Scalar-tensor theories in cosmology

Student Lecture 1 by Manuel Wittner

Motivation and Outline

- Today's standard model of cosmology ("ΛCDM model"):
  - 70% dark energy (Λ = cosmological constant) + 25% cold dark matter ("cold" = non-relativistic) + 5% baryonic matter (= normal matter that interacts with light)
  - general relativity as underlying theory
- Successes: Can explain very well a wide variety of cosmological observations as for example:
  - Accelerated background expansion first measured via type Ia supernovae
  - CMB power spectrum measured with high precision by Planck experiment and others
- Problems:
  - Cosmological constant problem: why do quantum corrections to the vacuum energy, which are naively of order $\Lambda_{\text{UV}} \sim M_P^4 \sim (10^{18}\,\text{GeV})^4$, add up to an effective cosmological constant of $\Lambda \sim (\text{meV})^4$ thus causing a fine-tuning of $\Lambda_{\text{UV}}/\Lambda \sim \mathcal{O}(10^{120})$?
  - Coincidence problem: why is $\Omega_{\Lambda 0} \approx 70\%$ of the same order as $\Omega_{c0} \approx 25\%$?
  - σ_8 tension: assuming ΛCDM, there is a tension between Planck measurements and LSS in the $\sigma_8 - \Omega_m$ plain, with the latter favouring smaller values.
  - Hubble tension: measurements of the cosmic microwave background (CMB) from Planck and others together with ΛCDM model imply a Hubble factor $H_0 \approx 67\,\text{km}\,(\text{s}\cdot\text{Mpc})^{-1}$ vs model-independent measurements from cepheids and supernovae that show $H_0 \approx 74\,\text{km}\,(\text{s}\cdot\text{Mpc})^{-1} \Rightarrow$ more than 3σ difference.

  $\Rightarrow$ The ΛCDM model seems not to be the final answer but needs modification.
- This course:
  1. Basic concepts of general relativity and standard cosmology
  2. Scalar-tensor theories as a modification of GR
  3. Horndeski theories and their basic properties
  4. An exemplary Horndeski theory: coupled dark energy
  5. Transient weak gravity in coupled dark energy

Basic concepts of GR and standard cosmology

- GR is a theory that identifies gravity with the curvature of a dynamic spacetime. Matter and energy content determine how spacetime curves and spacetime determines how the former move in it.

- Basic object to describe spacetime: metric tensor $g_{\mu\nu}$ $\rightarrow$ gives us a measure for distances in spacetime. For example:

  - Newtonian mechanics $\rightarrow$ 3D Euclidean space $\rightarrow$ $g_{\mu\nu} = \delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

    $\Rightarrow$ line element $\mathrm{d}s^2 = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$ invariant under Galilei transformations:

    $$t \rightarrow t + a, \quad \vec{x} \rightarrow \mathcal{R} \cdot \vec{x} + \vec{v} \cdot t + \vec{b}, \quad \mathcal{R} \in \mathrm{SO}(3)$$

  - Special relativity $\rightarrow$ 4D Minkowski space $\rightarrow$ $g_{\mu\nu} = \eta_{\mu\nu} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

    $\Rightarrow$ $\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ invariant under Poincaré transformations:

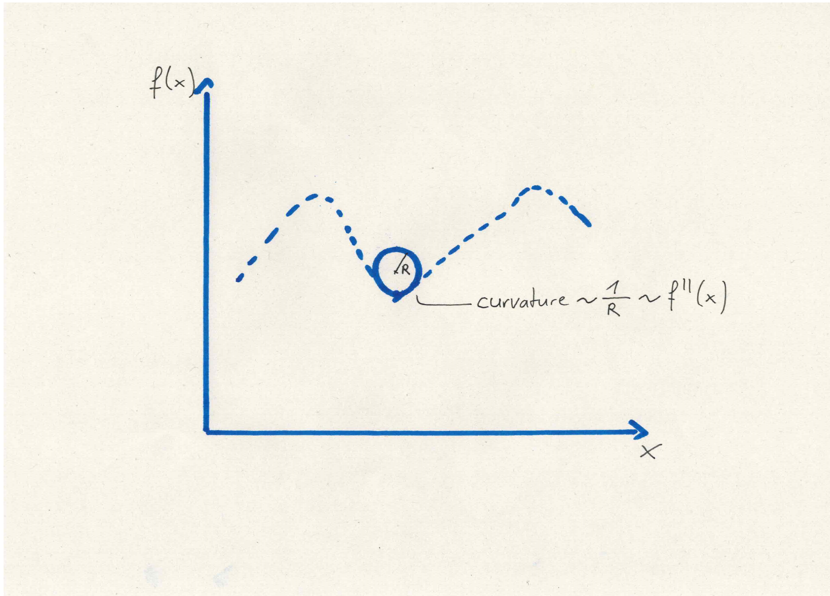
    $$x^\mu \rightarrow \Lambda^\mu_\nu x^\nu + a^\mu, \quad \Lambda \in \mathrm{O}(1,3)$$

  - General relativity $\rightarrow$ 4D pseudo-Riemannian manifold $\rightarrow$ $g_{\mu\nu} = g_{\mu\nu}(x)$
    $\Rightarrow$ $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ invariant under general, differentiable coordinate transformations: $x^\mu \rightarrow x'^\mu(x^\nu)$
    Why? $\rightarrow$ Metric transforms as covariant tensor:

    $$\begin{aligned} \mathrm{d}s'^2 = g'_{\mu\nu}\mathrm{d}x'^\mu\mathrm{d}x'^\nu &= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \frac{\partial x'^\mu}{\partial x^\rho}\mathrm{d}x^\rho \frac{\partial x'^\nu}{\partial x^\lambda}\mathrm{d}x^\lambda \\ &= \frac{\partial x^\alpha}{\partial x^\rho}\frac{\partial x^\beta}{\partial x^\lambda} g_{\alpha\beta}\mathrm{d}x^\rho\mathrm{d}x^\lambda \\ &= \delta^\alpha_\rho\delta^\beta_\lambda g_{\alpha\beta}\mathrm{d}x^\rho\mathrm{d}x^\lambda \\ &= g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta \\ &= \mathrm{d}s^2 \end{aligned}$$

    Here $\alpha^\mu_\nu \equiv \partial x'^\mu/\partial x^\nu$ is the transformation matrix whereas the metric $g_{\mu\nu}$ transforms with its inverse $(\alpha^{-1})^\mu_\nu \equiv \partial x^\mu/\partial x'^\nu$

- How can we measure curvature?
  $\Rightarrow$ for a function $f(x)$, the osculating circle at a specific point x_0 is the circle, that perfectly "touches" the function at x_0. Curvature is the inverse of its radius R.



⇒ analogously, curvature of spacetime, expressed in terms of the Ricci tensor, must be second-order derivatives of metric: $R_{\mu\nu} \sim (\partial^2 g) + (\partial g)^2$ (full expression in every GR textbook)

- How does matter influence spacetime?
  ⇒ We need an equation that relates matter content to curvature of spacetime.
  - Let us start with a "reasonable" action:

    $$S = \int \mathrm{d}^4x \sqrt{-g} (\mathcal{L}_{\text{gravity}} + \mathcal{L}_{\text{matter}}),$$

    where $\mathcal{L}_i$ transform as scalars, $\mathrm{d}^4x \to \alpha^4 \mathrm{d}^4x$ and $g \equiv \det(g_{\mu\nu}) \to (\alpha^{-1})^8 g$. Hence, both $\mathrm{d}^4x\sqrt{-g}$ and $\mathcal{L}_i$ are invariant under coordinate transformations ⇒ S is invariant as well.
  - What could $\mathcal{L}_{\text{gravity}}$ be?
    * We need scalar quantity that represents spacetime curvature: Ricci scalar $R \equiv g^{\mu\nu} R_{\mu\nu}$ ⇒ Einstein-Hilbert action:

      $$S_{\text{EH}} = \frac{M_P^2}{2} \int \mathrm{d}^4x \sqrt{-g} R$$

    * We can further add a constant Λ that couples to gravity due to $\sqrt{-g}$-term
  - In total we get

    $$S = \int \mathrm{d}^4x \sqrt{-g} \left(\frac{M_P^2}{2} (R - 2\Lambda) + \mathcal{L}_{\text{matter}} \right)$$

– Variation w.r.t. metric, a dynamical field, leads to the famous Einstein field equations:

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0$$
$$\Leftrightarrow \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

– Here $T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_{\text{matter}})}{\delta g^{\mu\nu}}$ is the energy-momentum tensor. This is generalisation of well-known EM-tensor from QFT (conserved Noether current w.r.t. time and space translations) to curved spacetimes. In flat spacetime, both are equivalent.

– Important: both, left and right handside are conserved: $\nabla_\mu G^{\mu\nu} = \nabla_\mu g^{\mu\nu} = \nabla_\mu T^{\mu\nu} = 0$, where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the so called "Einstein tensor". The conservation of $T_{\mu\nu}$ is the GR analogue to energy and momentum conservation.

- With the additional information that free-falling particles follow geodesics, the theory is complete.

- Application: cosmology!

  – Universe is homogeneous and isotropic on large scales and almost spatially flat → FRW-metric $\mathrm{d}s^2 = -c^2 \mathrm{d}t^2 + a(t)^2 \mathrm{d}\vec{x}^2$

  – Matter content can be described as a perfect fluid: $T_{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$

  – Einstein equations for this metric and EM tensor yield Friedmann equations

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3}$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2} + \frac{\Lambda c^2}{3}\right)$$

  – Furthermore, $\nabla_\mu T^{\mu\nu} = 0$ leads to the conservation equation

$$\dot{\rho} + 3H\left(\rho + \frac{p}{c^2}\right) = 0,$$

  which is not independent of the Friedmann equations so that we need only two of these 3 equations.

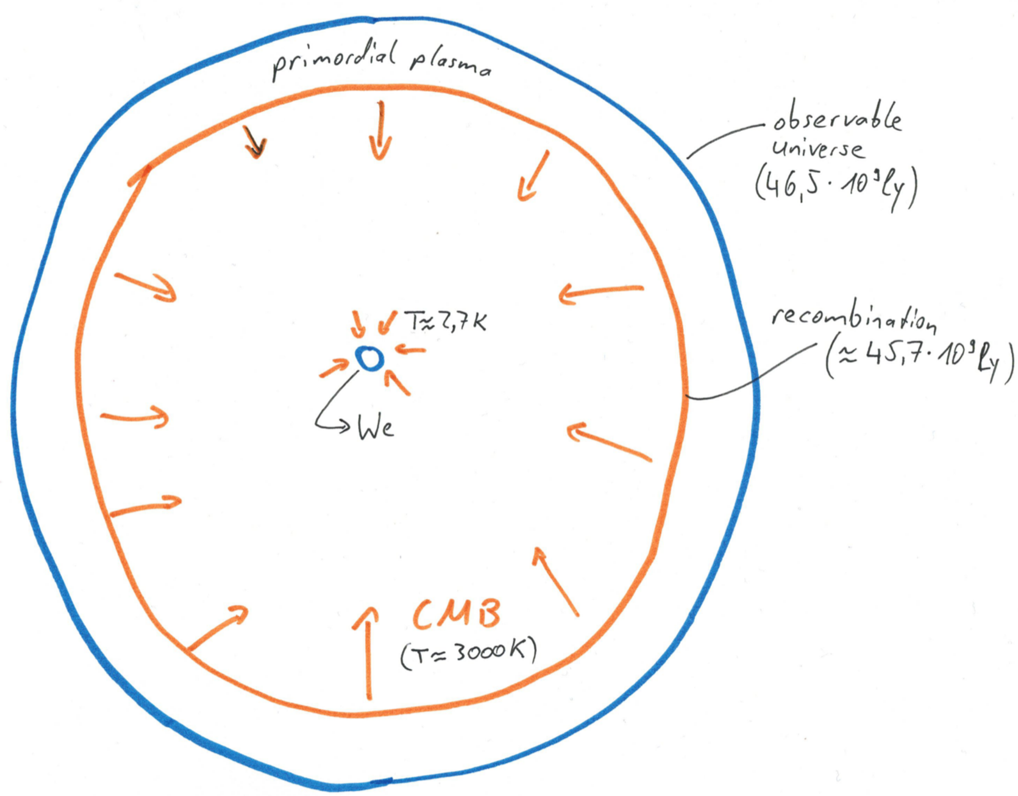
- Combined with the equation of state $w = p/\rho$, the background dynamics of the universe are fully determined. ($w = 0$ for matter, $w = 1/3$ for radiation, $w = -1$ for Λ)
- Examples:
  * Matter domination: $w = 0$

$$
\begin{aligned}
\dot{\rho} &= -3\frac{\dot{a}}{a}\rho \\
\Rightarrow \rho &= \rho_0 a^{-3} \\
\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho_0 a^{-3} \\
\Rightarrow a(t) &\propto t^{2/3}
\end{aligned}
$$

  * Λ domination:

$$
\begin{aligned}
\left(\frac{\dot{a}}{a}\right)^2 &= \frac{\Lambda c^2}{3} \\
\Rightarrow a &\propto \mathrm{e}^{\sqrt{\Lambda/3}\,t} = \mathrm{e}^{H\,t}
\end{aligned}
$$

- Excursion on Hubble tension: how can Hubble constant be inferred from CMB?



  * CMB almost isotropic but has small anisotropies ($\delta T/T \sim \mathcal{O}(10^{-5})$)

* Strongest effect are so called baryon acoustic oscillations (BAOs): interplay of gravity and pressure in high density regions leads to oscillations.
* At recombination they are frozen and define the so called "sound horizon"
* We can measure the angular distribution of the anisotropies and calculate the angular power spectrum $\rightarrow$ acoustic peak allows us to measure sound horizon during recombination precisely
* Since CMB anisotropies are seeds for structure formation, galaxy distribution in universe allows for measurement of sound horizon at later time
* Comparing them, lets us deduce how much the universe has expanded since recombination
  $\Rightarrow$ yields Hubble factor H_0