langle A_{\mu}(q) A_{\nu}(-q) \rangle = \delta_{\mu \nu} \Delta(q^2) P_{\mu \nu}(q), \]

where \( P_{\mu \nu}(q) = \delta_{\mu \nu} - q_{\mu} q_{\nu}/q^2 \), implies directly that \( \mathcal{G} \) is totally transverse: \( q \cdot \mathcal{G} = 0 \).

\[ \lambda_{\alpha \mu \nu}^{\text{tree}}(q, r, p) = \Gamma_{\alpha \mu \nu}^{(0)}(q, r, p) P_{\alpha \alpha}(q) P_{\mu \mu}(r) P_{\nu \nu}(p). \]

\[ \lambda_{\alpha \mu \nu}^{S}(q, r, p) = (r - p)_{\alpha} (p - q)_{\mu} (q - r)_{\nu}/r^2. \]

In Landau gauge and for particular kinematical configurations, transversality and Bose symmetry make possible a simple tensorial decomposition of the gluon Green's function.
The vertex and the three-gluon Green's function

\[
G^{abc}_{\alpha \mu \nu}(q, r, p) = \langle A^a_\alpha(q) A^b_\mu(r) A^c_\nu(p) \rangle = f^{abc} G_{\alpha \mu \nu}(q, r, p),
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where

\[
\Gamma^{\alpha \mu \nu}(q, r, p) = g \Gamma^{\alpha' \mu' \nu'}(q, r, p) \Delta^{\alpha' \alpha}(q) \Delta^{\mu' \mu}(r) \Delta^{\nu' \nu}(p),
\]

and

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G_{\alpha \mu \nu}(q, r, p) = T^{\text{sym}}(q^2) \lambda^{\text{tree}}_{\alpha \mu \nu}(q, r, p) + S^{\text{sym}}(q^2) \lambda^S_{\alpha \mu \nu}(q, r, p)
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Symmetric configuration:

\[
q^2 = r^2 = p^2 \text{ and } q \cdot r = q \cdot p = r \cdot p = -q^2/2;
\]

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where

\[ \Delta_{\mu\nu}(q) = \langle A_{\mu}^{a}(q)A_{\nu}^{a}(-q) \rangle = \delta^{ab} \Delta(p^2)P_{\mu\nu}(q), \]

and

\[ P_{\mu\nu}(q) = \delta_{\mu\nu} - q_{\mu}q_{\nu}/q^2, \]

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In Landau gauge and for particular kinematical configurations, transversality and Bose symmetry make possible a simple tensorial decomposition of the gluon Green's function.

Symmetric configuration:

\[ G_{\alpha\mu\nu}(q, r, p) = g \Gamma'_{\alpha'\mu'\nu'}(q, r, p) \Delta'_{\alpha'\alpha}(q) \Delta_{\mu'\mu}(r) \Delta_{\nu'\nu}(p), \]

where

\[ \Gamma_{\alpha\mu\nu}(q, r, p) = T_{\text{sym}}(q^2) \lambda^{\text{tree}}_{\alpha\mu\nu}(q, r, p) + S_{\text{sym}}(q^2) \lambda^{S}_{\alpha\mu\nu}(q, r, p) \]

\[ W_{\alpha\mu\nu} = \lambda^{\text{tree}}_{\alpha\mu\nu} + \frac{1}{2} \lambda^{S}_{\alpha\mu\nu} \]

\[ T_{\text{sym}}(q^2) = \frac{W_{\alpha\mu\nu}(q, r, p) G_{\alpha\mu\nu}(q, r, p)}{W_{\alpha\mu\nu}(q, r, p) W_{\alpha\mu\nu}(q, r, p)} \]

\[ \lambda^{\text{tree}}_{\alpha\mu\nu}(q, r, p) = \Gamma^{(0)}_{\alpha'\mu'\nu'}(q, r, p) P_{\alpha'\alpha}(q) P_{\mu'\mu}(r) P_{\nu'\nu}(p). \]

\[ \lambda^{S}_{\alpha\mu\nu}(q, r, p) = (r - p)_{\alpha}(p - q)_{\mu}(q - r)_{\nu}/r^2. \]
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Asymmetric configuration:
\[ q \to 0; \quad r^2 = p^2 = -p \cdot r \]

\[ G_{\alpha\mu\nu}(q, r, p) = g \Gamma_{\alpha'\mu'\nu'}(q, r, p) \Delta_{\alpha'\alpha}(q) \Delta_{\mu'\mu}(r) \Delta_{\nu'\nu}(p), \]

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- **Symmetric configuration:**
  \[ q^2 = r^2 = p^2 \text{ and } q \cdot r = q \cdot p = r \cdot p = -q^2/2; \]

\[ \Delta_R(q^2; \mu^2) = Z_A^{-1}(\mu^2) \Delta(q^2), \]
\[ T_R^{\text{sym}}(q^2; \mu^2) = Z_A^{-3/2}(\mu^2) T^{\text{sym}}(q^2), \]

- **MOM renormalization prescription:**
  \[ \Delta_R(q^2; q^2) = Z_A^{-1}(q^2) \Delta(q^2) = 1/q^2, \]
  \[ T_R^{\text{sym}}(q^2; q^2) = Z_A^{-3/2}(q^2) T^{\text{sym}}(q^2) = g_R^{\text{sym}}(q^2)/q^6. \]

\[ g^{\text{sym}}(q^2) = q^3 \frac{T_R^{\text{sym}}(q^2; \mu^2)}{[\Delta(q^2)]^{3/2}}, \]
\[ T^{\text{sym}}(q^2) = g \Gamma_T^{\text{sym}}(q^2) \Delta^3(q^2), \]
\[ g^{\text{sym}}(\mu^2) \Gamma_T^{\text{sym}}(q^2; \mu^2) = \frac{g^{\text{sym}}(q^2)}{[q^2 \Delta_R(q^2; \mu^2)]^{3/2}}. \]

After the required projection and the appropriate renormalization, one can define a QCD coupling from the Green's functions, and relate it to the 1PI vertex form factor, in both symmetric...
The vertex and the three-gluon Green's function

\[ \mathcal{G}_{\alpha\mu\nu}^{abc}(q, r, p) = \langle A_{\alpha}^{a}(q)A_{\mu}^{b}(r)A_{\nu}^{c}(p)\rangle = f^{abc} \mathcal{G}_{\alpha\mu\nu}(q, r, p), \]

Asymmetric configuration:
\[ q \rightarrow 0; \quad r^2 = p^2 = -p \cdot r \]

MOM renormalization prescription:

\[ \Delta_R(q^2; \mu^2) = Z_A^{-1}(\mu^2) \Delta(q^2), \]
\[ T_R^{\text{sym}}(q^2; \mu^2) = Z_A^{-3/2}(\mu^2)T^{\text{sym}}(q^2), \]

\[ \Delta_R(q^2; q^2) = Z_A^{-1}(q^2) \Delta(q^2) = 1/q^2, \]
\[ T_R^{\text{sym}}(r^2; r^2) = Z_A^{-3/2}(r^2)T^{\text{sym}}(r^2) = \Delta_R(0; q^2) g_R^{\text{asy}}(r^2)/r^4, \]

\[ \Delta_{\mu\nu}^{ab}(q) = \langle A_{\mu}^{a}(q)A_{\nu}^{b}(-q)\rangle = \delta^{ab} \Delta(p^2)P_{\mu\nu}(q), \]

\[ T^{\text{asy}}(r^2) = \left. \frac{W_{\alpha\mu\nu}(q, r, p)\mathcal{G}_{\alpha\mu\nu}(q, r, p)}{W_{\alpha\mu\nu}(q, r, p)W_{\alpha\mu\nu}(q, r, p)} \right|_{\text{asy}} \]

\[ g^{\text{asy}}(\mu^2) T^{\text{asy}}_{T, R}(q^2; \mu^2) = \frac{g^{\text{asy}}(q^2)}{\left[q^2 \Delta_R(q^2; \mu^2)\right]^{3/2}} \]

After the required projection and the appropriate renormalization, one can define a QCD coupling from the Green's functions, and relate it to the 1PI vertex form factor, in both symmetric and asymmetric kinematical configurations.
The zero-crossing of the three-gluon vertex

\[ g^i(\mu^2) \Gamma^i_{T,R}(q^2;\mu^2) = \frac{g^i(q^2)}{[q^2 \Delta_R(q^2;\mu^2)]^{3/2}} \]

\[ i = \text{sym}, \text{asym}. \]

\[ g_{\text{sym}}(q^2) = q^3 \frac{T_{\text{sym}}(q^2)}{[\Delta(q^2)]^3} \]

\[ g_{\text{asym}}(q^2) = q^3 \frac{T_{\text{asym}}(q^2)}{\Delta(0)[\Delta(q^2)]^{1/2}} \]

Let's then focus (again) on the symmetric case: the form factor appears to change its sign at very deep IR momenta and show then a zero-crossing. This appears to happen below \( \sim 0.2 \) GeV.
The zero-crossing of the three-gluon vertex

\[ g^i(\mu^2) \Gamma^i_{T,R}(q^2;\mu^2) = \frac{g^i(q^2)}{q^2 \Delta_R(q^2;\mu^2)^{3/2}} \]

\[ q = T^{\text{sym}}(q^2) \]

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Reference:
M. Tissier, N. Wschebor, PRD84(2011)045018
A.C Aguilar et al.; PRD89(2014)05008
A. Blum et al.; PRD89(2014)061703
G. Eichmann et al.; PRD89(2014)105014
The zero-crossing of the three-gluon vertex

\[ g_i(\mu^2) \Gamma_{T,R}^i(q^2;\mu^2) = \frac{g_i(q^2)}{q^2 \Delta_R(q^2;\mu^2)}^{3/2} \]

\( i = \text{sym}, \text{asym} \).

\[ g_{\text{sym}}(q^2) = q^3 \frac{T_{\text{sym}}(q^2)}{[\Delta(q^2)]^{3/2}} \]

\[ g_{\text{asym}}(q^2) = q^3 \frac{T_{\text{asym}}(q^2)}{\Delta(0)[\Delta(q^2)]^{1/2}} \]

Let's consider now the asymmetric case: the results are much noisier (surely because of the zero-momentum gluon field in the correlation function), although there appear to be strong indications for the happening of the zero-crossing.
The zero-crossing of the three-gluon vertex

\[ g^i(\mu^2) \Gamma_{T,R}^i(q^2;\mu^2) = \frac{g^i(q^2)}{[q^2 \Delta R(q^2;\mu^2)]^{3/2}} \]

\[ i = \text{sym, asym.} \]
The zero-crossing of the three-gluon vertex

\[ g^i(\mu^2) \]

\( i = \text{sym} \)

After leg amputation, the 1PI form factor for the tree-level tensor shows clearly the zero-crossing. The trend is the same for both Wilson and \( \text{tlSym} \) actions and symmetric and asymmetric configurations.

\[
\Gamma_T, R^i(q^2; \mu^2) = g^i(q^2) \left[ q^2 \Delta R(q^2; \mu^2) \right]^{3/2} \]

\( i = \text{sym}, \text{asym} \)
The zero-crossing of the three-gluon vertex

After leg amputation, the 1PI form factor for the tree-level tensor shows clearly the zero-crossing. The trend is the same for both Wilson and tSym actions and symmetric and asymmetric configurations.
The zero-crossing of the three-gluon vertex

DSE-based explanation:

\[
\Gamma^{i,(B)}_{T,R}(p^2; \mu^2) \overset{p^2/\mu^2 \ll 1}{\approx} F_R(0; \mu^2) \frac{\partial}{\partial p^2} \Delta^{-1}_R(p^2; \mu^2) + \ldots
\]

\[
\tilde{\Delta}^{-1}(q^2) = \sum_{B} -\Delta^{-1}(q^2) + \ldots
\]

\[
[1 + G(q^2)]^2 \Delta^{-1}(q^2) = \tilde{\Delta}^{-1}(q^2).
\]

\[
\Lambda_{\mu\nu}(q) = \mu \rightarrow \nu + \mu \rightarrow \nu
\]

\[
= G(q^2) g_{\mu\nu} + L(q^2) \frac{q_{\mu} q_{\nu}}{q^2}
\]

In PT-BFM truncation

cf. D. Binosi's talk!!!
The zero-crossing of the three-gluon vertex

DSE-based explanation:

\[
\Gamma_{T,R}^{(B)}(p^2;\mu^2) \approx \frac{\partial}{\partial p^2} \Delta_R^{-1}(p^2;\mu^2) + \ldots
\]

\[
[1 + G(q^2)]^2 \Delta^{-1}(q^2) = \Delta^{-1}(q^2).
\]

\[
\Pi_c(q^2) = \frac{g^2 C_A}{6} q^2 F(q^2) \int \frac{F(k^2)}{k^2(k+q)^2},
\]

In PT-BFM truncation

cf. D. Binosi's talk!!!
The zero-crossing of the three-gluon vertex

DSE-based explanation:

\[\Gamma_{T,R}^{i,(B)}(p^2;\mu^2) \approx F_R(0;\mu^2) \frac{\partial}{\partial p^2} \Delta_{R}^{-1}(p^2;\mu^2) + \ldots\]

In PT-BFM truncation

cf. D. Binosi's talk!!!

\[\Delta_{R}^{-1}(q^2) = \tilde{\Delta}_{-1}(q^2) = \frac{1}{q^2 + \mu^2} + \ldots\]

\[\Pi_{c}(q^2) = \frac{g^2 C_A}{6} q^2 F(q^2) \int_k \frac{F(k^2)}{k^2(k + q^2)}\]

\[\Lambda_{\mu\nu}(q) = \frac{1}{q^2} \left( \frac{g^2}{2} G(q^2) + \frac{L(q^2)}{q^2} \right)\]

\[\Delta_{R}^{-1}(q^2;\mu^2) \approx q^2 \left[ a + b \log \frac{q^2 + m^2}{\mu^2} + c \log \frac{q^2}{\mu^2} \right] + m^2,\]
The zero-crossing of the three-gluon vertex

DSE-based explanation:

\[ \Gamma_{T,R}^{(B)} (p^2; \mu^2) \approx F_R(0; \mu^2) \left( a + b \ln \frac{m^2}{\mu^2} + c \right) + c F_R(0; \mu^2) \ln \frac{p^2}{\mu^2} + \ldots \]

In PT-BFM truncation, the zero-crossing can be understood within a DSE framework.

A logarithmic divergent contribution at vanishing momentum, pulling down the 1PI form factor and generating a zero crossing, can be understood within a DSE framework.

\[ \Delta^{-1}(q^2) = [1 + G(q^2)] \Delta^{-1}(q^2) = \hat{\Delta}^{-1}(q^2) \]

\[ \Delta_R^{-1}(q^2; \mu^2) \bigg|_{q^2 \to 0} = q^2 \left[ a + b \log \frac{q^2 + m^2}{\mu^2} + c \log \frac{q^2}{\mu^2} \right] + m^2, \]

\[ \Pi_c(q^2) = \frac{g^2 C_A}{6} q^2 F(q^2) \int_k \frac{F(k^2)}{k^2(k + q)^2}, \]

D. Binosi's talk!!!
The zero-crossing of the three-gluon vertex

We can thus perform a fit, only over a deep IR domain, of our data to a DSE-grounded formula and describe the behaviour of the 1PI form factor.
The zero-crossing of the three-gluon vertex

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\[ g^i_R(\mu^2) \Gamma^i_R(p^2; \mu^2) = a^i_{ln}(\mu^2) \ln \frac{p^2}{\mu^2} + a^i_0(\mu^2) + a^i_2(\mu^2) p^2 \ln \frac{p^2}{M^2} + o(p^2) \]

\( i = \text{symmetric} \)

Consistent with direct large-volume lattice evaluations of the gluon and ghost two-point Green functions.

We can thus perform a fit, only over a deep IR domain, of our data to the DSE-grounded formula and describe the behaviour of the 1PI form factor.
The zero-crossing of the three-gluon vertex

\[ g_R^i(\mu^2) \Gamma_R^i(p^2; \mu^2) = a_{ln}^i(\mu^2) \ln \frac{p^2}{\mu^2} + a_0^i(\mu^2) + a_2^i(\mu^2) p^2 \ln \frac{p^2}{M^2} + o(p^2) \]

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Consistent with direct large-volume lattice evaluations of the gluon and ghost two-point Green functions.

\[ g^i_R(\mu^2) c F_R(0,\mu^2) \]
The zero-crossing of the three-gluon vertex

\[ g^i_R(\mu^2) \Gamma_R^i(p^2; \mu^2) = a^i_{ln}(\mu^2) \ln \frac{p^2}{\mu^2} + a^i_0(\mu^2) + a^i_2(\mu^2) p^2 \ln \frac{p^2}{M^2} + o(p^2) \]

\( i = \text{asymmetric} \)

Consistent with direct large-volume lattice evaluations of the gluon and ghost two-point Green functions.

The low-momenta asymptotic 1PI form factor obtained from DSE within the PT-BFM is fully consistent with lattice data for both symmetric and asymmetric kinematic configurations.

A.C Aguilar et al.; PRD89(2014)05008
Ph. Boucaud et al.; PRD95(2017)114503
Quark's gap equation: RGI interaction

\[ \hat{a}(k^2) = \alpha(\mu^2) \hat{\Delta}(k^2; \mu^2) \]

**Convert vertices/propagators into PT-BFM ones**
new RG invariant combination appears

\[ \hat{d}(k^2) = \alpha(\mu^2) \Delta(k^2; \mu^2) \]

**Use symmetry identity**
to identify the interaction strength

\[ I(k^2) = k^2 \hat{d}(k^2) = \left[ \frac{1}{1 - L(q^2) F(q^2)} \right]^2 \alpha_T(q^2) \]

**1+G and L determined by their own SDEs**
under simplifying assumptions:

\[ 1 + G(p^2) = Z_c - g^2 \int_k \left[ 2 + \frac{(k \cdot p)^2}{k^2 p^2} \right] B_1(k) \Delta(k) \frac{F((k + p)^2)}{(k + p)^2} \]

\[ L(p^2) = -g^2 \int_k \left[ 1 - 4 \frac{(k \cdot p)^2}{k^2 p^2} \right] B_1(k) \Delta(k) \frac{F((k + p)^2)}{(k + p)^2} \]

\[ F^{-1}(q^2) = Z_c - 3 g^2 \int_k \left[ 1 - \frac{(k \cdot p)^2}{k^2 p^2} \right] B_1(k) \Delta(k) \frac{F((k + p)^2)}{(k + p)^2} \]

- Main source of uncertainties:
  needs assumptions on ghost vertex behavior
- Parametrized by \( \delta \in [0,1] \)
lower bound (\( \delta = 0 \)): \( 1/F = 1 + G \)
Top-down vs. Bottom-up approaches

Let us now carefully examine the RGI Interaction:

\[ I(k^2) := k^2 \tilde{d}(k^2) = \frac{\alpha_T(k^2)}{[1 - L(k^2)F(k^2)]^2} \]

\[ \alpha_T(k^2) = \lim_{a \to 0} g^2(a) k^2 \Delta(k^2; a) F^2(k^2; a) \]

A running strong coupling in a particular scheme (Taylor), well-known in perturbation and easy-to-handle in Lattice QCD.
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\[ F(k^2; \zeta^2) \approx \frac{3g^2(\zeta^2)}{32\pi^2} \left( \ln \frac{k^2}{\Lambda_T^2} / \ln \frac{\zeta^2}{\Lambda_T^2} \right)^{-(\gamma_0 + \gamma_0)/\beta_0} \]

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RGI interaction kernel

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\]

\[
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\]

\[
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\[ \alpha_{\text{MS}}(k^2)(1 + 1.09 \alpha_{\text{MS}}(k^2) + \ldots) \]

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Zero-momentum freezing!
Flavor-dependent?

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Zero-momentum freezing!
Flavor-dependent?
Yes, but not when the quark thresholds are far above …

...As happens for

\[ N_f = n_f + \delta \]
\[ n_f = 2 \]
\[ \delta = 1(s) + 1(c) \]
Let us now carefully examine the RGI Interaction:

\[ I(k^2) := k^2 \hat{d}(k^2) = \frac{\alpha_T(k^2)}{[1 - L(k^2)F(k^2)]^2} \]

Low-momentum asymptotic expansion

\[ I(k^2) \approx \frac{k^2 \hat{d}(0)}{k^2/\Lambda_T^2 \ll 1} \left[ 1 - \left( \frac{\hat{d}(0)}{8\pi} + \frac{\ell_w}{m_g^2} \right) k^2 \ln \frac{k^2}{\Lambda_T^2} \right] \]

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\[ \hat{d}(0) = \frac{\alpha_0}{m_0^2} \approx \frac{0.9\pi}{(m_P/2)^2} \]

Ir mass scale of about one half of the proton mass (cf. C. Roberts' talk!!)

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A divergent ghost-loop contribution to the gluon vacuum polarization in its DSE

A.C. Aguilar et al., PRD89(2014)05008
A.K. Cyrol et al. PRD94(2016)054005
Ph. Boucaud et al., PRD95(2017)114503
QCD effective charge

Let us first carefully examine the RGI Interaction:

\[ I(k^2) := k^2 \hat{d}(k^2) = \frac{\alpha_T(k^2)}{[1 - L(k^2)F(k^2)]^2} \]

Remarkable QCD feature: saturation of the RG key ingredient \( \hat{d}(0) \)

Define then the RGI invariant function

\[ D(k^2) = \frac{\Delta(k^2; \mu^2)}{\Delta(0; \mu^2) m_0^2} = \frac{\Delta(k^2; \xi_0^2)}{z(\xi_0^2)} = \begin{cases} 1/m_0^2 & k^2 \ll m_0^2 \\ 1/k^2 & k^2 \gg m_0^2 \end{cases} \]

Extract the (process-independent) coupling
Using the quark gap equation

\[ \Sigma(p) = Z_2 \int_{dq} 4\pi \hat{\alpha}_{\text{PI}}(k^2) D_{\mu\nu}(k^2) \gamma_\mu S(q) \hat{\Gamma}_\nu(q,p) \]

Conclusions

- Lattice contemporary results for the three-gluon Green's functions provide, as a main feature, a zero-crossing at very low-momenta...
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Conclusions

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- ... and applied to define a process-independent effective charge.

Thank you!!!
QCD effective charge

\[ \alpha(k^2) = \frac{\hat{d}(k^2)}{D(k^2)} \xrightarrow{k^2 \gg m_0^2} \mathcal{I}(k^2) \]

- Parameter free
  completely determined from 2-points sector
- No Landau pole
  physical coupling showing an IR fixed point
- Smoothly connects IR and UV domains
  no explicit matching procedure
- Essentially non-perturbative result
  continuum/lattice results plus setting of single mass scale (from the gluon)
- Ghost gluon dynamics critical
  enhancement at intermediate momenta

QCD effective charge: comparison

- Process dependent effective charges fixed by the leading-order term in the expansion of a given observable

- Bjorken sum rule defines such a charge
  - Bjorken, PR 148 (1966); PRD 1 (1970)
  \[ \int_0^1 dx \left[ g_1^p(x, k^2) - g_1^n(x, k^2) \right] = \frac{g_A}{6} \left[ 1 - \alpha_{g_1}(k^2)/\pi \right] \]

  - \( g_1^p, n \) spin dependent p/n structure functions extracted from measurements using unpolarized targets
  - \( g_A \) nucleon flavour-singlet axial charge

- Many merits
  - Existence of data for a wide momentum range
  - Tight sum rules constraints on the integral at IR and UV extremes
  - Isospin non-singlet suppress contributions from hard-to-compute processes

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