Student lectures on:

Dualities in Field and String Theory

Christian Reichelt

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Summary of important points from the first lecture:

- Maxwell's equations have an electromagnetic duality $(\mathbf{E}, \mathbf{B}) \mapsto (\mathbf{B}, -\mathbf{E})$ when introducing magnetic monopoles.
- Dirac quantisation (quantisation of electric and magnetic charge) is a directs consequence of magnetic sources in a quantum theory.
- The Georgi-Glashow $SU(2)$ Higgs model also has monopole solutions, as finite energy solutions to the classical equations of motion (solitons).
- These 't Hooft-Polyakov monopoles are different from the Dirac monopole (they are non-singular), and are characterised by their topology.

For completeness we note that when extending the analysis to other topological solutions one finds a set of possible magnetic charges:

$$g = \frac{-4\pi}{e} n_m, \qquad n_m \in \mathbb{Z}. \tag{1}$$

where n_m is the degree of the map describing the topological solution.

It is important to note that the existence of 't Hooft-Polyakov monopoles is not specific to this model. It is for example a generic prediction for any model which embeds the standard model into a simple gauge group, in Grand Unified Theories (The requirement is that the homotopy group $\pi_2\left(G/H\right)$ is non-trivial). As Polchinski wrote [1]:

> *The existence of magnetic monopoles seems like one of the safest bets that one can make about physics not yet seen.*

Whether they are within observational reach is another question, but direct searches are for example being carried out at the LHC by the MoEDAL experiment [2].

1 | Montonen-Olive Duality

Given that we have found monopole solutions in the Georgi-Glashow monopole, we will then continue to explore the possibility that it also enjoys a duality. In order to do so we have to comment on the masses of particles in the spectrum.

1.1 Bogomol'nyi bound and Bogomol'nyi-Prasad-Sommerfield limit

In a more general treatment when allowing for non-static solutions one will find the possibility of dyonic solutions as well. From estimating a lower bound of the total energy field configuration in the center of mass for such solutions one can arrive at the Bogomol'nyi bound on the mass

$$M(q, g)_{\text{Monopole}} \geq v\sqrt{q^2 + g^2}, \tag{1.1}$$

and in fact this equations turns out to hold true for all particles in the spectrum. A solution which saturates this bound we will call **BPS** (it is the Bogomol'nyi-Prasad-Sommerfield limit).

An interesting thing happens when one considers the option to saturate the BPS bound. Here we will merely state that this happens when we let the Higgs self-interaction $\lambda \to 0$ while still maintaining the boundary condition $\Phi^2 = v^2$. Recalling the equation for the mass of the monopole we see that in this limit

$$M = \frac{4\pi v}{e} f(0) = |g| v \tag{1.2}$$

which indeed saturates the Bogomol'nyi bound. Including the monopole solution and taking the BPS limit ($\lambda \to 0$) then gives a bosonic spectrum of the Georgi-Glashow model which is:

Field	Mass	Electric charge	Magnetic charge	Spin/Helicity
A_μ	0	0	0	± 1
ϕ	0	0	0	0
$W_\mu^\pm$	ve	$\pm e$	0	1
$M(onopole)_\pm$	vg	0	$\pm g$	?

1.2 Montonen-Olive duality

After having sketched the 't Hooft-Polyakov solution we are ready to look at a famous duality in physics conjectured by Montonen and Olive [3]:

Montonen-Olive duality conjecture - $\mathbb{Z}_2$ **(Not correct)**
The Georgi-Glashow $SU(2)$ model has an electromagnetic $\mathbb{Z}_2$ duality that leaves the spectrum invariant by taking the charges

$$(q, g) \mapsto (g, -q) \tag{1.3}$$

and interchanging magnetic monopoles with massive vector bosons.

This offers two dual perspectives on the model, and we can use the Higgs field vev v as a natural scale of comparison:

Weak coupling ($e \ll 1$): Here the gauge bosons $W_\mu^\pm$ are very light perturbative fields, while the monopoles are heavy ($M = vq \sim v/e$) and are non-perturbative solitonic solutions.

Strong coupling ($e \gg 1$): The gauge bosons $W_\mu^\pm$ become heavy, and the monopoles become the light perturbative objects.

Under the duality transformation a coupling constant e transforms as

$$e \to g = \frac{4\pi}{e} \tag{1.4}$$

and thus relates the behaviour at strong coupling of a gauge theory to the weakly coupled dual. At this point the conjecture is however not on solid ground yet.

Why should we be sceptical about this conjecture?
A few obvious objections are: How can the magnetic monopoles be interchanged with a spin 1 field? What is the role of dyons? Might radiative corrections spoil the Bogomol'nyi mass bound through renormalisation?

To avoid such problems seems to require more symmetry, and supersymmetry can provide that. But first we will see that the $\mathbb{Z}_2$ duality conjecture can be enhanced to a larger symmetry group.

1.2.1 Extending to an SL(2,$\mathbb{Z}$) duality

Let us consider adding a θ-term to the Lagrangian, as is familiar from the CP violating sector of QCD

$$\mathcal{L}_\theta = -\frac{\theta e^2}{32\pi^2} F^a_{\mu\nu} \star F^{a\mu\nu}, \tag{1.5}$$

which is a total derivative and does not affect the equation of motion. By rescaling the gauge field, $A_\mu \to e^{-1}A_\mu$, we then rewrite the Lagrangian

$$\mathcal{L}_{\text{GG}} + \mathcal{L}_\theta = -\frac{1}{4e^2}F^a_{\mu\nu}F^{a\mu\nu} - \frac{\theta}{32\pi^2}F^a_{\mu\nu}\star F^{a\mu\nu} + \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a - \frac{\lambda}{4}\left(\Phi^a\Phi^a - v^2\right)^2 \tag{1.6}$$

$$= -\frac{1}{32}\text{Im}\left[\left(\frac{\theta}{2\pi} + i\frac{4\pi}{e^2}\right)\left(F^a_{\mu\nu} + i\star F^a_{\mu\nu}\right)^2\right] + \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a - \frac{\lambda}{4}\left(\Phi^a\Phi^a - v^2\right)^2 \tag{1.7}$$

where all dependence on e and θ is shown explicitly. It is custom to then define the complex coupling constant

$$\tau \equiv \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}. \tag{1.8}$$

Now recall that at $\theta = 0$ the conjectured $\mathbb{Z}_2$ duality took $e \mapsto -4\pi/e$ which in the language of τ is the transformation:

$$S : \tau \mapsto -\frac{1}{\tau}. \tag{1.9}$$

In addition, since θ is an angular variable it is invariant under 2π-shifts, which in the language of τ is the transformation:

$$T : \tau \mapsto \tau + 1. \tag{1.10}$$

These two transformations are known to generate the group of $SL(2,\mathbb{Z})$ transformations

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \qquad (\text{i.e. } a,b,c,d \in \mathbb{Z} \quad \text{and} \quad ad - bc = 1), \tag{1.11}$$

where in particular

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{1.12}$$

We will return to elaborate on the structure of this group later. In 1979 Witten [4] showed that the presence of a θ-term shifts the allowed electric charges for a dyon, in what has become known as the Witten effect

$$q = e\left(n_e + \frac{\theta}{2\pi}n_m\right), \qquad g = \frac{4\pi}{e}n_m \qquad n_e, n_m \in \mathbb{Z}. \tag{1.13}$$

Although we will not derive this result, it is an important result and we will now classify a dyonic state by a charge vector $\mathbf{n} = (n_e, n_m)$. The BPS mass formula in this formulation becomes:

$$M^2 = 4\pi v^2 \left(n_e, n_m\right)\frac{1}{\text{Im}\tau}\begin{pmatrix} 1 & \text{Re}\tau \\ \text{Re}\tau & |\tau|^2 \end{pmatrix}\begin{pmatrix} n_e \\ n_m \end{pmatrix} \tag{1.14}$$

which turns out to be $SL(2,\mathbb{Z})$ invariant if the charged states at the same time transform as:

$$\mathbf{n} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\mathbf{n}. \tag{1.15}$$

All these things combined leads to a bolder statement of the Montonen-Olive duality:

Montonen-Olive duality conjecture - SL(**2**,**$\mathbb{Z}$**) **(Not correct)**
The physics of the Georgi-Glashow model has an $SL(2,\mathbb{Z})$ duality under which the complex coupling τ and the charge vector $\mathbf{n}$ transforms as:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \qquad \mathbf{n} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\mathbf{n}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{1.16}$$

This extension of the Montonen-Olive duality to the full $SL(2,\mathbb{Z})$ is what is often referred to as *S-duality*. An immediate consequence of this symmetry would be the existence of an infinite lattice of charge states generate by transforming a charge vector $\mathbf{n}$.

1.2.2 Montonen-Olive duality and Supersymmetry

In order to put the Montonen-Olive duality conjecture on solid ground one has to study it in the setting of supersymmetric field theories.

🔔 **Supersymmetry (SUSY)**
An extension of the Poincaré space-time symmetry, $\mathbb{R}^{1,d} \rtimes O(1,d)$, by a $\mathbb{Z}_2$ grading to the Lie algebra by the introduction of fermionic symmetry generators (supercharges)

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P^{IJ}_\mu, \qquad I, J = 1, \ldots, \mathcal{N} \tag{1.17}$$

$$\{Q^I_\alpha, Q^J_\beta\} = 2\epsilon_{\alpha\beta} Z^{IJ} \tag{1.18}$$

$$\vdots \tag{1.19}$$

In other words, a symmetry relating fermionic states to bosonic states, and vice versa.

In particular the Montonen-Olive conjecture has so far held up to all test in the case of $N = 4$ SYM [5]. This is the maximally supersymmetric extension of Yang-Mills theory with spin ≤ 1 (rigid) in four dimensions with the vector multiplet

$$\left(A_\mu, \lambda^a_\alpha, X^i\right) = (\text{Vector field,}4\times\text{Weyl Spinor, } 6\times \text{ real scalar}). \tag{1.20}$$

We will not have time to go into details, but simply list the heuristic arguments for why $\mathcal{N} = 4$ supersymmetry strengthens the credibility of the Montonen-Olive duality:

- Again a Higgs potential for scalars can be added to the theory, and one has to consider the BPS limit of $\lambda \to 0$ to restore supersymmetry.
- The states are BPS, and by supersymmetry there masses are protected from renormalisation.
- It is a superconformal theory, meaning that the β-function for the coupling vanishes $\beta\left(g_{YM}\right)$=0.
- A short multiplet containing the BPS-monopole and a short multiplet containing the massive vector bosons are isomorphic, thus explaining the problem of spin for the monopoles.

We therefore end by stating the modern formulation of Montonen-Olive duality which also includes a transformation of a possible gauge group:

Supersymmetric Montonen-Olive duality conjecture
Four dimensional $\mathcal{N} = 4$ SYM with gauge group G has an $SL(2, \mathbb{Z})$ duality, and under S-duality the gauge group G is replaced with the so-called Langlands dual ${}^LG$ e.g:

G	${}^LG$
$U(N)$	$U(N)$
$SU(N)$	$SU(N)/\mathbb{Z}_N$
$SO(2N)$	$SO(2N)$
$SO(2N+1)$	$Sp(2N)$

One strength of string theory is that it often provides very effective tools to study such theories. In the case of Montonen-Olive duality of $N = 4$ SYM it can be studied as the theory on a stack of D3-branes inheriting the $SL(2, \mathbb{Z})$ symmetry from Type IIB string theory. It is exactly the ingredients of

- an $SL(2, \mathbb{Z})$ duality transformation
- Dp-branes
- transformations/variations of τ

that carries over to a discussion of Type IIB string theory, and in particular F-theory.

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