Motivation

Why String theory?

"Popular argument": Candidate for Quantum Gravity

However: many unresolved issues

Undisputed: String theory uses physics to uncover mathematical mysteries !!

Teaser: "Moonshine"

'Unexpected' (from a mathematical point of view) relation between complex analysis and finite groups:

- well-known: j-invariant $j(\tau)$ is a meromorphic function (related to geometry of tori)

$\longrightarrow$ Fourier expansion

$$j(\tau) = \underbrace{1}_{a_1} \cdot e^{-2\pi i \tau} + 744 + \underbrace{196884}_{a_2} e^{2\pi i \tau} + \underbrace{21493760}_{a_3} e^{2\pi i (2\tau)} + \underbrace{864299970}_{a_4} e^{2\pi i (3\tau)} + \dots$$

- "monster group" $\mathbb{M}$ (largest "exceptional" finite simple group) has $\approx 8 \cdot 10^{53}$ elements

$\mathbb{M}$ has irreducible representations of dimensions

$r_1 = 1, \quad r_2 = 196883, \quad r_3 = 21296876, \quad r_4 = 842609326, \dots$

observation:

$\Longrightarrow a_1 = r_1, \quad a_2 = r_1 + r_2, \quad a_3 = r_1 + r_2 + r_3$

$a_4 = 2r_1 + 2r_2 + r_3 + r_4, \quad \dots$

Relations between all coefficients and all representations! → not just coincidence!

Original math proof already uses theorems discovered in the context of string theory, but still "technical"

Physical explanation provided by Witten in 2007:

There is a QFT (in 2 dimensions) constructed with ST, with partition function $Z[q] = j(\tau)$, $q = 2\pi i \tau$

In path integral formulation of QFT, all data is encoded in the partition function

$$Z[q] = \int \mathcal{D}\phi \, \exp\left(i S[\phi] + i \int d^d x \, q \, \phi \right)$$

Correlation functions are computed as

$$\langle \phi(x_1) \ldots \phi(x_n) \rangle = \left. \frac{\delta^n Z[q]}{\delta q(x_1) \ldots \delta q(x_n)} \right|_{q=0}$$

This QFT has $\mathbb{M}$ as a symmetry
→ Hilbet space $\mathcal{H} = V_1 \oplus V_2 \oplus \ldots$
where V_i are irreducible repesentations of $\mathbb{M}$

$Z \cong$ trace over Hilbert space ⇝ Z counts the dimension of V_i!
⇒ physics can provide link between areas of mathematics with a priori no known connection!

By now: many other examples of Moonshine constructed from String theory

Intro to String theory

Basic idea : replace point particles by extended objects, strings

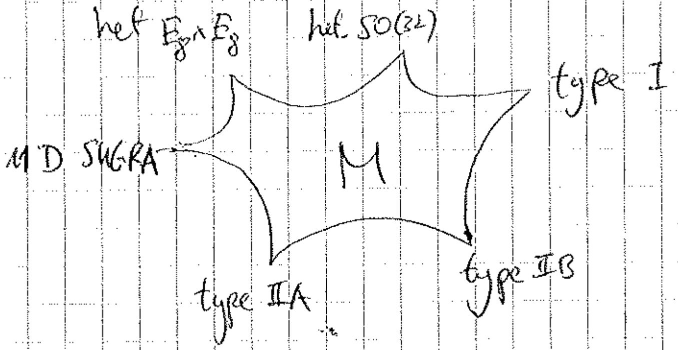
If small, we only perceive them ~~as~~ as point-like.
Different vibrations $\longleftrightarrow$ different "particles"

Quantize string oscillations : obtain massless excitations and massive modes, latter to heavy to be observed $\rightsquigarrow$ focus on massless modes
two curiosities : 1. massless modes contain a spin 2 particle $\rightarrow$ graviton?
2. scattering amplitudes are UV-finite
$\rightarrow$ quantum gravity ?

massless modes form representations of Lorentz-group
$\rightarrow$ effective field theory description including gravity ("effective" $\rightarrow$ non-renormalisable)

"Supergravity (SUGRA)" $\rightarrow$ supersymmetric

Consistency of string theory (not just of SUGRA!) require 9+1 spacetime dimensions !
Only five possible formulations in 10D
($\Rightarrow$ different field contents of corresponding SUGRA)
In fact, all 5 S.T.'s are related to a unique theory in 11D, whose existence is conjectured ; but we know its effective field theory :
11D SUGRA (unique SUGRA in 11D !!)



Full theory: M-theory
corners $\hat{=}$ certain limits of M-theory

Analogy: M-th $\hat{=}$ Standard Model
corners $\hat{=}$ QED, Fermi theory, etc.

Branes:

super gravity: have metric as a dynamical field governed by Einstein-Hilbert action:

$$S \supset \int d^{10}x \, \sqrt{-g} \, \underbrace{R(g)}_{\text{Ricci-scalar}}$$

$\rightarrow$ can have non-trivial vacuum solutions
in 4D: black holes, i.e. point-like mass concentration

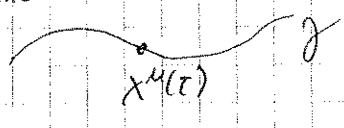
in higher dimensions: can also have higher dimensional "black" objects, e.g. membranes

$\rightsquigarrow$ branes are gravitational solutions

SUGRAs also have gauge fields!
Recall electromagnetism:

gauge potential: $A_\mu \xrightarrow[\text{transf.}]{\text{gauge}} A_\mu + \partial_\mu f$

coupling of a charged particle: $S \supset \underset{\uparrow \text{charge}}{q} \int dx^\mu A_\mu$
moving on trajectory $x^\mu(\tau)$
world line

geometrically: 

$\longrightarrow \int_\gamma A$

A is a 1-form
(a tensor with 1 index)

In SUGRAs : higher "dimensional" gauge fields (5)

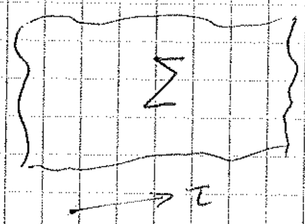
e.g. $B_{\mu\nu} \equiv B_2 \xrightarrow[\text{transf.}]{\text{gauge}} B_2 + d\Lambda_1$

$B_{\mu\nu}$: anti-symm $\leftrightarrow$ 2-form

$\Lambda_1 \equiv \Lambda_\mu$ a 1-form

$(d\Lambda)_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$ 2-form

can be integrated over a 2-dimensional world sheet :



$\longrightarrow \int_\Sigma B_{\mu\nu}\, dx^\mu dx^\nu$

Σ is the worldsheet of a string, that is charged under $B_2 \longleftrightarrow B_2$ is sourced by string !

Higher dimensional analogues:

n-form field $C_n \xrightarrow[\text{transf.}]{\text{gauge}} C_n + d\Lambda_{n-1}$

C_n: n-form

$\rightarrow$ can be ~~source~~ sourced by a (mem-) brane with (n-1) spatial dimensions!

These branes are also the gravity solutions !
(compare to Reissner-Nordström black holes in 4D!!)

$\rightarrow$ String theory also contains branes. Branes are themselves dynamical objects.

Important: Dynamics is the same as that of an 'ordinary' QFT with gauge symmetries !!

Compactifications:

In order to obtain physics in 4D from String theory, we "curl up" 6 of the 10 dimensions:

$$M_{10} = M_4 \times X_6$$

M_4: extended, non-compact, spacetime

X_6: compact space → size small → need high energies to resolve

intuitive example: a wire

Moreover, geometry of X determines physics in M_4! Historic example: Kaluza-Klein theory

GR in 5D: degrees of freedom is metric G_{MN}

$$G_{MN} = \left(\begin{array}{cccc|c} & & & & G_{04} \\ & G_{\mu\nu} & & & G_{14} \\ & \equiv g_{\mu\nu} & & & G_{24} \\ & & & & G_{34} \\ \hline G_{40} & G_{41} & G_{42} & G_{43} & G_{44} \end{array}\right) \qquad \begin{array}{l} \mu, \nu = 0, \dots, 3 \\ x^M = (x^\mu, y) \\ A_\mu := (G_{04}, G_{14}, G_{24}, G_{34}) \end{array}$$

Now compactify fifth dimension: $y \cong y + 2\pi r \rightarrow$ circle

i.e. $M_5 = M_4 \times S^1 \Rightarrow$ can Fourier expand any (x^μ, y) function $f(x^\mu, y) = \sum_k f_k(x^\mu) e^{iky/r}$

5D action: $S_{5D} \sim \int d^5x \sqrt{-G}\, R(G) = \int d^4x\, dy \dots$

can explicitly perform y-integration, end up with 4D action

$$S_{4D} \sim \frac{1}{16\pi G_N^{(4D)}} \int d^4x \sqrt{-g}\, R(g) + \frac{1}{4} \int d^4x\, F_{\mu\nu} F^{\mu\nu}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \longrightarrow$ Electromagnetism

Furthermore: $G_N^{(4D)} \sim \frac{1}{r} G_N^{(5D)} \Rightarrow$ radius (= geometry) of S^1 determines relative strength of gravity & EM in 4D.