# Matching

(Lecture # 3)

The concept of matching can be broken down to a single question

- How does the physics at high energies show up in the low energy theory?

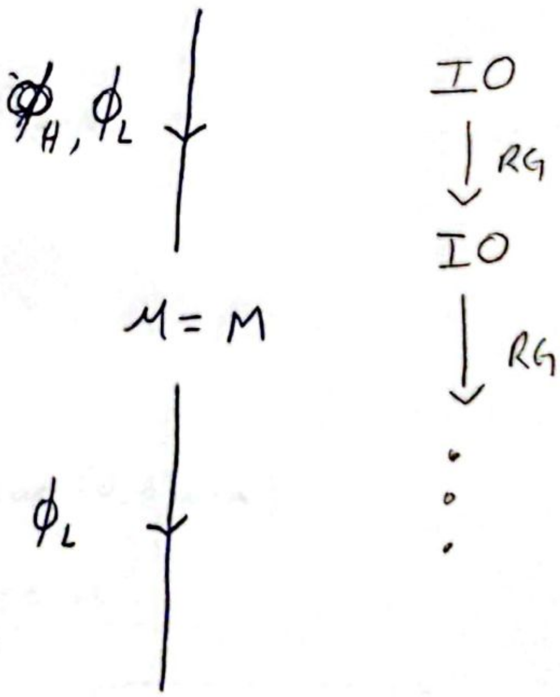
* We need to keep track of physical effects in the low and high energy theories in order to have consistency between them. Matching is nothing else but the right establishment of that connection.

In a nutshell, performing the matching consist of the following steps.

1) Computing the amplitudes in both the full and EFT theories up to a certain loop level

2) Equate the resulting amplitudes order by order in the perturbation theory

3) Solve for the wilson coefficients $C_i$

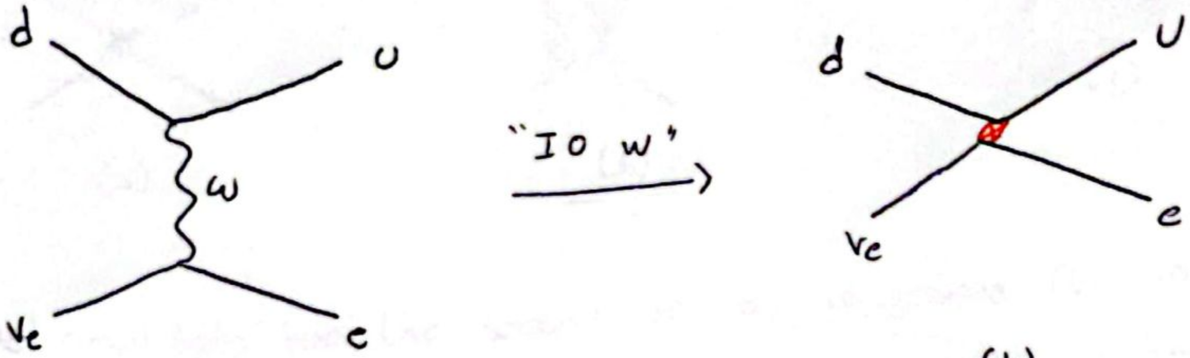



• Matching is specially useful as it is not always feasible to integrate out heavy modes formally (as shown in Lecture # 1)

• Matchin is also an excelent tool to deal with large logarithms as they dissapear when performing the matching.

## Fermis theory of weak interactions

**Tree level** $(E \ll m_W)$




(a) (b)

The amplitud in the full theory is:

$$\mathcal{M}^a_{UV} = -\frac{g^2 V_{ud}}{8} \frac{1}{p^2 - m_W^2} (\bar{e}, \nu_e)_{V-A} (\bar{u}, d)_{V-A}$$

$$= -\frac{g^2 V_{ud}}{8\, m_W^2} (\bar{e}, \nu_e)_{V-A} (\bar{u}, d)_{V-A} + \mathcal{O}\left(\frac{p^2}{m_W^2}\right)$$

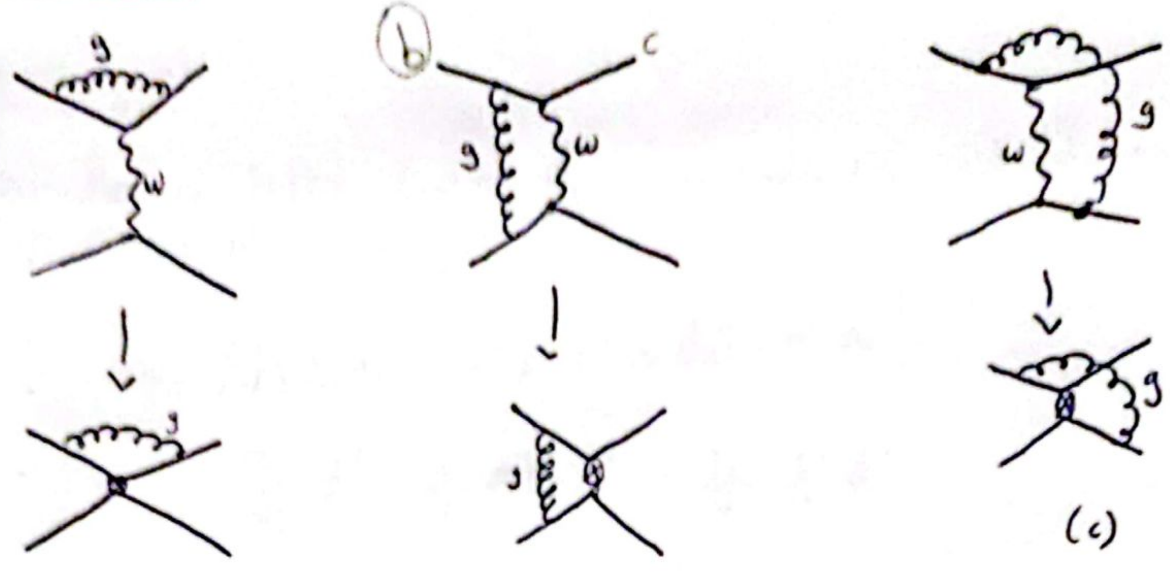
where $(\bar{e}, \nu_e)_{V-A} = \bar{e}\gamma^\mu(1-\gamma^5)\nu_e$ (same for $(\bar{u}, d)_{V-A}$)

On the other hand the EFT amplitud is

$$\mathcal{M}^b_{EFT} = \frac{C}{\Lambda^2} (\bar{u}, d)_{V-A} (\bar{e}, \nu_e)_{V-A} \quad \Rightarrow \quad \boxed{\begin{array}{c} \Lambda^2 = M_W^2 \\ C = -\dfrac{g^2 V_{ud}}{8} \end{array}}$$

$$\frac{G_F}{\sqrt{2}} = \frac{g^2 V_{ud}}{8\, m_W^2} \Rightarrow m_W \sim \left(\frac{g^2\sqrt{2}}{8\, G_F}\right)^{\frac{1}{2}} \sim \mathcal{O}(10^2\ \text{GeV}$$

## Loop - level




(a) (b) (c)

The amplitude for the second set of diagrams (b) in the full theory reads

$$\mathcal{M}^b_{UV} = \lambda_1 \frac{\alpha_s(\mu)}{4\pi} \ln\left(\frac{m_W^2}{m_b^2}\right)$$

In general for large disparity of scales

$$G_{log} = \sum_{n=1}^{\infty} \left(\frac{\alpha_s(\mu)}{4\pi}\right)^n \sum_{m=0}^{n} C_m^n \ln(s)^m$$

In the EFT the amplitude reads

$$\mathcal{M}^b_{EFT} = \lambda_1 \frac{\alpha_s(\mu)}{4\pi} \left(\cancel{\frac{1}{\varepsilon}}^{\to 0 \ \overline{MS}} - \ln\left(\frac{m_b^2}{\mu^2}\right)\right) + \kappa$$

Equating the full and EFT amplitudes we get

$$\underbrace{\frac{C}{\Lambda^2}}_{C'(\mu)} \left(1 + \lambda_1 \frac{\alpha_s(\mu)}{4\pi} \left(1 - \ln\left(\frac{m_b^2}{\mu^2}\right)\right) + \kappa\right) = \underbrace{\frac{g^2}{8 m_W^2}}_{\frac{G_F}{\sqrt{2}}} \left(1 + \lambda_1 \frac{\alpha(\mu)}{4\pi} \ln\left(\frac{m_W^2}{m_b^2}\right)\right)$$

$C(\mu)$ = full - EFT

$$C'(\mu) = \frac{G_F}{\sqrt{2}} \left(1 + \lambda_1 \frac{\alpha_s(\mu)}{4\pi} \left(\ln\left(\frac{m_W^2}{m_b^2}\right) + \ln\left(\frac{m_b^2}{\mu^2}\right) + \rho\right)\right)$$

Setting $\mu = m_W$ $\qquad$ $\rho = \kappa = cte$

$$\boxed{C'(\mu) = \frac{G_F}{\sqrt{2}} \left(1 + \lambda_1 \frac{\alpha_s(\mu)}{4\pi} \rho\right)}$$

No logs !! ∪

## General example

### Tree level

Consider a light field $\phi_L$ with mass $m_L$ and a heavy field $\phi_H$ with mass $M$. then

$$\mathcal{L}_{UV} = \frac{1}{2}\left[(\partial_\mu\phi_L)^2 - m_L^2\phi_L^2 + (\partial_\mu\phi_H)^2 - M^2\phi_H^2\right]$$

$$- \frac{\lambda_0}{4!}\phi_L^4 - \frac{\lambda_1}{2} M\phi_H\phi_L^2 - \frac{\lambda_2}{4}\phi_L^2\phi_H^2$$

• we assume $\phi_H^3$ and $\phi_H^4$ irrelevant for the discussion and impose a $Z_2$ parity symmetry such that $\phi_L \xrightarrow{Z_2} -\phi_L$

After integrating out the heavy field the lagrangian reads.

$$\mathcal{L}_{EF} = \frac{1}{2}\left[(\partial_\mu\phi_L)^2 - m^2\phi_L^2\right] - \frac{C_4}{4!}\phi^4 - \cancel{\frac{C_6}{M^2}\frac{\phi_L^6}{6!}} + \ldots$$

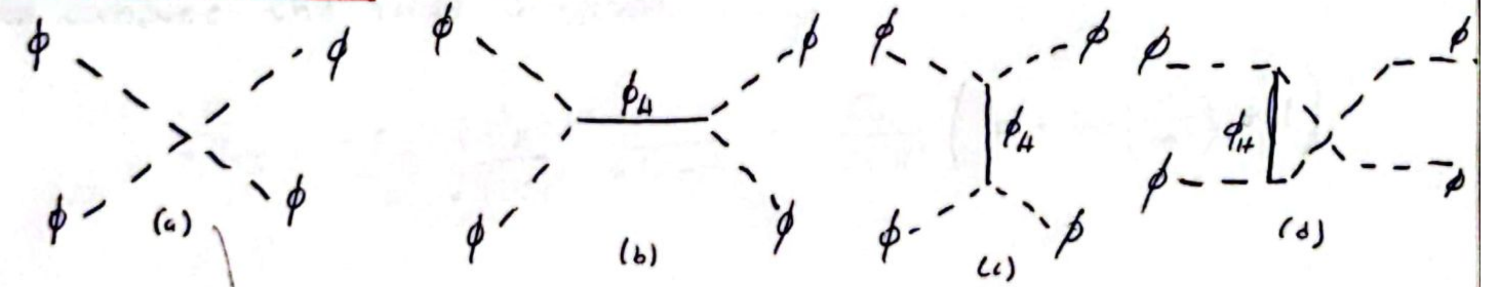
Ignore in the following

still generic case (no matching)

$m$ and $C_4$ are free parameters

one loop — tree level

### 2-to-2 scattering



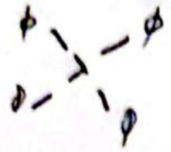

$$\mathcal{M}_{UV} = -\lambda_0 - \lambda_1 M^2\left[\frac{1}{s-M^2} + \frac{1}{t-M^2} + \frac{1}{u-M^2}\right]$$

$$\boxed{\approx -\lambda_0 + 3\lambda_1^2 + \frac{4m_L^2\lambda_1^2}{M^2} + \mathcal{O}(M^{-4})}$$

$s+t+u = 4m_L^2$

$\frac{\lambda_1^2}{M^2}(s+t+u)$

In the EFT we only have the contribution coming from (a)




$\Rightarrow$ $\mathcal{M}_{eff} = -C_4$ $\qquad \frac{C_4 \phi^4}{4!}$

Equating the UV and EFT we get

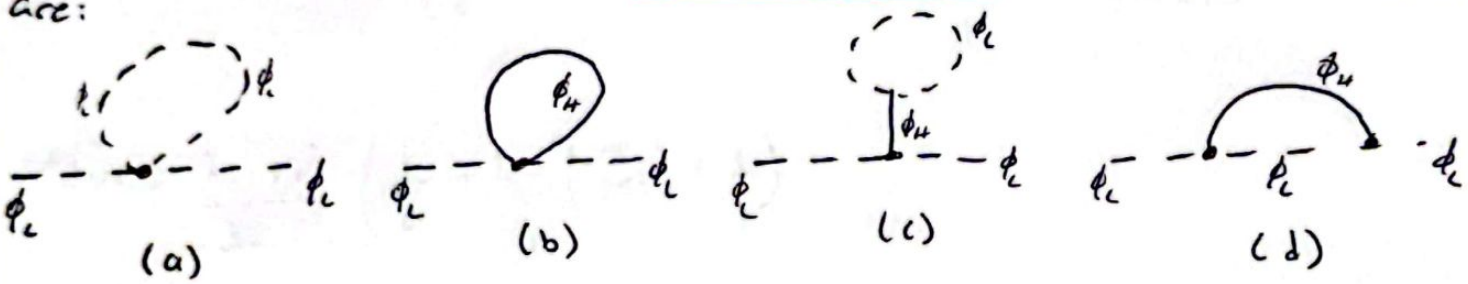
$$\boxed{C_4 = \lambda_0 - 3\lambda_1^2 - \frac{4\lambda_1^2 m_L^2}{M^2}}$$

$\phi^4$ $(\phi\phi \to \phi\phi)$

## Loop level

$\phi_L^2 \phi_H$ $\qquad m^2 = m_L^2$

We start with the 1PI 2-point function. The one loop contributions are:




Let us start with the EFT calculation now. Here we just need to compute the first diagram.

$$\delta\Pi_{eff} = \frac{-C_4}{2}\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2} = \frac{C_4}{32\pi^2}\left(\cancel{\frac{1}{\varepsilon}}\xrightarrow{\overline{MS}} 0 + \log\left(\frac{\mu^2}{m^2}\right) + 1\right)$$

• $\mu$ is a dimension full parameter introduced trough the dimensional regularization prescription and can be identified with the renormalization scale

The physical mass in the EFT reads.

$$m^2_{Phys.} = m^2 - \frac{C_4}{32\pi^2}\left(\log\left(\frac{\mu^2}{m^2}\right) + 1\right) \qquad (\star)$$

$m_{phys}$ is an observable an thus should not depend on the arbitrary scale $\mu$. To solve this, the mass term in the one loop contribution should be promoted to a scale dependent object $m^2(\mu)$. To ensure it the parameter $m^2$ must satisfy the RG equation

$$\frac{dm^2}{d\log\mu} = \frac{C_4 m^2}{16\pi^2}$$

Now let us move to the UV theory.

$$\mathcal{M}^a_{UV} = \lambda_0 \frac{m_L^2}{32\pi^2}\left(\frac{1}{\varepsilon} + \log\left(\frac{\mu^2}{m_L^2}\right) + 1\right) \to \text{Same as EFT}$$

$$\mathcal{M}^b_{UV} = \frac{\lambda_2 M^2}{32\pi^2}\left(\frac{1}{\varepsilon} + \log\left(\frac{\mu^2}{M^2}\right) + 1\right)$$

$$\mathcal{M}^c_{UV} = (-i)(-i\lambda_1 M)^2 \frac{1}{-M^2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2} = \frac{-\lambda_1^2 m_L^2}{32\pi^2}\left[\frac{1}{\varepsilon} + \log\left(\frac{\mu^2}{m_b^2}\right) + 1\right)$$

$$\mathcal{M}^d_{UV} = (-i)(-i\lambda_1 M)^2 \int\frac{d^dk}{(2\pi)^d}\frac{i^2}{(k^2-M^2)(k+p)^2-m_L^2}$$

$$\to \frac{\lambda_1 M^2}{16\pi^2}\left(\frac{1}{\varepsilon} + \log\left(\frac{\mu^2}{M^2}\right) + 1\right) + \frac{\lambda_1^2 m_L^2}{32\pi^2}\left(-2\log\left(\frac{M^2}{m_L^2}\right) + 1\right)$$

$$+ \frac{\lambda_1^2 m_L^4}{48\pi^2 M^2}\left(-6\log\left(\frac{M^2}{m_L^2}\right) + 5\right)$$

Accounting all the contributions at one-loop in the $\overline{MS}$ scheme, the physical mass of the light field is given by

$$m^2_{phys} = m_L^2 - \underbrace{\left(\lambda_0 - 3\lambda_1^2 - 4\lambda_1^2 \frac{m_L^2}{M^2}\right)}_{c_4} \frac{m_L^2}{32\pi^2}\left(\log\left(\frac{\mu^2}{m_L^2}\right) + 1\right)$$

$$- \frac{1}{32\pi^2}\log\left(\frac{\mu^2}{M^2}\right)\left(M^2(\lambda_2 + 2\lambda_1^2) + 2\lambda_1^2 m_L^2 + 4\lambda_1^2 \frac{m_L^4}{M^2}\right) \qquad (*)$$

$$- \frac{1}{32\pi^2}\left(M^2(\lambda_2 + 2\lambda_1^2) + 3\lambda_1^2 m_L^2 + \frac{22}{3}\lambda_1^2 \frac{m_L^4}{M^2}\right)$$

Equating the UV and EFT equations $(*)$ we find

$$m^2(\mu) = m_L^2(\mu) - \frac{1}{32\pi^2}\overset{\to 0\ \mu = M}{\log\left(\frac{\mu^2}{M^2}\right)}\left(M^2(\lambda_2 + 2\lambda_1^2) + 2\lambda_1^2 m_L^2 + 4\lambda_1^2 \frac{m_L^4}{M^2}\right)$$

$$- \frac{1}{32\pi^2}\left(M^2(\lambda_2 + 2\lambda_1^2) + 3\lambda_1^2 m_L^2 + \frac{22}{3}\lambda_1^2 \frac{m_L^4}{M^2}\right)$$

We can see at this point the cancellation of the $\log\left(\frac{\mu^2}{m_L^2}\right)$. → IR physics is the same in both theor[ies]

Now, choosing $\mu = M$ we get the matching equation

$$\boxed{m^2(M) = m_L^2(M) - \frac{1}{32\pi^2}\left(M^2(\lambda_2 + 2\lambda_1^2) + 3\lambda_1^2 m_L^2 + \frac{22}{3}\lambda_1^2 \frac{m_L^4}{M^2}\right)}$$

where there are No logarithm dependence!

Final
## Take away

→ The fermion mass spectrum spans over several orders of magnitude and it give rise to still unanswered questions

→ EFTs are excellent for multi-scale problems
- model independent
- Offer a modern view on renormalization

→ One can formulate new frameworks like eDMEFT < EFT / Simp. models and impose some symmetries (like $Z_2$) to create a specific model tha explains for example, $m_1 \ll m_2 m_3$ and be testable at colliders

→ By performing matching one gets rid of large logarithms

→ The wilson coefficients keep the info of the UV theory and ensure the consistency between both UV and EFT theories.