Student lectures on:

Dualities in Field and String Theory

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1 | Dualities in String Theory

For anyone unfamiliar with string theory, we will not give an introduction to the subject, but rather state a minimum amount of facts necessary to appreciate the role of dualities.

What is string theory? What you get when you quantize a theory where the fundamental objects are 1-dimensional (strings).

Why string theory? It is arguably the leading theory which unifies gravity with quantum field theory, and provides an anomaly free and UV complete description for quantum gravity.

Uniqueness of string theory? A supersymmetric string theory requires 10 spacetime dimensions, but there exists five consistent formulations:

Type I, Type IIA, Type IIB, $E_8 \times E_8$ Heterotic and $SO(32)$ Heterotic.

In the early nineties important lessons where however learned about the connection between these five theories:

The second superstring revolution (Early 90's):
All the superstring theories are related by dualities, and they are all limits of an 11-dimensional theory (M-theory) (See figure 1.1).

Another type of duality which has its origin in string theory, but which we will not discuss is the celebrated AdS/CFT correspondence conjectured by Maldacena [1]:

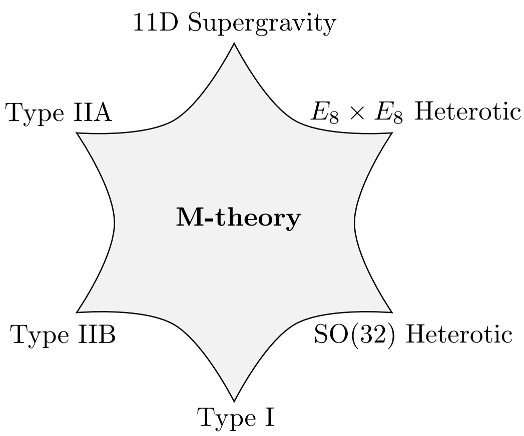


Figure 1.1: String theories in 10-dimensions and 11-dimensional supergravity are all conjectured to arise from limits of an 11-dimensional M-theory.

AdS/CFT Correspondence (Gauge/gravity duality):
Type IIB superstring theory on $AdS_5 \times S^5$ is dynamically equivalent to $\mathcal{N} = 4$ Super Yang-Mills theory with gauge group $SU(N)$ in 3+1 dimensions, with maps between parameters

$$g^2_{\text{YM}} = 2\pi g_s \qquad \textit{and} \qquad 2g^2_{\text{YM}} N = \frac{L^4}{\alpha'^2} \tag{1.1}$$

with string coupling g_s, Yang-Mills coupling g_{YM} and AdS/S^5 radius L.

1.1 Type IIB string theory

As already mentioned Type IIB string theory is defined on a 9+1 dimensional spacetime, and the low-energy limit which is the chiral $\mathcal{N} = (2, 0)$ maximal supergravity in 10 dimensions has a bosonic spectrum given by the following set of fields:

$$\text{Graviton: } G_{\mu\nu} \tag{1.2}$$
$$\text{Dilatino: } \Phi \tag{1.3}$$
$$\text{Form fields: } B_2, C_0, C_2, C_4 \tag{1.4}$$

where one can define the so-called axio-dilaton:

$$\text{Axio-dilaton:} \quad \tau \equiv C_0 + i\mathrm{e}^{-\Phi}, \qquad \langle \mathrm{e}^{\Phi} \rangle = g_s \text{ (String coupling).} \tag{1.5}$$

Conventionally the Type IIB low energy effective action (Type IIB supergravity) for the bosonic sector can be split in three terms

$$\mathcal{S}_{\text{IIB}} = \mathcal{S}_{\text{NS}-\text{NS}} + \mathcal{S}_{\text{R}-\text{R}} + \mathcal{S}_{\text{CS}} \tag{1.6}$$

where each sector (NS = Neveu-Schwarz, R=Ramond and CS = Chern-Simons) is given by

$$\mathcal{S}_{\text{NS}-\text{NS}} = \frac{1}{2\kappa^2_{10}} \int \mathrm{d}^{10}x\sqrt{-G}\,\mathrm{e}^{-2\Phi}\left(\mathcal{R} + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right) \tag{1.7a}$$
$$\mathcal{S}_{\text{R}-\text{R}} = -\frac{1}{4\kappa^2_{10}} \int \mathrm{d}^{10}x\sqrt{-G}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right) \tag{1.7b}$$
$$\mathcal{S}_{\text{CS}} = -\frac{1}{4\kappa^2_{10}} \int_{S.T.} C_4 \wedge H_3 \wedge F_3. \tag{1.7c}$$

Here $\mathcal{R}$ is the Ricci scalar, κ_{10} is the gravitational coupling constant in 10 dimensions, the superstring coupling constant is encoded by $g_s = \langle \mathrm{e}^{\Phi} \rangle$ and the field strengths follow the definitions:

$$F_{n+1} \equiv \mathrm{d}C_n \tag{1.8a}$$
$$H_3 \equiv \mathrm{d}B_2 \tag{1.8b}$$
$$\tilde{F}_3 \equiv F_3 - C_0 \wedge H_3 \tag{1.8c}$$
$$\tilde{F}_5 \equiv F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3. \tag{1.8d}$$

However, this action does not automatically include the self-duality of the 5-form field strength, as it can not be included covariantly. Thus we have to impose at the level of the equations of motion that

$$\tilde{F}_5 = \star\tilde{F}_5. \tag{1.9}$$

The Type IIB action is invariant under $SL(2, \mathbb{Z})$ transformations, where in particular the axio-dilaton transforms as:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{1.10}$$

Since the dilaton determines the string coupling, the transformation $\tau \mapsto -1/\tau$ in Type IIB is also a weak/strong coupling duality:

$$g_s = \langle \mathrm{e}^{\Phi} \rangle \quad \xleftrightarrow{\text{dual}} \quad \langle \mathrm{e}^{-\Phi} \rangle = \frac{1}{g_s}. \tag{1.11}$$

1.1.1 A rough guide to form fields and branes

Due to the higher dimensionality of string theory, higher form fields are natural, as opposed to simply the vector potential and its field strength in four dimensions. In order to discuss their interpretation, we will start by a formal definition:

Differential forms:
A differential form of order r (r-form) is a totally anti-symmetric $(0,r)$-tensor.

Using the **wedge product** $\wedge$ of one-forms $\mathrm{d}x^{\mu_i}$ defined by

$$\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_r} = \sum_{P \in S_r} \mathrm{sgn}(P) \mathrm{d}x^{\mu_{P(1)}} \wedge \mathrm{d}x^{\mu_{P(2)}} \wedge \cdots \wedge \mathrm{d}x^{\mu_{P(r)}} \tag{1.12}$$

then any element C_r in the set of r-forms on a manifold M, $\Omega^r(M)$, can be written as:

$$C_r = \frac{1}{r!} C_{\mu_1 \mu_2 \cdots \mu_r} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_r}. \tag{1.13}$$

For the familiar Yang-Mills gauge potential, we say that A is a 1-form:

$$A_1 = A_\mu \mathrm{d}x^\mu. \tag{1.14}$$

To define the field strength in this language, we will need the notion of an exterior derivative:

Exterior derivative:
The exterior derivative d is a map from $\Omega^r(M)$ to $\Omega^{r+1}(M)$ with an action on an r-form

$$C_r = \frac{1}{r!} C_{\mu_1 \mu_2 \cdots \mu_r} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_r}, \tag{1.15}$$

defined by

$$\mathrm{d}C_r = \frac{1}{r!} \left(\frac{\partial}{\partial x^\nu} C_{\mu_1 \mu_2 \cdots \mu_r} \right) \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_r}, \tag{1.16}$$

with the important property that

$$\mathrm{d}^2 = 0. \tag{1.17}$$

We can now write the field strength as the 2-form obtained when acting with the exterior derivative on the gauge potential

$$F_2 = \mathrm{d}A_1 = (\partial_\mu A_\nu - \partial_\nu A_\mu) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{1.18}$$

and gauge invariance is the statement that the field strength is invariant under transformations:

$$A_1 \to A_1' = A_1 + \mathrm{d}\Lambda_0 \qquad \text{since} \qquad F_2' = \mathrm{d}(A_1') = \mathrm{d}(A_1) = F_2, \tag{1.19}$$

using that $d^2 = 0$. Also recall that a charge particle interacting with the gauge field A, in the action is described by integrating along the world-line γ of the particle

$$S \supset q \int A_\mu \mathrm{d}x^\mu = q \int_\gamma A_1. \tag{1.20}$$

This is a particle solution, but now we can equally call it a D0-brane (because a particle has 0 spatial dimensions).

Now for a p-form potential C_p we can define a field strength $F_{p+1} = \mathrm{d}C_p$, and write down an electric coupling

$$S \supset \mu_p \int_{\Sigma_p} C_p \qquad \Longrightarrow \qquad \mathrm{D}(p-1)\text{brane}. \tag{1.21}$$

Thus, we see that the C_p form fields are generalisations of the vector potentials, and the electrically charged objects under these field are D($p-1$)-branes. These are dynamical, but non-perturbative, objectives in string theory.

In addition one can define the hodge $\star$ operator on an m-dimensional manifold M, which is a linear map

$$\star_m : \Omega(M)^r \to \Omega(M)^{m-r} \tag{1.22}$$

and the dual field strength

$$\tilde{F}_{m-p-1} \equiv \star F_{p+1} = \mathrm{d}\tilde{C}_{m-p-2} \tag{1.23}$$

then couples magnetically to an extended object by

$$S \supset \mu_{m-p-2} \int_{\Sigma_{m-p-2}} \tilde{C}_{m-p-2} \qquad \Longrightarrow \qquad \text{D}(m-p-3)\text{brane}. \tag{1.24}$$

i.e. the magnetically charged objects under a form field C_p are D($m-p-3$)-branes. We leave it as an exercise to find the brane content of Type IIB string theory ($m = 10$), and focus on one particular form field

$$C_0 \Rightarrow \begin{cases} \text{D}(-1)\text{-brane (i.e. an instanton)} \\ \text{D7-brane} \end{cases} \tag{1.25}$$

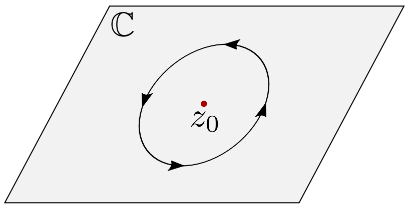
because, heuristically due to their size, 7-branes play an important role in Type IIB string theory, and leads to the definition of F-theory to deal with them, as we shall demonstrate.

1.1.2 7-branes and monodromies

We now consider Type IIB string theory on a manifold

$$\mathcal{M}^{10} = \mathbb{R}^{1,7} \times \mathbb{C} \tag{1.26}$$

and analyse the situation of encircling a D7-brane which extends along the $\mathbb{R}^{1,7}$ directions, i.e. it sits at a point z_0 in the complex plane $\mathbb{C}$ described by the complex coordinate $z = x_8 + ix_9$:



Now the D7-brane is a magnetic source for the C_0 field, which was part of the axio-dilaton, $\tau \equiv C_0 + i\mathrm{e}^{-\Phi}$, and one can show that the solution for τ in the transverse space to the D7-brane has to be of the form

$$\tau(z) = \frac{1}{2\pi i} \ln(z - z_0) + (\text{regular at } z_0). \tag{1.27}$$

Due to the branch cut in the logarithmic function, it means that encircling the brane induces a **monodromy** of τ

$$\tau \to \tau + 1, \tag{1.28}$$

thus we see that placing a 7-brane in the spacetime has non-trivial effects, a back-reaction on the space, which has to be accounted for. That τ seems to be multivalued is not at problem since $\tau \to \tau + 1$ is part of the $SL(2,\mathbb{Z})$ duality symmetry of Type IIB. What we have seen is that a D7-brane introduces a monodromy identified by the monodromy matrix:

$$M_{D7} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{1.29}$$

If we considered N D7-branes located at the same point we would have a monodromy

$$M_{N\times D7} = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{1.30}$$

and by applying the $SL(2,\mathbb{Z})$ duality transformations one is automatically required to also include so-called $[p,q]$-7-branes with monodromy matrix:

$$M_{[p,q]} = \begin{pmatrix} 1-pq & p^2 \\ -q^2 & 1+pq \end{pmatrix} \in SL(2,\mathbb{Z}). \tag{1.31}$$

Thus we have seen that 7-branes have direct implications for the variation of τ, and in particular the presence of 7-branes necessarily makes τ vary over the spacetime, such that generically one can not control the weak coupling limit everywhere on the manifold.
Let us try and see it from the opposite perspective and conclude what the τ profile tells us about 7-branes:

- The divergence of τ, from $\ln(z-z_0)$, indicates the location of a 7-brane.
- The monodromy of τ indicates which 7-branes are located there.

Given the importance of τ and its variations under $SL(2,\mathbb{Z})$ transformations we have to comment on the connection with the geometry of the torus, which we have so far neglected.

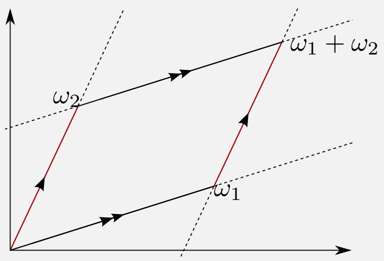
SL(2,$\mathbb{Z}$) as the symmetry group of the torus:
Consider the complex plane $\mathbb{C}$ with a lattice $\Lambda(\omega_1,\omega_2)$, $\omega_1,\omega_2 \in \mathbb{C}$, defined by

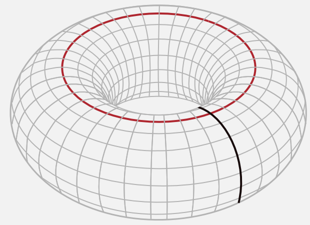
$$\Lambda = \{k_1\omega_1 + k_2\omega_2 \mid k_1,k_2 \in \mathbb{Z}\}, \tag{1.32}$$

where ω_1 and ω_2 considered as vectors in the plane should be none-parallel, i.e $\mathrm{Im}(\omega_2/\omega_1) > 0$. By identifying points on the lattice by an equivalence

$$z \sim z+\omega_1, \qquad z \sim z+\omega_2, \qquad \forall z \in \mathbb{C} \tag{1.33}$$

we obtain the structure of the torus by a homeomorphism from the quotient space $\mathbb{C}/\Lambda \simeq T^2$:





To find the moduli space of the torus we need to identify equivalences of the complex structure determined by (ω_1,ω_2). We introduce the modular parameter $\tau \equiv \omega_2/\omega_1$ such that when scaling equation (1.33) by a factor of $1/\omega_1$ it takes the form

$$z' \sim z'+1, \qquad z' \sim z'+\tau, \qquad \forall z' \in \mathbb{C}. \tag{1.34}$$

where $\tau \in \mathbb{C}|\mathrm{Im}(\tau) \geq 0$. Hence, we can consider a lattice generated by 1 and τ, and consider the transformations which merely generates a different choice of fundamental domain for the torus. These operations preserve the complex structure, and they can all be generated by the transformations:

$$S: \quad \tau \to -\frac{1}{\tau} \qquad\qquad T: \quad \tau \to \tau+1 \tag{1.35}$$

i.e. they generate the group SL(2,$\mathbb{Z}$) - the modular group of the torus, with transformations of τ in a general form given by a Möbius transformation (or linear fractional transformation)

$$\tau \mapsto \frac{a\tau+b}{c\tau+d} \qquad \text{with } a,b,c,d \in \mathbb{Z} \text{ and } ad-bc=1. \tag{1.36}$$

Thus we see that it seems reasonable to interpret the axio-dilaton as the complex structure of a torus.

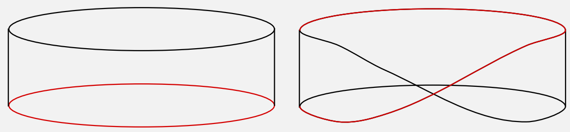
1.2 Defining F-theory

The heuristic idea/definition of F-theory[1][2], is to describe Type IIB string theory in which one deals consistently with the back-reaction from 7-branes on the spacetime. This is done by encoding the information of the axio-dilaton, τ, in an auxiliary torus T^2 attached over every point of the spacetime. Attaching a geometry at every point over a space in an appropriate way defines a **fibration**.

Fibrations
A fibration is obtained by fibering a space F (the fiber) over each point of another space B (the base). The total space space we will call X, and there must then exist a map

$$\pi : X \to B \tag{1.37}$$

such that for (almost) any points $b, b' \in B$ we have $\pi^{-1}(b) \cong \pi^{-1}(b')$. A trivial example is the cylinder which is an interval $[a, b]$ trivially fibered over S^1, but if the fibration of the interval is defined with a twist, one instead obtains the Möbius strip:



In this language, F-theory is defined by a torus (elliptic) fibration over the Type IIB spacetime (See Figure 1.2) and our previous statements about τ and 7-branes can be rephrased in a geometrical language:

- A degeneration (singularity) of the torus indicates the location of 7-branes.
- The type of degeneration (singularity) determines the monodromy, i.e. the type of 7-brane configuration (by the so-called Kodaira classification).

What makes this so useful is, that the objective to write down a consistent model with 7-branes in Type IIB string theory, then translates into studying well defined elliptic fibrations. In particular, for phenomenology one would want to consider F-theory compactifications on manifolds of the type

$$\mathbb{R}^{1,3} \times CY_4 \tag{1.38}$$

where CY_4 is an elliptically fibered Calabi-Yau manifold, $\pi : CY_4 \to B_3$, of dimension

$$\dim_{\mathbb{R}}(CY_4) = 2\dim_{\mathbb{C}}(CY_4) = 2 \times 4 = 8. \tag{1.39}$$

In this sense, F-theory studies a 12-dimensional geometry, even though only 10 of them are physical, and the remaining two belong to the auxiliary torus.

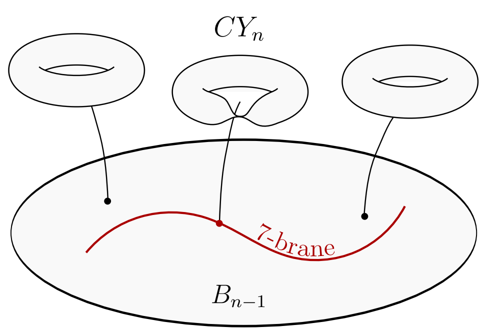


Figure 1.2: Illustration of the elliptic fibration in an F-theory compactification. The total space CY_n has to be of the Calabi-Yau type, and the elliptic fibration is over a base space B_{n-1}.

[1] We unfortunately do not have time to comment on the duality with M-theory in this lecture.

In discussing phenomenology, it is a known fact in string theory, that gauges theories are located on branes. The type of gauge group is determined by the number and type of branes, and is dictated by the singularity type of the torus. In particular, F-theory has the virtue of being able to describe exceptional Lie groups, useful for the realisation of Grand Unified Theories (GUTs).

The mathematically consistent elliptic fibrations in F-theory therefore encodes all the information about the 7-brane content, the geometric singularities, and gauge symmetries as illustrated in Figure 1.3. Behind all this lies the $SL(2,\mathbb{Z})$ duality symmetry, and research continues into ways to utilise this symmetry to the fullest in describing F-theory compactifications.

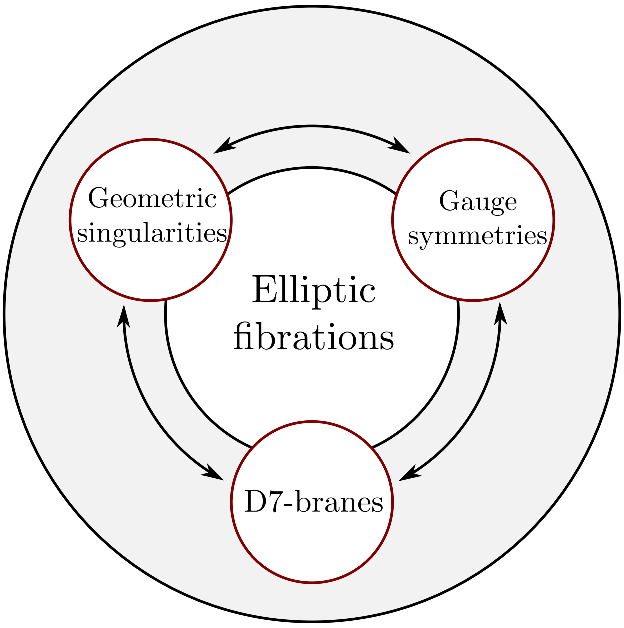


Figure 1.3: Illustration of the connection between elements in F-theory, which are all encoded in the study of elliptic fibrations.

Bibliography

1. Maldacena, J. M. The Large N limit of superconformal field theories and supergravity. *Int. J. Theor. Phys.* **38.** [Adv. Theor. Math. Phys.2,231(1998)]. arXiv: `hep-th/9711200 [hep-th]` (1999).
2. Vafa, C. Evidence for F theory. *Nucl. Phys.* **B469.** arXiv: `hep-th/9602022 [hep-th]` (1996).