Scalar-tensor theories in cosmology

Student Lecture 3 by Manuel Wittner

Horndeski theories and some basic properties

- Often, stability is tied to second-order equations of motion (eom):

$$\mathcal{L} = \frac{1}{2}\ddot{\phi}^2 - V(\phi)$$
$$\Rightarrow \quad \ddot{\ddot{\phi}} = \frac{\partial V}{\partial \phi}$$

  - Fourth-order eom requires four initial values
  - Corresponds two canonical field variables (incl. their momenta)
  - One of those is a ghost (= wrong sign in kinetic term)

- Horndeski theories: most general 4D scalar-tensor theory with 2nd-order derivatives in equations of motion

- They are specified by four functions $G_i(\phi, X)$ where $X \equiv -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$ is kinetic term of scalar field ϕ:

$$\mathcal{L}_{\mathrm{H}} = \sum_{i=2}^{5} \mathcal{L}_i$$

  where

$$\begin{aligned}
\mathcal{L}_2 &= G_2(\phi, X), \\
\mathcal{L}_3 &= -G_3(\phi, X)\Box\phi, \\
\mathcal{L}_4 &= G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla^\mu\nabla^\nu\phi)(\nabla_\mu\nabla_\nu\phi)\right], \\
\mathcal{L}_5 &= G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5X}}{6}\left[(\Box\phi)^3 - 3\Box\phi(\nabla^\mu\nabla^\nu\phi)(\nabla_\mu\nabla_\nu\phi) + 2\phi^\mu_\nu\phi^\nu_\lambda\phi^\lambda_\mu\right]
\end{aligned}$$

- Horndeski contains a plethora of well-known theories, e.g.:
  - ΛCDM: $G_2 = -2\Lambda$, $G_4 = M_P^2/2$, $G_{3,5} = 0$
  - Quintessence: $G_2 = X - V$, $G_4 = M_P^2/2$, $G_{3,5} = 0$
  - Brans-Dicke theory: $G_2 = \omega X/\phi$, $G_4 = \phi M_P^2/2$, $G_{3,5} = 0$
  - ...

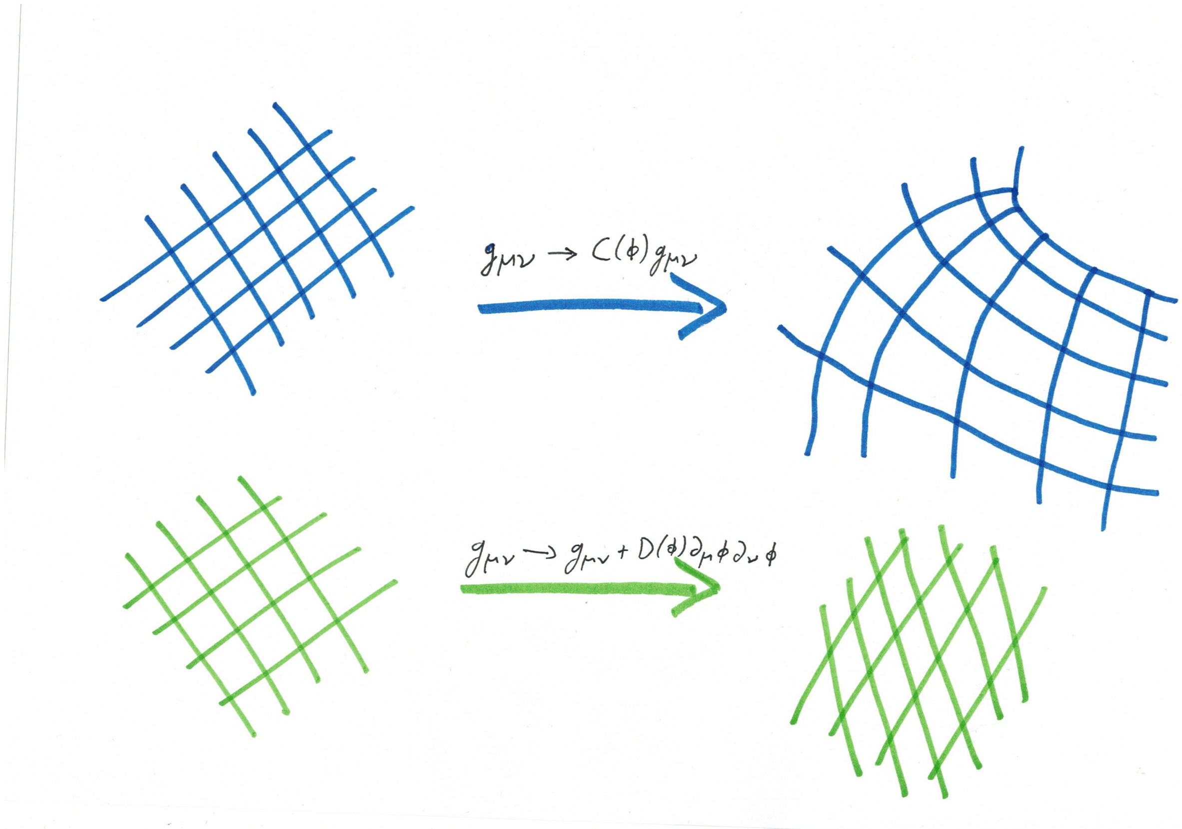
- Conditions of stability: restrictions on G_i's. For the simple case of quintessence:

$$\frac{XG_{2X}}{H^2} = \frac{X}{H^2} > 0$$

  $\Rightarrow$ kinetic term positive!

- Horndeski theories are form-invariant under disformal transformations of the metric:

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = C(\phi) g_{\mu\nu} + D(\phi) \partial_\mu \phi \partial_\nu \phi$$



That is, if a scalar-tensor theory $\mathcal{L}_1(\phi, g_{\mu\nu}) \subset \mathcal{L}_\mathrm{H}$ is a Horndeski theory, another theory given by $\mathcal{L}_2 = \mathcal{L}_1(\phi, \tilde{g}_{\mu\nu})$ will also be $\subset \mathcal{L}_\mathrm{H}$. Or in other words: the second-order nature of the eoms is preserved under a disformal transformation.

An exemplary Horndeski theory: Coupled Dark Energy

- Consider following theory:

$$S_\mathrm{CDE} = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{M_P^2}{2} R + \mathcal{L}_\phi(g_{\mu\nu}, \phi) + \mathcal{L}_\mathrm{b}(g_{\mu\nu}, \psi_\mathrm{b}) \right] + \int \mathrm{d}^4x \sqrt{-\tilde{g}} \tilde{\mathcal{L}}_\mathrm{c}(\tilde{g}_{\mu\nu}, \psi_\mathrm{c}),$$

where

$$\begin{aligned} \mathcal{L}_\phi &= -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi), \\ \psi_\mathrm{b} &= \text{baryonic matter}, \\ \psi_\mathrm{c} &= \text{(cold) dark matter}, \\ \tilde{g}_{\mu\nu} &= C(\phi) g_{\mu\nu}. \end{aligned}$$

- We consider theory in terms of $g_{\mu\nu}$ ("Einstein frame"), not $\tilde{g}_{\mu\nu}$ ("Dark-matter frame"). That is, gravity is considered standard but dark matter feels additional fifth force ϕ.

- Let us calculate Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}\left(T^{\phi}_{\mu\nu} + T^{\rm b}_{\mu\nu} + T^{\rm c}_{\mu\nu}\right),$$

where

$$T^{\phi}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\phi})}{\delta g^{\mu\nu}}$$
$$T^{\rm b}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\rm b})}{\delta g^{\mu\nu}}$$
$$T^{\rm c}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-\tilde{g}}\tilde{\mathcal{L}}_{\rm c})}{\delta g^{\mu\nu}}$$

- Conservation equations:

$$\nabla^{\mu}T^{\rm b}_{\mu\nu} = 0,$$
$$\nabla^{\mu}\left(T^{\phi}_{\mu\nu} + T^{\rm c}_{\mu\nu}\right) = 0$$

$\Rightarrow$ only total EM tensor of dark sector conserved whereas individual components:

$$\nabla^{\mu}T^{\phi}_{\mu\nu} = -\nabla^{\mu}T^{\rm c}_{\mu\nu} \equiv -Q(\phi)T^{\rm c}\partial_{\nu}\phi,$$

where $T^{\rm c} \equiv g^{\mu\nu}T^{\rm c}_{\mu\nu}$ and

$$Q(\phi) = -\frac{1}{2C(\phi)}\frac{{\rm d}C(\phi)}{{\rm d}\phi}$$

is so called coupling function.

- Background equations:
  - Friedmann equation and baryonic-matter conservation remain standard

$$H^2 = \frac{8\pi G}{3}(\rho_{\phi} + \rho_{\rm b} + \rho_{\rm c})$$
$$\rho_{\rm b}' + 3H\rho_{\rm b} = 0$$

  - Dark matter and ϕ-conservation equation get modified:

$$\rho_{\phi}' + 3(1 + w_{\phi})\rho_{\phi} = Q\rho_{\rm c}\phi'$$
$$\rho_{\rm c}' + 3\rho_{\rm c} = -Q\rho_{\rm c}\phi'$$

  - Let us choose

$$C(\phi) \propto {\rm e}^{\beta\phi},$$

    so that $Q \sim$ const. Of course, if $Q > 0$, energy flows from DM to DE and, if $Q < 0$, from DE to DM.
    In our case, we choose $Q > 0$, i.e. $\beta < 0$, and for the sake of clarity $V(\phi) = V_0\phi^{-\alpha}$, with $\alpha > 0$ ("Peebles-Ratra potential") so that $\phi' > 0$.

- Can then solve conservation equation:

$$\rho_{\rm c} = \frac{\rho_{\rm c0}}{a^3} {\rm e}^{Q(\phi_0 - \phi)}$$

  ⇒ DM density decays exponentially with ϕ

- Solution to Hubble tension?
  - Due to exponential, DM energy density was larger in early times than in ΛCDM
  - Since during recombination era $H^2 \sim \rho_{\rm c}$, this implies larger Hubble function at early times and therefore smaller comoving sound horizon:

$$r_s = \int_0^{t_{\rm rec}} \frac{c_s {\rm d}t}{a} = \int_0^{a_{\rm rec}} \frac{c_s {\rm d}a}{a^2 H}$$

  - Remembering that Hubble factor is extracted from measurement of angular diameter distance:

$$H_0 \propto D_A^{-1} = \frac{\theta_s}{r_s},$$

    this might potentially increase the Hubble value measured from CMB
  - However, data analysis shows that this model can only slightly alleviate Hubble tension: $H_0 \approx 69\,{\rm km\,s^{-1}\,Mpc^{-1}}$

  ⇒ Generalise: $C(\phi)$? Non-canonical kinetic term? Disformal coupling?

Transient weak gravity in Coupled Dark Energy

- σ_8-parameter = "clustering strength"
  - σ_8-tension: σ_8 measured via CMB assuming ΛCDM larger than from measurements using large scale structure

  ⇒ want to weaken gravity

- However, typically in CDE with $Q \sim$ const:

$$\delta_{\rm c}'' + F\delta_{\rm c}' = \frac{3}{2}\Omega_{\rm c}\frac{G_{\rm eff}}{G_{\rm N}}\delta_{\rm c},$$

  with

$$G_{\rm eff} = G_{\rm N}\left(1 + 2M_P^2 Q^2 \frac{k^2}{k^2 + m_\phi^2}\right)$$

  Leads to real-space potential:

$$V(r) = -\frac{G_{\rm N} m}{r}\left(1 + 2Q^2 {\rm e}^{-m_\phi r}\right)$$

  ⇒ Yukawa correction that makes gravity even stronger ⇒ σ_8-tension gets worse.

- Our approach: allow ϕ-dependence of Q, e.g. consider

$$C(\phi) = \mathrm{e}^{m_C^{-2}\phi^2}.$$

Then close to minimum of C at $\phi = 0$, we have new mass scale:

$$\frac{\mathrm{d}Q}{\mathrm{d}\phi} = -\frac{1}{m_C^2}$$

This enters the equations in such a way that the resulting potential is

$$V(r) = -\frac{G_{\mathrm{N}}m}{r}\left[1 - \frac{2M_P^2(Q')^2}{\bar{M}^2}\left(1 - \mathrm{e}^{-\bar{M}r}\right)\right]$$

$\Rightarrow$ weakens gravity on large scales and could potentially alleviate σ_8-tension

Summary and Conclusions

- ΛCDM is good model but not perfect
- Hubble tension: $H_{0,\mathrm{CMB}+\Lambda\mathrm{CDM}} < H_{0,\mathrm{local}}$
- Scalar-tensor theories modify gravity via additional scalar degree of freedom
- Stability is a delicate issue
- Coupled Dark Energy can perhaps solve problems but needs more research
- There are many other good ST theories!

References and Further Reading

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