Student lectures on:

Dualities in Field and String Theory

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1 | Introduction

Dualities seem to play an increasingly important role in theoretical physics, for example in the pursuit of understanding quantum gravity. It is however not a new phenomena, and has historical roots in for example electromagnetism. So what are dualities in physics:

> **Dualities in physics** (Heuristic definition):
> When two seemingly distinct theories, either approximately or exactly, describe the same dynamics, and there exists a map from one to the other.

These dualities most often reveal themselves as an underlying mathematical structure (symmetry) of the theories.

The plan of these lectures is to describe prominent examples of dualities in field theory and their implications. This will lead us to finally describe examples of the appearance and use of dualities in superstring theory. The main references on which the beginning of these notes are build are [1–3].

2 | Electromagnetic Duality and Monopoles

Maxwell equations (in differential form) in vacuum with units $c = 1$:

(i)	$\nabla \cdot \mathbf{E} = 0$	Gauss's law
(ii)	$\nabla \cdot \mathbf{B} = 0$	Gauss's law for magnetism
(iii)	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	Faraday's law
(iv)	$\nabla \times \mathbf{B} = \dfrac{\partial \mathbf{E}}{\partial t}$	Ampère's law

This set of equations are covariant under Lorentz transformations and conformal transformations and invariant under a duality transformation:

$$\mathbf{E} \mapsto \cos\phi \mathbf{E} - \sin\phi \mathbf{B} \tag{2.1}$$

$$\mathbf{B} \mapsto \cos\phi \mathbf{B} + \sin\phi \mathbf{E} \tag{2.2}$$

In particular, we shall refer to the invariance under restriction of $\phi = -\pi/2$ as the **electromagnetic duality**, which is of the discrete form:

$$(\mathbf{E}, \mathbf{B}) \mapsto (\mathbf{B}, -\mathbf{E}) \tag{2.3}$$

We will comment on this restriction when discussing the Dirac quantisation condition. The covariance under Lorentz transformations is only really manifest when we introduce the electromagnetic field strength tensor $F_{\mu\nu}$ with components defined by

$$E^i = F^{0i} = -F^{i0}, \qquad B^i = \frac{1}{2}\epsilon^{ijk}F^{jk} \tag{2.4}$$

where we use 4-vector notation $\mu = 0, i$ for $i = 1, 2, 3$. Note that $F^{\mu\nu}$ is anti-symmetric and therefore has precisely $\binom{4}{2} = 6$ independent components. The Maxwell equations in vacuum then take the simple form:

$$\partial_\mu F^{\mu\nu} = 0 \qquad \text{Reproduces (i)+(iv) - Electric source equations} \tag{2.5}$$

$$\partial_\mu \star F^{\mu\nu} = 0 \qquad \text{Reproduces (ii)+(iii) - Magnetic source equations} \tag{2.6}$$

where we use the definition of the hodge star operation

$$\star F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}. \tag{2.7}$$

In this language the electromagnetic duality is simply:

$$F^{\mu\nu} \mapsto \star F^{\mu\nu} \qquad \star F^{\mu\nu} \mapsto -F^{\mu\nu}. \tag{2.8}$$

This duality can be preserved in the presence of sources as longs as both electric and magnetic sources are introduced with appropriate transformations:

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad \text{Transformation: } F^{\mu\nu} \mapsto \star F^{\mu\nu} \text{ and } j^\nu \mapsto k^\nu \tag{2.9}$$

$$\partial_\mu \star F^{\mu\nu} = k^\nu \qquad \text{Transformation: } \star F^{\mu\nu} \mapsto -F^{\mu\nu} \text{ and } k^\nu \mapsto -j^\nu \tag{2.10}$$

As to this day, no experimental evidence for magnetic monopoles exists, but neither is there any theoretical reason for the non-existence available. Electrodynamics is fully compatible with the introduction of magnetic monopoles, and restoring the electromagnetic duality leads to new conclusions about electromagnetism.

2.1 The Dirac quantisation condition

If we take the existence of magnetic monopoles seriously it immediately leads to non-trivial consequences for the possible particle spectrum. A magnetic monopole would generate a magnetic field in analogy with that of a point electric charge:

$$\mathbf{B} = \frac{g}{4\pi r^2}\hat{\mathbf{r}}. \tag{2.11}$$

But now that the magnetic field is no longer divergence free, we can no longer globally define a vector potential

$$\mathbf{B} \neq \boldsymbol{\nabla} \times \mathbf{A}. \tag{2.12}$$

In geometrical terms, this is a consequence of now having to define a U(1) gauge-bundle over $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ which is homotopically equivalent to S^2. We can however define two vector potentials on separate patches of the space in spherical coordinates

$$\mathbf{A}^+ = \frac{g}{4\pi r}\frac{1-\cos\theta}{\sin\theta}\hat{\mathbf{e}}_\phi, \qquad \text{for } \theta \neq \pi \text{ (Negative } z\text{-axis)} \tag{2.13}$$

$$\mathbf{A}^- = \frac{-g}{4\pi r}\frac{1+\cos\theta}{\sin\theta}\hat{\mathbf{e}}_\phi, \qquad \text{for } \theta \neq 0 \text{ (Positive } z\text{-axis)} \tag{2.14}$$

which on each of their non-singular region gives the magnetic field of the monopole. This inevitable line (which could be in any direction) where the vector potential is singular is called the **Dirac string**. On the common domain where the potentials are defined ($\mathbb{R}$ without the z-axis) they are related by a gauge transformation:

$$\mathbf{A}^- = \mathbf{A}^+ + \nabla\alpha, \qquad \alpha = \frac{g}{2\pi}\varphi. \tag{2.15}$$

Now place a particle with electric charge, q, in the presence of the field of the magnetic monopole. For the wave function, ψ, the Schrödinger equation is invariant under a gauge transformation

$$\mathbf{A} \to \mathbf{A} + \nabla\alpha, \qquad \psi \to e^{iq\alpha}\psi \tag{2.16}$$

But for a sensible interpretation of the wave-function, it must be a single-valued function. That, is we must demand that

$$e^{iq\alpha}|_{\varphi=0} = e^{iq\alpha}|_{\varphi=2\pi} \tag{2.17}$$

which gives a condition on the electric and magnetic charge first given by Dirac [4]:

Dirac quantization condition:
Assuming the existence of magnetic monopoles of charge, g, leads to a consistency equation involving the electric charge q:

$$qg = 2\pi n, \qquad n \in \mathbb{Z} \tag{2.18}$$

Note that this condition is only invariant under the *discrete* electromagnetic duality.

The existence of magnetic monopoles could therefore potentially explain the observation that all particles have charges which are integer multiples of the electric charge of the electron. Note that quarks are exempt from this condition due to confinement.

The quantisation condition also easily generalises to particles with both electric and magnetic charge, so-called dyons:

Dirac-Zwanziger-Schwinger quantization for dyons:
Two dyonic point charges, i.e. with combined electric and magnetic charges (q, g) and (q', g'), satisfy the constraint

$$qg' - q'g = 2\pi n, \qquad n \in \mathbb{Z}. \tag{2.19}$$

2.2 Georgi-Glashow Model

In 1974, 't Hooft [5] and Polyakov [6] independently studied an alternative to our current theory of electroweak symmetry breaking, namely the Georgi-Glashow $SU(2)$ model, and showed that it contains magnetic monopole solutions. These solutions are however regular, in contrast to the Dirac monopole solution. Here we shall first briefly describe the perturbative constituents of the Georgi-Glashow model.

Consider an $SU(2)$ gauge theory with a Higgs field Φ^a in the adjoint representation of $SU(2)$

$$\mathcal{L}_{\rm GG} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a - \frac{\lambda}{4}\left(\Phi^a\Phi^a - v^2\right)^2 \tag{2.20}$$

with the usual definitions of the field strength and covariant derivative

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e\epsilon^{abc}A^b_\mu A^c_\nu, \qquad D_\mu\Phi^a = \partial_\mu\Phi^a - e\epsilon^{abc}A^b_\mu\phi^c. \tag{2.21}$$

The choice of a Higgs vacuum, $\Phi^2 = v^2$, then familiarly gives rise to a spontaneous breaking of the gauge group to $U(1)$. An expansion around the vacuum with a vacuum expectation value along the third direction ($\phi_a = v\delta_{a3}$) reveals, in accordance with Goldstone's theorem, a massless gauge field A^3_μ and two massive vector bosons:

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\left(A^1_\mu \pm iA^2_\mu\right). \tag{2.22}$$

The masses of the fields after symmetry breaking can then be read of from the quadratic terms to be:

Field	Mass	Charge
A_μ	0	0
φ	$v\sqrt{2\lambda}$	0
$W^\pm_\mu$	$v\lvert e\rvert$	$\pm e$

2.2.1 't Hooft-Polyakov Monopoles

Above we examined the trivial vacuum solution where the orientation of the Higgs field was not dependent on the spatial coordinates

$$\Phi = (0, 0, v). \tag{2.23}$$

However, the Georgi-Glashow model admits other localised finite energy solutions to the classical equations of motion (so called solitons), where the directions of the Higgs field varies over space. The energy of a field configurations in the static (time independent) case is given by

$$E = \int d^3x \frac{1}{2} \left(E_n^a E_n^a + B_n^a B_n^a + D_n \Phi^a D_n \Phi^a \right) + V(\Phi) \tag{2.24}$$

where

$$E_n^a = F_{0n}^a, \qquad \text{and} \qquad B_n^a = \frac{1}{2} \epsilon_{nmk} F_{mk}^a. \tag{2.25}$$

Evidently, for a finite energy solution we must impose that the vacuum condition is obtained at the boundary. Such a spherically symmetric solution was found by 't Hooft and Polyakov, and can be described by the ansatz:

$$\Phi^a = vh(r) \frac{r^a}{r}, \qquad A_a^i = -\left(1 - k(r)\right) \epsilon_{aij} \frac{r^j}{er^2}, \qquad A_a^0 = 0. \tag{2.26}$$

with functions h and k satisfying boundary conditions to avoid a singularity at the origin and a finite energy solution:

$$h(0) = 0, \qquad k(0) = 1, \qquad \lim_{r\to\infty} h(r) = 1, \qquad \text{and} \qquad \lim_{r\to\infty} k(r) = 0. \tag{2.27}$$

In this case we have that $\lim_{r\to\infty} \Phi(\mathbf{r}) = v\hat{\mathbf{r}}$, and everywhere the Higgs field points radially outwards to form the so-called *hedgehog* solution. It is important to emphasize that this solution is **topologically protected** as it is impossible to continuously transform it into the trivial vacuum solution.

Extra: On the topology of the 't Hooft-Polyakov solutions
Requiring that the field configuration obtains a vacuum solution on the boundary defines a map from the sphere at infinity to the Higgs vacuum manifold, which in this case is also a sphere with radius v:

$$\varphi : S_\infty^2 \to S_v^2. \tag{2.28}$$

The classification of such maps is given by the homotopy group $\pi_2(S^2) = \mathbb{Z}$. Hence they are classified by an integer, the Brouwer degree, n, giving the wrapping number of the domain manifold around the image manifold. The ansatz above corresponds to $n = 1$, whereas the trivial solutions corresponds to $n = 0$.

For a general gauge group G broken to a subgroup H we would be interested in the structure of the second homotopy group of the vacuum manifold:

$$\pi_2(G/H). \tag{2.29}$$

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The explicit solutions of h and k for the classical equations of motion can be found numerically, but here we will only comment on the features of this solution. By definition of the ansatz it has an asymptotic behaviour as $r \to \infty$ which looks like

$$\Phi^a = v\frac{r^a}{r}, \qquad A_a^i = -\epsilon_{aij} \frac{r^j}{er^2}, \qquad A_a^0 = 0, \tag{2.30}$$

and solutions of the shape function h tells us that the asymptotic value is obtained very fast. This means that there is a core of energy centered at the origin with a characteristic radius and mass:

$$R_c \sim \frac{1}{ev}, \qquad M = \frac{4\pi v}{e} f\left(\frac{\lambda}{e^2}\right) \tag{2.31}$$

where f is function with $f(0) = 1$, and $f(\infty) = 1.787$. Now we would like to know how this solution is charged under the unbroken $U(1)$ gauge group, and a calculation shows that the field strength has the asymptotic behaviour

$$F_{ij} \sim \epsilon_{ijk} \frac{r^k}{er^3} \tag{2.32}$$

which tells us that asymptotically there is a magnetic field of a monopole

$$\mathbf{B} = -\frac{1}{er^2}\hat{\mathbf{r}}. \tag{2.33}$$

A comparison with equation (2.11) or direct calculation shows that it has magnetic charge

$$g = \int_{S^2_\infty} \mathbf{B} \cdot d\mathbf{S} = -\frac{4\pi}{e}. \tag{2.34}$$

and satisfies the Dirac quantization (with a subtlety for the factor of 2[1]). Thus, we have found that the 't Hooft-Polyakov solution indeed behaves like a magnetic monopole in the Georgi-Glashow model.

Bibliography

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[1]The obtained charge is twice ($n = 1$) the Dirac charge, because we are only considering field in the adjoint. Adding matter fields in the fundamental accounts for $n = 1$,