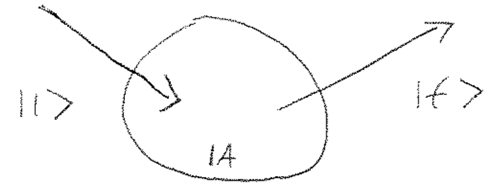
1) Decay rates and phase-space

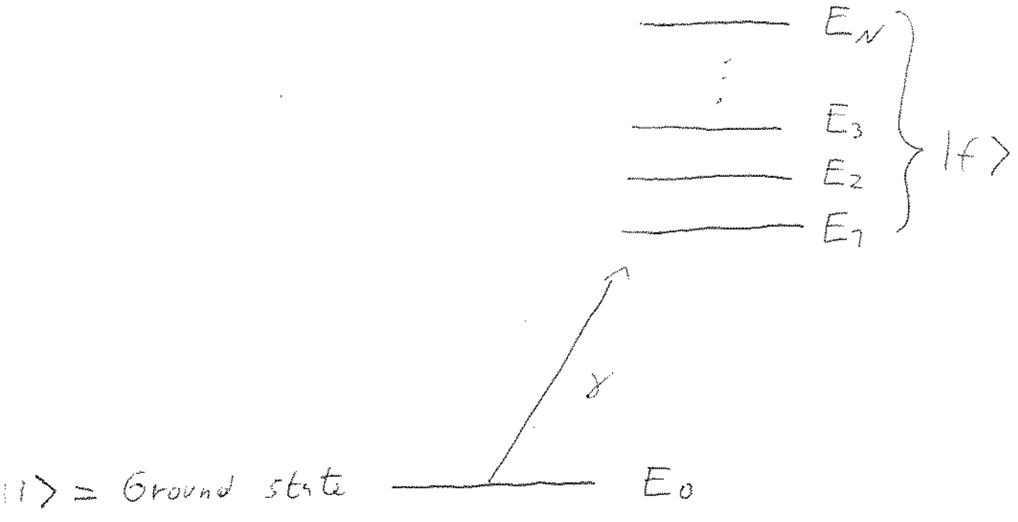
a) Fermi's Golden Rule

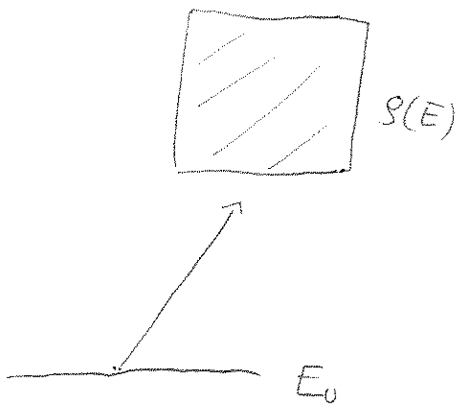


Transition rate from QM state |i> to final state |f>:

$$\underbrace{W_{fi}}_{\frac{\text{Transition prob.}}{\text{time}}} = \underbrace{|A_{fi}|^2}_{\substack{\text{QM amplitude describing}\\ \text{the interaction}\\ \text{(matrix element)}}} \quad \underbrace{\rho_f}_{\substack{\text{density of final}\\ \text{states}}}$$

Example: Hydrogen atom

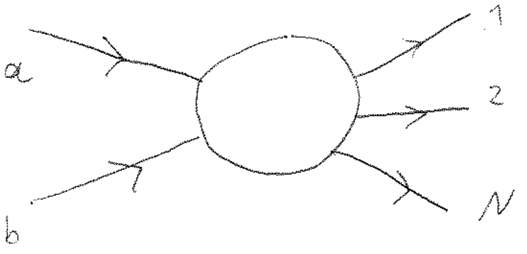


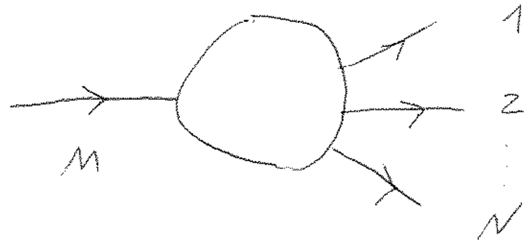


$$P(i \to f) \propto N \underbrace{|A_{fi}|^2}_{\substack{\text{QED effects}\\ \text{e.g. spin-orbit coupling}\\ \Rightarrow \text{fine structure}}}$$

$$P(i \to f) = \int |A(E)|^2 \rho(E)\, dE$$

Collider Experiments :





Scattering

$\Rightarrow$ Cross-section σ

Decay

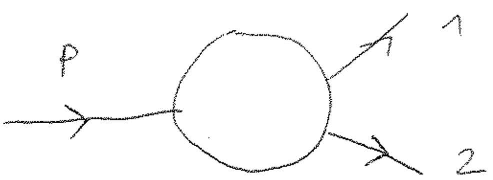
$\Rightarrow$ Decay rate Γ

$$\Gamma(M \to 12 \cdots N) = \frac{1}{M} \int |A_{fi}|^2 \underbrace{\rho \; d^4p_1 \cdots d^4p_j}_{d\phi_N \,=\, \text{phase-space element}}$$

$$\rho = 1 \begin{cases} E, \vec{p} \text{ conserved} \\ \text{particles are on-shell} \\ E \text{ positive} \end{cases} \quad \text{or} \quad \rho = 0 \quad \text{otherwise}$$

$$\rho = \delta\left(P^\mu - \sum_f p_f^\mu\right) \prod_f \delta(p_f^2 - m_f^2) \; \theta(p_f^0)$$

b) 2-body decay



$$P = P_1 + P_2 \overset{\text{CMS}}{=} \begin{pmatrix} M \\ \vec{0} \end{pmatrix}$$

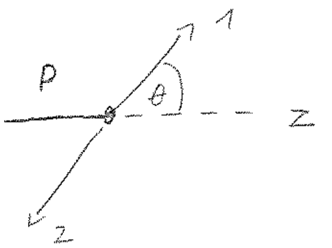
$$m_1 = m_2 = m$$

$$d\phi_2 = d^4p_1\, d^4p_2\ \delta(P - P_1 - P_2)\ \delta(P_1^2 - m^2)\ \delta(P_2^2 - m^2)\ \theta(P_1^0)\ \theta(P_2^0)$$

$$\overset{\text{Proof 1}}{=} \frac{d^3p_1}{E_1}\ \frac{d^3p_2}{E_2}\ \underbrace{\delta(P - P_1 - P_2)}_{\delta(M - E_1 - E_2)\ \delta(\vec{p}_1 + \vec{p}_2) \quad \text{in CMS}}$$

$$= \frac{d^3p_1}{m^2 + \vec{p}_1^{\,2}}\ \delta\left(M - 2\sqrt{m^2 + \vec{p}^{\,2}}\right)$$

spherical coordinates : $d^3p_1 = |\vec{p}_1|^2\, d|\vec{p}_1|\ \underbrace{d\Omega}_{d\cos\theta\ d\varphi}$



$$E_1 = E_2 = \frac{M}{2}$$

$$\vec{p}_1 = -\vec{p}_2$$

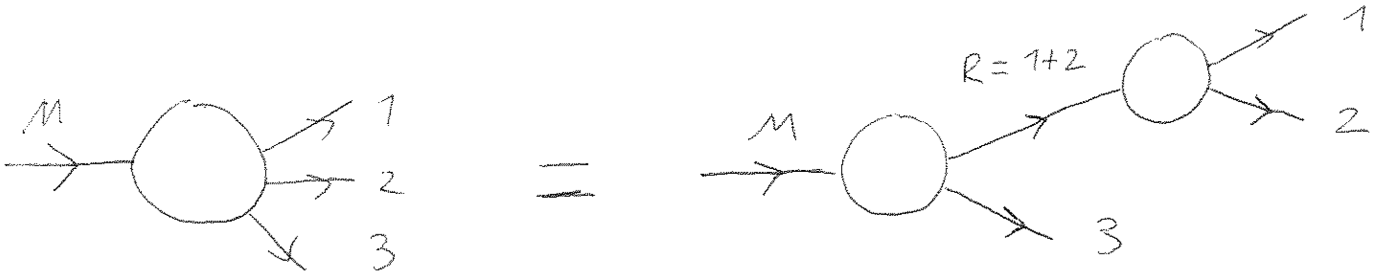
$$\Rightarrow d\phi_2 \overset{\text{Proof 2}}{=} \sqrt{1 - \frac{4m^2}{M^2}}\ d\Omega$$

$$\Rightarrow \Gamma(M \to 12) = \frac{1}{M}\sqrt{1 - \frac{4m^2}{M^2}}\ \underbrace{\int |A|^2\, d\Omega}_{= |A|^2 \text{ if } M, 1, 2 \text{ spin-0}}$$

General solution : $\dfrac{d\Gamma(M \to 12)}{d\Omega} = \dfrac{\sqrt{\lambda(M^2, m_1^2, m_2^2)}}{M^3}\ |A|^2$

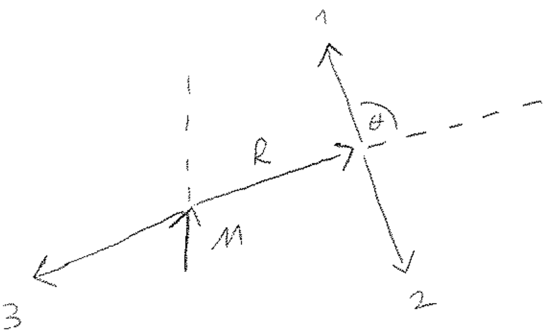
$$\lambda(a, b, c) := a^2 + b^2 + c^2 - 2(ab + ac + bc)$$

c) 3-body decay



Decompose 3-body phase-space into product of 2-body phase-space elements:

$$d\phi_3(M\to 123) = \underbrace{d\phi_2(M\to R3)}_{\frac{\lambda(M^2, m_{12}^2, m_3^2)^{\frac{1}{2}}}{M^2} d\Omega_0} \; \underbrace{d\phi_2(R\to 12)}_{\frac{\lambda(m_{12}^2, m_1^2, m_2^2)^{\frac{1}{2}}}{m_{12}^2} d\Omega_1} \; \underbrace{dm_{12}^2}_{\text{Inv. mass of effective particle } R:\; m_{12}^2 = (P_1+P_2)^2}$$



CMS: $\vec{P_1} + \vec{P_2} + \vec{P_3} = \vec{0}$

HF: $\vec{P_2} + \vec{P_3} = \vec{0}$

$m_i = 0$

In HF: $P_1 = E_1 \begin{pmatrix} 1 \\ \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$; $P_2 = E_1 \begin{pmatrix} 1 \\ -\sin\theta\cos\varphi \\ -\sin\theta\sin\varphi \\ -\cos\theta \end{pmatrix}$; $P_3 = E_3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$

$$p^2 = M^2 = \left((P_1+P_2)+P_3\right)^2 = m_{12}^2 + 2(P_1+P_2)P_3 + \cancel{P_3^2}$$
$$\overset{HF}{=} m_{12}^2 + 4E_1E_3$$

$$m_{13}^2 = (P_1+P_3)^2 = 2P_1P_3 \overset{HF}{=} 2E_1E_3(\cos\theta+1)$$
$$= \frac{1}{2}(M^2 - m_{12}^2)(1+\cos\theta) \left\} \begin{array}{l} = 0, \quad \cos\theta = -1 \\ = M^2 - m_{12}^2, \quad \cos\theta = 1 \end{array} \right.$$

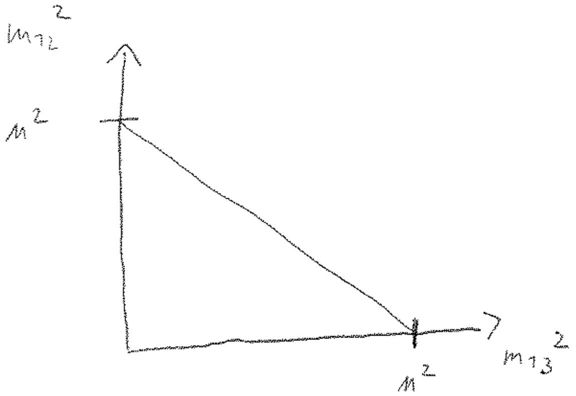
$$\Rightarrow \frac{\partial m_{13}^2}{\partial\cos\theta} = M^2 - m_{12}^2$$

$$d\Phi_3 = \frac{M^2 - m_{12}^2}{M^2} \, dm_{12}^2 \, d\cos\theta \, d\varphi \, d\Omega_0$$

$$= \frac{1}{M^2} \, dm_{12}^2 \, dm_{13}^2 \, d\Omega_0 \, d\varphi$$

$$\Rightarrow \Gamma(M \to 123) = \frac{1}{M^3} \int_0^{M^2} \int_0^{M^2 - m_{12}^2} |A|^2 \, dm_{12}^2 \, dm_{13}^2$$

(if $M, 1, 2, 3$ spin-0)



= Dalitz - Plot

- If matrix element constant $\Rightarrow$ Dalitz-Plot uniformly populated
- If non-uniform $\Rightarrow$ Information on underlying physics
- Phase-space boundaries:

Maximum of m_{13}^2 reached when particle 2 at rest in CMS:

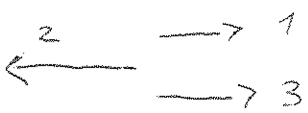
$$m_{13}^2 = (P_1 + P_3)^2 = (P - P_2)^2 \overset{\text{CMS}}{=} (M - |\vec{P}_2|)^2$$

$$\Rightarrow m_{13}^2\big|_{max} = M^2$$

Minimum of m_{13}^2 when $\vec{P}_2$ is maximal:

$$m_{13}^2 \overset{\text{CMS}}{=} 2(|\vec{P}_1||\vec{P}_3| - \vec{P}_1 \vec{P}_3) = 0 \text{ , if } \vec{P}_1, \vec{P}_3 \text{ collinear; } \vec{P}_2 = \vec{P}_2\big|_{max}$$

$$\Rightarrow m_{13}^2\big|_{min} = 0$$



Proof 1: $d^4p\ \delta(p^2-m^2)\ \theta(p^0) = \frac{d^3p}{2E}$

$$d^4p\ \delta(\underbrace{p^2-m^2}_{p^{0\,2}-\underbrace{\vec{p}^2-m^2}_{E^2} = p^{0\,2}-E^2})\ \theta(p^0)$$

$$= \frac{d^3p\, dp^0}{2E}\left(\delta(p^0-E)+\underbrace{\delta(\underbrace{p^0}_{>0}+\underbrace{E}_{>0})}_{=0}\right)\theta(p^0) = \frac{d^3p}{2E} \quad \checkmark$$

Using: $\delta(x^2-a^2) = \frac{1}{2|a|}\left(\delta(x-a)+\delta(x+a)\right)$

Proof 2: $\int dp\ \frac{|p|^2}{m^2+p^2}\ \delta\left(M-2\sqrt{m^2+p^2}\right) = \sqrt{1-\frac{4m^2}{M^2}}$

$$\delta(g(x)) = \frac{\delta(x-x_i)}{|g'(x_i)|} \quad ; \quad g(x_i)=0$$

$$g = M-2\sqrt{m^2+p^2} \stackrel{!}{=} 0 \Rightarrow p = \sqrt{\frac{M^2}{4}-m^2}$$

$$g' = \frac{2p}{\sqrt{m^2+p^2}}$$

$$\Rightarrow \delta\left(M-2\sqrt{m^2+p^2}\right) = \frac{\sqrt{m^2+p^2}}{2p}\ \delta\left(p-\left(\frac{M^2}{4}-m^2\right)^{\frac{1}{2}}\right)$$